A NON-COHERENT RING

Brian Conrad gave me a beautiful example of a non-coherent ring (a ring not coherent over itself).

Consider the ring $\mathcal{O}_0$ of germs of smooth (C^∞) functions at $0 \in \mathbb{R}$, with coordinate x.

Lemma. The maximal ideal $\mathfrak{m}$ is generated by x. (You might recall that this is how one inductively produces the Taylor series for an analytic function.)

Proof. Suppose $f \in \mathfrak{m}$. Choose a representative of f, i.e. consider f as defined on $(-\epsilon, \epsilon)$. Now $f(0) = 0$. Then for $t \in (-\epsilon, \epsilon)$,

$$f(t) = \int_0^t f'(u)\ du = t \int_0^1 f'(tv)\ dv,$$

using $u = tv$. By "differentiating under the integral sign" as much as we want, we see that $\int_0^1 f'(tv)\ dv$ is smooth. (As always, the case $t = 0$ must be dealt with slightly differently.) $\square$

Now we show that $\mathcal{O}_0$ is not coherent over itself. Let $\phi \in \mathcal{O}_0$ be a smooth function that is 0 for $x \leq 0$ and > 0 for $x > 0$. (Example: e^{-1/x^2} for $x > 0$.) Consider the map $\times\phi : \mathcal{O}_0 \to \mathcal{O}_0$. The kernel is the ideal I_ϕ of functions vanishing for $x > 0$. It is clearly not 0 (e.g. $\phi(-x) \in I_\phi$). But by the lemma $I_\phi = xI_\phi$, which would contradict Nakayama's Lemma if I_ϕ were finitely generated.

Remark. One can use the same argument to show that the sheaf of smooth functions on $\mathbb{R}$ is not coherent. Alternatively, one could show this by showing that the stalks of coherent sheaves are coherent.

Remark. This shows that coherence has no useful meaning for smooth manifolds, unlike the case of real-analytic manifolds, complex manifolds, varieties, or (most reasonable) schemes. Note that coherence is trivial in the locally Noetherian situation, but very difficult in the (real- and complex-)analytic cases (because Oka's theorem is hard).

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