

Elementary proof of generic freeness

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30 July 2011

The following proof was taught me by Madhav Nori.

Lemma. Let A be a ring (commutative, with unit), and M an A -module that is an increasing union of submodules M_n . (Assume $M_0 = 0$.) Suppose that for all n , the quotient M_{n+1}/M_n is free. Then M is free.

Proof. First, we inductively construct compatible isomorphisms

$$\phi_n : \bigoplus_{i=1}^n M_i/M_{i-1} \rightarrow M_n.$$

The base case $n = 1$ is obvious.

For the induction step, consider the short exact sequence

$$0 \rightarrow M_{n-1} \rightarrow M_n \rightarrow M_n/M_{n-1} \rightarrow 0.$$

Since M_n/M_{n-1} is free, the sequence splits. Thus, there is a (non-natural) isomorphism

$$M_{n-1} \oplus M_n/M_{n-1} \rightarrow M_n.$$

Applying the isomorphism ϕ_{n-1} that exists by inductive hypothesis, we obtain the desired ϕ_n .

Since the ϕ_n are compatible, we may take

$$\phi := \varinjlim \phi_n : \bigoplus_{i=1}^{\infty} M_i/M_{i-1} \rightarrow M.$$

Since the \varinjlim is a functor, it takes isomorphisms to isomorphisms. Since $\{\phi_n\}$ is an isomorphism of directed systems, the direct limit ϕ is an isomorphism. (Alternately, it is easy to verify injectivity and surjectivity by looking at elements.) \square

Remark. The argument above actually shows that if the M_{n+1}/M_n are all projective, then M is (non-naturally) isomorphic to their direct sum.

We now progress to the main result. Throughout the following, let A be a fixed Noetherian integral domain. Call an A -algebra B “nice” if for every finite B -module M , there exists nonzero $f \in A$ such that M_f is free as an A_f -module.

Theorem. (Generic freeness) Every finitely generated A -algebra is nice.

Proof. We prove this theorem via a series of exercises.

Exercise 1. Show that A is nice as an A -algebra.

Exercise 2. Reduce to the following statement: If B is a nice Noetherian A -algebra, so is the polynomial ring $B[T]$. (Hint: induct on the number of generators of B as an A -algebra.)

Assume B is a nice A -algebra, and let M be a finite $B[T]$ -module. Let $S \subset M$ be a finite set that generates M as a $B[T]$ -module. Let M_1 be the sub- B -module of M generated by S . Inductively define

$$M_{n+1} = M_n + TM_n,$$

a sub- B -module of M . Observe that, as a B -module, M is the increasing union of the M_n .

Exercise 3. For $n \gg 0$, the B -module M_n/M_{n-1} is isomorphic to M_{n+1}/M_n . (Hint: Note that for all n , T defines a surjective B -module morphism $M_n/M_{n-1} \rightarrow M_{n+1}/M_n$. Apply the ascending chain condition on M_1 .)

Exercise 4. There exists nonzero $f \in A$ such that $(M_{n+1}/M_n)_f$ is free as an A_f -module, for all n . (As n ranges over the natural numbers, M_{n+1}/M_n only hits finitely many isomorphism classes.)

Applying the Lemma, we see that M_f is free as an A_f -module. □