# MATH 216: FOUNDATIONS OF ALGEBRAIC GEOMETRY

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# Contents

CHAPTER 1

# Introduction

*I can illustrate the .... approach with the ... image of a nut to be opened. The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you let time pass. The shell becomes more flexible through weeks and months — when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado!*

*A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration ... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it ... yet finally it surrounds the resistant substance.*

*— Grothendieck,* Récoltes et Semailles *p. 552-3, translation by Colin McLarty*

## 1.1 Goals

These are an updated version of notes accompanying a hard year-long class taught at Stanford in 2009-2010. I am currently editing them and adding a few more sections, and I hope a reasonably complete (if somewhat rough) version over the 2010-11 academic year at the site `http://math216.wordpress.com/`.

In any class, choices must be made as to what the course is about, and who it is for — there is a finite amount of time, and any addition of material or explanation or philosophy requires a corresponding subtraction. So these notes are highly inappropriate for most people and most classes. Here are my goals. (I do not claim that these goals are achieved; but they motivate the choices made.)

These notes currently have a very particular audience in mind: Stanford Ph.D. students, postdocs and faculty in a variety of fields, who may want to use algebraic geometry in a sophisticated way. This includes algebraic and arithmetic geometers, but also topologists, number theorists, symplectic geometers, and others.

The notes deal purely with the algebraic side of the subject, and completely neglect analytic aspects.

They assume little prior background (see §1.2), and indeed most students have little prior background. Readers with less background will necessarily have to work harder. It would be great if the reader had seen varieties before, but many students haven't, and the course does not assume it — and similarly for category theory, homological algebra, more advanced commutative algebra, differential geometry, .... Surprisingly often, what we need can be developed quickly from scratch. The cost is that the course is much denser; the benefit is that more people can follow it; they don't reach a point where they get thrown. (On the other hand, people who already have some familiarity with algebraic geometry, but want to understand the foundations more completely should not be bored, and will focus on more subtle issues.)

The notes seek to cover everything that one should see in a first course in the subject, including theorems, proofs, and examples.

They seek to be complete, and not leave important results as black boxes pulled from other references.

There are lots of exercises. I have found that unless I have some problems I can think through, ideas don't get fixed in my mind. Some are trivial — that's okay, and even desirable. As few necessary ones as possible should be hard, but the reader should have the background to deal with them — they are not just an excuse to push material out of the text.

There are optional (starred $\star$) sections of topics worth knowing on a second or third (but not first) reading. You should not read double-starred sections ($\star\star$) unless you really really want to, but you should be aware of their existence.

The notes are intended to be readable, although certainly not easy reading.

In short, after a year of hard work, students should have a broad familiarity with the foundations of the subject, and be ready to attend seminars, and learn more advanced material. They should not just have a vague intuitive understanding of the ideas of the subject; they should know interesting examples, know why they are interesting, and be able to prove interesting facts about them.

I have greatly enjoyed thinking through these notes, and teaching the corresponding classes, in a way I did not expect. I have had the chance to think through the structure of algebraic geometry from scratch, not blindly accepting the choices made by others. (Why do we need this notion? Aha, this forces us to consider this other notion earlier, and now I see why this third notion is so relevant...) I have repeatedly realized that ideas developed in Paris in the 1960's are simpler than I initially believed, once they are suitably digested.

**1.1.1. Implications.** We will work with as much generality as we need for most readers, and no more. In particular, we try to have hypotheses that are as general as possible without making proofs harder. The right hypotheses can make a proof easier, not harder, because one can remember how they get used. As an inflammatory example, the notion of quasiseparated comes up early and often. The cost is that one extra word has to be remembered, on top of an overwhelming number of other words. But once that is done, it is not hard to remember that essentially every scheme anyone cares about is quasiseparated. Furthermore, whenever the hypotheses "quasicompact and quasiseparated" turn up, the reader will likely immediately see a key idea of the proof.

Similarly, there is no need to work over an algebraically closed field, or even a field. Geometers needn't be afraid of arithmetic examples or of algebraic examples; a central insight of algebraic geometry is that the same formalism applies without change.

**1.1.2. Costs.** Choosing these priorities requires that others be shortchanged, and it is best to be up front about these. Because of our goal is to be comprehensive, and to understand everything one should know after a first course, it will necessarily take longer to get to interesting sample applications. You may be misled into thinking that one has to work this hard to get to these applications — it is not true!

## 1.2 Background and conventions

All rings are assumed to be commutative unless explicitly stated otherwise. All rings are assumed to contain a unit, denoted $1$. Maps of rings must send $1$ to $1$. We don't require that $0 \neq 1$; in other words, the "$0$-ring" (with one element) is a ring. (There is a ring map from any ring to the $0$-ring; the $0$-ring only maps to itself. The $0$-ring is the final object in the category of rings.) We accept the axiom of choice. In particular, any proper ideal in a ring is contained in a maximal ideal. (The axiom of choice also arises in the argument that the category of $A$-modules has enough injectives, see Exercise 23.2.E.)

The reader should be familiar with some basic notions in commutative ring theory, in particular the notion of ideals (including prime and maximal ideals) and localization. For example, the reader should be able to show that if $S$ is a multiplicative set of a ring $A$ (which we assume to contain $1$), then the primes of $S^{-1}A$ are in natural bijection with those primes of $A$ not meeting $S$ (§4.2.6). Tensor products and exact sequences of $A$-modules will be important. We will use the notation $(A, \mathfrak{m})$ or $(A, \mathfrak{m}, k)$ for local rings — $A$ is the ring, $\mathfrak{m}$ its maximal ideal, and $k = A/\mathfrak{m}$ its residue field. We will use (in Proposition 14.7.1) the structure theorem for finitely generated modules over a principal ideal domain $A$: any such module can be written as the direct sum of principal modules $A/(a)$.

**1.2.1.** *Caution about on foundational issues.* We will not concern ourselves with subtle foundational issues (set-theoretic issues involving universes, etc.). It is true that some people should be careful about these issues. (If you are one of these rare people, a good start is [**KS**, §1.1].)

**1.2.2.** *Further background.* It may be helpful to have books on other subjects handy that you can dip into for specific facts, rather than reading them in advance. In commutative algebra, Eisenbud [**E**] is good for this. Other popular choices are Atiyah-Macdonald [**AM**] and Matsumura [**M-CRT**]. For homological algebra, Weibel [**W**] is simultaneously detailed and readable.

Background from other parts of mathematics (topology, geometry, complex analysis) will of course be helpful for developing intuition.

Finally, it may help to keep the following quote in mind.

*Algebraic geometry seems to have acquired the reputation of being esoteric, exclusive, and very abstract, with adherents who are secretly plotting to take over all the rest of mathematics. In one respect this last point is accurate.*

*— David Mumford*

# Part I

# Preliminaries

CHAPTER 2

# Some category theory

*That which does not kill me, makes me stronger. — Nietzsche*

## 2.1 Motivation

Before we get to any interesting geometry, we need to develop a language to discuss things cleanly and effectively. This is best done in the language of categories. There is not much to know about categories to get started; it is just a very useful language. Like all mathematical languages, category theory comes with an embedded logic, which allows us to abstract intuitions in settings we know well to far more general situations.

Our motivation is as follows. We will be creating some new mathematical objects (such as schemes, and certain kinds of sheaves), and we expect them to act like objects we have seen before. We could try to nail down precisely what we mean by "act like", and what minimal set of things we have to check in order to verify that they act the way we expect. Fortunately, we don't have to — other people have done this before us, by defining key notions, such as *abelian categories*, which behave like modules over a ring.

Our general approach will be as follows. I will try to tell what you need to know, and no more. (This I promise: if I use the word "topoi", you can shoot me.) I will begin by telling you things you already know, and describing what is essential about the examples, in a way that we can abstract a more general definition. We will then see this definition in less familiar settings, and get comfortable with using it to solve problems and prove theorems.

For example, we will define the notion of *product* of schemes. We could just give a definition of product, but then you should want to know why this precise definition deserves the name of "product". As a motivation, we revisit the notion of product in a situation we know well: (the category of) sets. One way to define the product of sets $U$ and $V$ is as the set of ordered pairs $\{(u, v) : u \in U, v \in V\}$. But someone from a different mathematical culture might reasonably define it as the set of symbols $\{{u \atop v} : u \in U, v \in V\}$. These notions are "obviously the same". Better: there is "an obvious bijection between the two".

This can be made precise by giving a better definition of product, in terms of a *universal property*. Given two sets $M$ and $N$, a product is a set $P$, along with maps $\mu : P \to M$ and $\nu : P \to N$, such that for *any set* $P'$ *with maps* $\mu' : P' \to M$ *and*

$\nu' : P' \to N$, these maps must factor *uniquely* through $P$:

(2.1.0.1)
$$\begin{tikzcd} P' \arrow[dr, dotted, "\exists!"] \arrow[drr, "\nu'"] \arrow[ddr, "\mu'"'] & & \\ & P \arrow[r, "\nu"'] \arrow[d, "\mu"] & N \\ & M & \end{tikzcd}$$

(The symbol $\exists$ means "there exists", and the symbol ! here means "unique".) Thus a **product** is a *diagram*

$$\begin{tikzcd} P \arrow[r, "\nu"] \arrow[d, "\mu"'] & N \\ M & \end{tikzcd}$$

and not just a set $P$, although the maps $\mu$ and $\nu$ are often left implicit.

This definition agrees with the traditional definition, with one twist: there isn't just a single product; but any two products come with a *unique* isomorphism between them. In other words, the product is unique up to unique isomorphism. Here is why: if you have a product

$$\begin{tikzcd} P_1 \arrow[r, "\nu_1"] \arrow[d, "\mu_1"'] & N \\ M & \end{tikzcd}$$

and I have a product

$$\begin{tikzcd} P_2 \arrow[r, "\nu_2"] \arrow[d, "\mu_2"'] & N \\ M & \end{tikzcd}$$

then by the universal property of my product (letting $(P_2, \mu_2, \nu_2)$ play the role of $(P, \mu, \nu)$, and $(P_1, \mu_1, \nu_1)$ play the role of $(P', \mu', \nu')$ in (2.1.0.1)), there is a unique map $f : P_1 \to P_2$ making the appropriate diagram commute (i.e. $\mu_1 = \mu_2 \circ f$ and $\nu_1 = \nu_2 \circ f$). Similarly by the universal property of your product, there is a unique map $g : P_2 \to P_1$ making the appropriate diagram commute. Now consider the universal property of my product, this time letting $(P_2, \mu_2, \nu_2)$ play the role of both $(P, \mu, \nu)$ and $(P', \mu', \nu')$ in (2.1.0.1). There is a unique map $h : P_2 \to P_2$ such that

$$\begin{tikzcd} P_2 \arrow[dr, "h"] \arrow[drr, "\nu_2"] \arrow[ddr, "\mu_2"'] & & \\ & P_2 \arrow[r, "\nu_2"'] \arrow[d, "\mu_2"] & N \\ & M & \end{tikzcd}$$

commutes. However, I can name two such maps: the identity map $\mathrm{id}_{P_2}$, and $g \circ f$. Thus $g \circ f = \mathrm{id}_{P_2}$. Similarly, $f \circ g = \mathrm{id}_{P_1}$. Thus the maps $f$ and $g$ arising from the universal property are bijections. In short, there is a unique bijection between $P_1$ and $P_2$ preserving the "product structure" (the maps to $M$ and $N$). This gives us the right to name any such product $M \times N$, since any two such products are uniquely identified.

This definition has the advantage that it works in many circumstances, and once we define categories, we will soon see that the above argument applies verbatim in any category to show that products, if they exist, are unique up to unique isomorphism. Even if you haven't seen the definition of category before, you can verify that this agrees with your notion of product in some category that you have seen before (such as the category of vector spaces, where the maps are taken to be linear maps; or the category of smooth manifolds, where the maps are taken to be smooth maps).

This is handy even in cases that you understand. For example, one way of defining the product of two manifolds $M$ and $N$ is to cut them both up into charts, then take products of charts, then glue them together. But if I cut up the manifolds in one way, and you cut them up in another, how do we know our resulting manifolds are the "same"? We could wave our hands, or make an annoying argument about refining covers, but instead, we should just show that they are "categorical products" and hence canonically the "same" (i.e. isomorphic). We will formalize this argument in §2.3.

Another set of notions we will abstract are categories that "behave like modules". We will want to define kernels and cokernels for new notions, and we should make sure that these notions behave the way we expect them to. This leads us to the definition of *abelian categories*, first defined by Grothendieck in his Tôhoku paper **[Gr]**.

In this chapter, we'll give an informal introduction to these and related notions, in the hope of giving just enough familiarity to comfortably use them in practice.

## 2.2 Categories and functors

We begin with an informal definition of categories and functors.

**2.2.1. Categories.**

A **category** consists of a collection of **objects**, and for each pair of objects, a set of maps, or **morphisms** (or **arrows**), between them. The collection of objects of a category $\mathcal{C}$ are often denoted $\mathrm{obj}(\mathcal{C})$, but we will usually denote the collection also by $\mathcal{C}$. If $A, B \in \mathcal{C}$, then the morphisms from $A$ to $B$ are denoted $\mathrm{Mor}(A, B)$. A morphism is often written $f : A \to B$, and $A$ is said to be the **source** of $f$, and $B$ the **target** of $f$. (Of course, $\mathrm{Mor}(A, B)$ is taken to be disjoint from $\mathrm{Mor}(A', B')$ unless $A = A'$ and $B = B'$.)

Morphisms compose as expected: there is a composition $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$, and if $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, then their composition is denoted $g \circ f$. Composition is associative: if $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, and $h \in \mathrm{Mor}(C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$. For each object $A \in \mathcal{C}$, there is always an **identity morphism** $\mathrm{id}_A : A \to A$, such that when you (left- or right-)compose a morphism with the identity, you get the same morphism. More precisely, if $f : A \to B$ is a morphism, then $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$. (If you wish, you may check that "identity morphisms are unique": there is only one morphism deserving the name $\mathrm{id}_A$.)

If we have a category, then we have a notion of **isomorphism** between two objects (a morphism $f : A \to B$ such that there exists some — necessarily unique — morphism $g : B \to A$, where $f \circ g$ and $g \circ f$ are the identity on $B$ and $A$ respectively), and a notion of **automorphism** of an object (an isomorphism of the object with itself).

**2.2.2.** *Example.* The prototypical example to keep in mind is the category of sets, denoted *Sets*. The objects are sets, and the morphisms are maps of sets. (Because Russell's paradox shows that there is no set of all sets, we did not say earlier that there is a set of all objects. But as stated in §1.2, we are deliberately omitting all set-theoretic issues.)

**2.2.3.** *Example.* Another good example is the category $\mathit{Vec}_k$ of vector spaces over a given field $k$. The objects are $k$-vector spaces, and the morphisms are linear transformations. (What are the isomorphisms?)

**2.2.A.** UNIMPORTANT EXERCISE. A category in which each morphism is an isomorphism is called a **groupoid**. (This notion is not important in these notes. The point of this exercise is to give you some practice with categories, by relating them to an object you know well.)
(a) A perverse definition of a group is: a groupoid with one object. Make sense of this.
(b) Describe a groupoid that is not a group.

**2.2.B.** EXERCISE. If $A$ is an object in a category $\mathcal{C}$, show that the invertible elements of $\mathrm{Mor}(A, A)$ form a group (called the **automorphism group of** $A$, denoted $\mathrm{Aut}(A)$). What are the automorphism groups of the objects in Examples 2.2.2 and 2.2.3? Show that two isomorphic objects have isomorphic automorphism groups. (For readers with a topological background: if $X$ is a topological space, then the fundamental groupoid is the category where the objects are points of $x$, and the morphisms $x \to y$ are paths from $x$ to $y$, up to homotopy. Then the automorphism group of $x_0$ is the (pointed) fundamental group $\pi_1(X, x_0)$. In the case where $X$ is connected, and $\pi_1(X)$ is not abelian, this illustrates the fact that for a connected groupoid — whose definition you can guess — the automorphism groups of the objects are all isomorphic, but not canonically isomorphic.)

**2.2.4.** *Example: abelian groups.* The abelian groups, along with group homomorphisms, form a category *Ab*.

**2.2.5.** *Important example: modules over a ring.* If $A$ is a ring, then the $A$-modules form a category $\mathit{Mod}_A$. (This category has additional structure; it will be the prototypical example of an *abelian category*, see §2.6.) Taking $A = k$, we obtain Example 2.2.3; taking $A = \mathbb{Z}$, we obtain Example 2.2.4.

**2.2.6.** *Example: rings.* There is a category *Rings*, where the objects are rings, and the morphisms are morphisms of rings (which send 1 to 1 by our conventions, §1.2).

**2.2.7.** *Example: topological spaces.* The topological spaces, along with continuous maps, form a category *Top*. The isomorphisms are homeomorphisms.

In all of the above examples, the objects of the categories were in obvious ways sets with additional structure. This needn't be the case, as the next example shows.

**2.2.8.** *Example: partially ordered sets.* A **partially ordered set**, or **poset**, is a set $S$ along with a binary relation $\geq$ on $S$ satisfying:

(i) $x \geq x$ (reflexivity),
(ii) $x \geq y$ and $y \geq z$ imply $x \geq z$ (transitivity), and
(iii) if $x \geq y$ and $y \geq x$ then $x = y$.

A partially ordered set $(S, \geq)$ can be interpreted as a category whose objects are the elements of $S$, and with a single morphism from $x$ to $y$ if and only if $x \geq y$ (and no morphism otherwise).

A trivial example is $(S, \geq)$ where $x \geq y$ if and only if $x = y$. Another example is

(2.2.8.1)
$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Here there are three objects. The identity morphisms are omitted for convenience, and the two non-identity morphisms are depicted. A third example is

(2.2.8.2)
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Here the "obvious" morphisms are again omitted: the identity morphisms, and the morphism from the upper left to the lower right. Similarly,

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

depicts a partially ordered set, where again, only the "generating morphisms" are depicted.

**2.2.9.** *Example: the category of subsets of a set, and the category of open sets in a topological space.* If $X$ is a set, then the subsets form a partially ordered set, where the order is given by inclusion. Similarly, if $X$ is a topological space, then the *open* sets form a partially ordered set, where the order is given by inclusion.

**2.2.10.** *Example.* A **subcategory** $\mathcal{A}$ of a category $\mathcal{B}$ has as its objects some of the objects of $\mathcal{B}$, and some of the morphisms, such that the morphisms of $\mathcal{A}$ include the identity morphisms of the objects of $\mathcal{A}$, and are closed under composition. (For example, (2.2.8.1) is in an obvious way a subcategory of (2.2.8.2).)

**2.2.11. Functors.**

A **covariant functor** $F$ from a category $\mathcal{A}$ to a category $\mathcal{B}$, denoted $F : \mathcal{A} \to \mathcal{B}$, is the following data. It is a map of objects $F : \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$, and for each $A_1$, $A_2 \in \mathcal{A}$, and morphism $m : A_1 \to A_2$, a morphism $F(m) : F(A_1) \to F(A_2)$ in $\mathcal{B}$. We require that $F$ preserves identity morphisms (for $A \in \mathcal{A}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$), and that $F$ preserves composition ($F(m_1 \circ m_2) = F(m_1) \circ F(m_2)$). (You may wish to verify that covariant functors send isomorphisms to isomorphisms.)

If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are covariant functors, then we define a functor $G \circ F : \mathcal{A} \to \mathcal{C}$ in the obvious way. Composition of functors is associative in an evident sense.

**2.2.12.** *Example: a forgetful functor.* Consider the functor from the category of vector spaces (over a field $k$) $Vec_k$ to *Sets*, that associates to each vector space its underlying set. The functor sends a linear transformation to its underlying map of sets. This is an example of a **forgetful functor**, where some additional structure is forgotten. Another example of a forgetful functor is $Mod_A \to Ab$ from $A$-modules to abelian groups, remembering only the abelian group structure of the $A$-module.

**2.2.13.** *Topological examples.* Examples of covariant functors include the fundamental group functor $\pi_1$, which sends a topological space $X$ with choice of a point $x_0 \in X$ to a group $\pi_1(X, x_0)$ (what are the objects and morphisms of the source category?), and the $i$th homology functor $Top \to Ab$, which sends a topological space $X$ to its $i$th homology group $H_i(X, \mathbb{Z})$. The covariance corresponds to the fact that a (continuous) morphism of pointed topological spaces $f : X \to Y$ with $f(x_0) = y_0$ induces a map of fundamental groups $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, and similarly for homology groups.

**2.2.14.** *Example.* Suppose $A$ is an object in a category $\mathcal{C}$. Then there is a functor $h^A : \mathcal{C} \to Sets$ sending $B \in \mathcal{C}$ to $\text{Mor}(A, B)$, and sending $f : B_1 \to B_2$ to $\text{Mor}(A, B_1) \to \text{Mor}(A, B_2)$ described by

$$[g : A \to B_1] \mapsto [f \circ g : A \to B_1 \to B_2].$$

This seemingly silly functor ends up surprisingly being an important concept, and will come up repeatedly for us. (Warning only for experts: this is strictly speaking a lie: why should $\text{Mor}(A, B)$ be a set? But as stated in Caution 1.2.1, we will deliberately ignore these foundational issues, and we will in general pass them by without comment. Feel free to patch the problem on your time, perhaps by working in a *small category*, defined in §2.4.1. But don't be distracted from our larger goal.)

**2.2.15. Full and faithful functors.** A covariant functor $F : \mathcal{A} \to \mathcal{B}$ is **faithful** if for all $A, A' \in \mathcal{A}$, the map $\text{Mor}_{\mathcal{A}}(A, A') \to \text{Mor}_{\mathcal{B}}(F(A), F(A'))$ is injective, and **full** if it is surjective. A functor that is full and faithful is **fully faithful**. A subcategory $i : \mathcal{A} \to \mathcal{B}$ is a **full subcategory** if $i$ is full. Thus a subcategory $\mathcal{A}'$ of $\mathcal{A}$ is full if and only if for all $A, B \in \text{obj}(\mathcal{A}')$, $\text{Mor}_{\mathcal{A}'}(A, B) = \text{Mor}_{\mathcal{A}}(A, B)$.

**2.2.16. Definition.** A **contravariant functor** is defined in the same way as a covariant functor, except the arrows switch directions: in the above language, $F(A_1 \to A_2)$ is now an arrow from $F(A_2)$ to $F(A_1)$. (Thus $\mathcal{F}(m_2 \circ m_1) = \mathcal{F}(m_1) \circ \mathcal{F}(m_2)$, not $\mathcal{F}(m_2) \circ \mathcal{F}(m_1)$.)

It is wise to always state whether a functor is covariant or contravariant. If it is not stated, the functor is often assumed to be covariant.

(Sometimes people describe a contravariant functor $\mathcal{C} \to \mathcal{D}$ as a covariant functor $\mathcal{C}^{\text{opp}} \to \mathcal{D}$, where $\mathcal{C}^{\text{opp}}$ is the same category as $\mathcal{C}$ except that the arrows go in the opposite direction. Here $\mathcal{C}^{\text{opp}}$ is said to be the **opposite category** to $\mathcal{C}$.)

**2.2.17.** *Linear algebra example.* If $\mathit{Vec}_k$ is the category of $k$-vector spaces (introduced in Example 2.2.12), then taking duals gives a contravariant functor $\cdot^\vee : \mathit{Vec}_k \to \mathit{Vec}_k$. Indeed, to each linear transformation $f : V \to W$, we have a dual transformation $f^\vee : W^\vee \to V^\vee$, and $(f \circ g)^\vee = g^\vee \circ f^\vee$.

**2.2.18.** *Topological example (cf. Example 2.2.13).* The $i$th cohomology functor $H^i(\cdot, \mathbb{Z}) : \mathit{Top} \to \mathit{Ab}$ is a contravariant functor.

**2.2.19.** *Example.* There is a contravariant functor $\mathit{Top} \to \mathit{Rings}$ taking a topological space $X$ to the real-valued continuous functions on $X$. A morphism of topological spaces $X \to Y$ (a continuous map) induces the pullback map from functions on $Y$ to maps on $X$.

**2.2.20.** *Example (cf. Example 2.2.14).* Suppose $A$ is an object of a category $\mathcal{C}$. Then there is a contravariant functor $h_A : \mathcal{C} \to \mathit{Sets}$ sending $B \in \mathcal{C}$ to $\mathrm{Mor}(B, A)$, and sending the morphism $f : B_1 \to B_2$ to the morphism $\mathrm{Mor}(B_2, A) \to \mathrm{Mor}(B_1, A)$ via

$$[g : B_2 \to A] \mapsto [g \circ f : B_1 \to B_2 \to A].$$

This example initially looks weird and different, but Examples 2.2.17 and 2.2.19 may be interpreted as special cases; do you see how? What is $A$ in each case?

**2.2.21.** $\star$ **Natural transformations (and natural isomorphisms) of functors, and equivalences of categories.**

(This notion won't come up in an essential way until at least Chapter 7, so you shouldn't read this section until then.) Suppose $F$ and $G$ are two functors from $\mathcal{A}$ to $\mathcal{B}$. A **natural transformation of functors** $F \to G$ is the data of a morphism $m_a : F(a) \to G(a)$ for each $a \in \mathcal{A}$ such that for each $f : a \to a'$ in $\mathcal{A}$, the diagram

$$\begin{tikzcd} F(a) \arrow[r, "F(f)"] \arrow[d, "m_a"'] & F(a') \arrow[d, "m_{a'}"] \\ G(a) \arrow[r, "G(f)"'] & G(a') \end{tikzcd}$$

commutes. A **natural isomorphism** of functors is a natural transformation such that each $m_a$ is an isomorphism. The data of functors $F : \mathcal{A} \to \mathcal{B}$ and $F' : \mathcal{B} \to \mathcal{A}$ such that $F \circ F'$ is naturally isomorphic to the identity functor $I_{\mathcal{B}}$ on $\mathcal{B}$ and $F' \circ F$ is naturally isomorphic to $I_{\mathcal{A}}$ is said to be an **equivalence of categories**. "Equivalence of categories" is an equivalence relation on categories. The right meaning of when two categories are "essentially the same" is not isomorphism (a functor giving bijections of objects and morphisms) but an equivalence. Exercises 2.2.C and 2.2.D might give you some vague sense of this. Later exercises (for example, that "rings" and "affine schemes" are essentially the same, once arrows are reversed, Exercise 7.3.D) may help too.

Two examples might make this strange concept more comprehensible. The double dual of a finite-dimensional vector space $V$ is *not* $V$, but we learn early to say that it is canonically isomorphic to $V$. We can make that precise as follows. Let $\mathit{f.d.Vec}_k$ be the category of finite-dimensional vector spaces over $k$. Note that this category contains oodles of vector spaces of each dimension.

**2.2.C.** EXERCISE. Let $\cdot^{\vee\vee} : \textit{f.d.Vec}_k \to \textit{f.d.Vec}_k$ be the double dual functor from the category of finite-dimensional vector spaces over $k$ to itself. Show that $\cdot^{\vee\vee}$ is naturally isomorphic to the identity functor on $\textit{f.d.Vec}_k$. (Without the finite-dimensional hypothesis, we only get a natural transformation of functors from $\mathrm{id}$ to $\cdot^{\vee\vee}$.)

Let $\mathcal{V}$ be the category whose objects are $k^n$ for each $n$ (there is one vector space for each $n$), and whose morphisms are linear transformations. This latter space can be thought of as vector spaces with bases, and the morphisms are honest matrices. There is an obvious functor $\mathcal{V} \to \textit{f.d.Vec}_k$, as each $k^n$ *is* a finite-dimensional vector space.

**2.2.D.** EXERCISE. Show that $\mathcal{V} \to \textit{f.d.Vec}_k$ gives an equivalence of categories, by describing an "inverse" functor. (Recall that we are being cavalier about set-theoretic assumption, see Caution 1.2.1, so feel free to simultaneously choose bases for each vector space in $\textit{f.d.Vec}_k$. To make this precise, you will need to use Godel-Bernays set theory or else replace $\textit{f.d.Vec}_k$ with a very similar small category, but we won't worry about this.)

**2.2.22.** ⋆⋆ *Aside for experts.* Your argument for Exercise 2.2.D will show that (modulo set-theoretic issues) this definition of equivalence of categories is the same as another one commonly given: a covariant functor $F : A \to B$ is an equivalence of categories if it is fully faithful and every object of $B$ is isomorphic to an object of the form $F(a)$ ($F$ is *essentially surjective*). One can show that such a functor has a *quasiinverse*, i.e., that there is a functor $G : B \to A$, which is also an equivalence, and for which there exist natural isomorphisms $G(F(A)) \cong A$ and $F(G(B)) \cong B$.

## 2.3 Universal properties determine an object up to unique isomorphism

Given some category that we come up with, we often will have ways of producing new objects from old. In good circumstances, such a definition can be made using the notion of a *universal property*. Informally, we wish that there were an object with some property. We first show that if it exists, then it is essentially unique, or more precisely, is unique up to unique isomorphism. Then we go about constructing an example of such an object to show existence.

Explicit constructions are sometimes easier to work with than universal properties, but with a little practice, universal properties are useful in proving things quickly and slickly. Indeed, when learning the subject, people often find explicit construction more appealing, and use them more often in proofs, but as they become more experienced, find universal property arguments more elegant and insightful.

**2.3.1. Products were defined by universal property.** We have seen one important example of a universal property argument already in §2.1: products. You should go back and verify that our discussion there gives a notion of product in any category, and shows that products, *if they exist*, are unique up to unique isomorphism.

**2.3.2. Initial, final, and zero objects.** Here are some simple but useful concepts that will give you practice with universal property arguments. An object of a category $\mathcal{C}$ is an **initial object** if it has precisely one map to every object. It is a **final object** if it has precisely one map from every object. It is a **zero object** if it is both an initial object and a final object.

**2.3.A.** EXERCISE. Show that any two initial objects are uniquely isomorphic. Show that any two final objects are uniquely isomorphic.

In other words, *if* an initial object exists, it is unique up to unique isomorphism, and similarly for final objects. This (partially) justifies the phrase "*the* initial object" rather than "*an* initial object", and similarly for "*the* final object" and "*the* zero object".

**2.3.B.** EXERCISE. What are the initial and final objects in *Sets*, *Rings*, and *Top* (if they exist)? How about in the two examples of §2.2.9?

**2.3.3. Localization of rings and modules.** Another important example of a definition by universal property is the notion of *localization* of a ring. We first review a constructive definition, and then reinterpret the notion in terms of universal property. A multiplicative subset $S$ of a ring $A$ is a subset closed under multiplication containing $1$. We define a ring $S^{-1}A$. The elements of $S^{-1}A$ are of the form $a/s$ where $a \in A$ and $s \in S$, and where $a_1/s_1 = a_2/s_2$ if (and only if) *for some* $s \in S$, $s(s_2a_1 - s_1a_2) = 0$. (This implies that $S^{-1}A$ is the $0$-ring if $0 \in S$.) We define $(a_1/s_1) \times (a_2/s_2) = (a_1a_2)/(s_1s_2)$, and $(a_1/s_1) + (a_2/s_2) = (s_2a_1 + s_1a_2)/(s_1s_2)$. We have a canonical ring map $A \to S^{-1}A$ given by $a \mapsto a/1$.

There are two particularly important flavors of multiplicative subsets. The first is $\{1, f, f^2, \dots\}$, where $f \in A$. This localization is denoted $A_f$. The second is $A - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. This localization $S^{-1}A$ is denoted $A_\mathfrak{p}$. (Notational warning: If $\mathfrak{p}$ is a prime ideal, then $A_\mathfrak{p}$ means you're allowed to divide by elements not in $\mathfrak{p}$. However, if $f \in A$, $A_f$ means you're allowed to divide by $f$. This can be confusing. For example, if $(f)$ is a prime ideal, then $A_f \neq A_{(f)}$.)

Warning: sometimes localization is first introduced in the special case where $A$ is an integral domain and $0 \notin S$. In that case, $A \hookrightarrow S^{-1}A$, but this isn't always true, as shown by the following exercise. (But we will see that noninjective localizations needn't be pathological, and we can sometimes understand them geometrically, see Exercise 4.2.H.)

**2.3.C.** EXERCISE. Show that $A \to S^{-1}A$ is injective if and only if $S$ contains no zero-divisors. (A **zero-divisor** of a ring $A$ is an element $a$ such that there is a nonzero element $b$ with $ab = 0$. The other elements of $A$ are called **non-zero-divisors**. For example, a unit is never a zero-divisor. Counter-intuitively, $0$ is a zero-divisor in a ring $A$ if and only if $A$ is not the $0$-ring.)

If $A$ is an integral domain and $S = A \setminus \{0\}$, then $S^{-1}A$ is called the **fraction field** of $A$, which we denote $K(A)$. The previous exercise shows that $A$ is a subring of its fraction field $K(A)$. We now return to the case where $A$ is a general (commutative) ring.

**2.3.D.** EXERCISE. Verify that $A \to S^{-1}A$ satisfies the following universal property: $S^{-1}A$ is initial among $A$-algebras $B$ where every element of $S$ is sent to a unit in

B. (Recall: the data of "an A-algebra B" and "a ring map $A \to B$" the same.) Translation: any map $A \to B$ where every element of S is sent to a unit must factor uniquely through $A \to S^{-1}A$.

In fact, it is cleaner to *define* $A \to S^{-1}A$ by the universal property, and to show that it exists, and to use the universal property to check various properties $S^{-1}A$ has. Let's get some practice with this by *defining* localizations of modules by universal property. Suppose M is an A-module. We define the A-module map $\phi : M \to S^{-1}M$ as being initial among A-module maps $M \to N$ such that elements of S are invertible in N ($s \times \cdot : N \to N$ is an isomorphism for all $s \in S$). More precisely, any such map $\alpha : M \to N$ factors uniquely through $\phi$:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "\alpha"'] & S^{-1}M \arrow[d, dotted, "\exists!"] \\ & N \end{tikzcd}$$

Translation: $M \to S^{-1}M$ is universal (initial) among A-module maps from M to modules that are actually $S^{-1}A$-modules.

Notice: (i) this determines $\phi : M \to S^{-1}M$ up to unique isomorphism (you should think through what this means); (ii) we are defining not only $S^{-1}M$, but also the map $\phi$ at the same time; and (iii) essentially by definition the A-module structure on $S^{-1}M$ extends to an $S^{-1}A$-module structure.

**2.3.E.** EXERCISE. Show that $\phi : M \to S^{-1}M$ exists, by constructing something satisfying the universal property. Hint: define elements of $S^{-1}M$ to be of the form $m/s$ where $m \in M$ and $s \in S$, and $m_1/s_1 = m_2/s_2$ if and only if for some $s \in S$, $s(s_2m_1 - s_1m_2) = 0$. Define the additive structure by $(m_1/s_1) + (m_2/s_2) = (s_2m_1 + s_1m_2)/(s_1s_2)$, and the $S^{-1}A$-module structure (and hence the A-module structure) is given by $(a_1/s_1) \circ (m_2/s_2) = (a_1m_2)/(s_1s_2)$.

**2.3.F.** EXERCISE. Show that localization commutes with finite products. In other words, if $M_1, \dots, M_n$ are A-modules, describe an isomorphism $S^{-1}(M_1 \times \cdots \times M_n) \to S^{-1}M_1 \times \cdots \times S^{-1}M_n$. Show that localization does not necessarily commute with infinite products. (Hint: $(1, 1/2, 1/3, 1/4, \dots) \in \mathbb{Q} \times \mathbb{Q} \times \cdots$.)

**2.3.4. Tensor products.** Another important example of a universal property construction is the notion of a **tensor product** of A-modules

$$\otimes_A : \quad \mathrm{obj}(Mod_A) \times \mathrm{obj}(Mod_A) \longrightarrow \mathrm{obj}(Mod_A)$$

$$(M, N) \longmapsto M \otimes_A N$$

The subscript A is often suppressed when it is clear from context. The tensor product is often defined as follows. Suppose you have two A-modules M and N. Then elements of the tensor product $M \otimes_A N$ are finite A-linear combinations of symbols $m \otimes n$ ($m \in M$, $n \in N$), subject to relations $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$, $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$, $a(m \otimes n) = (am) \otimes n = m \otimes (an)$ (where $a \in A$, $m_1, m_2 \in M$, $n_1, n_2 \in N$). More formally, $M \otimes_A N$ is the free A-module generated by $M \times N$, quotiented by the submodule generated by $(m_1 + m_2) \otimes n -$

$m_1 \otimes n - m_2 \otimes n$, $m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2$, $a(m \otimes n) - (am) \otimes n$, and $a(m \otimes n) - m \otimes (an)$ for $a \in A$, $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$.

If $A$ is a field $k$, we recover the tensor product of vector spaces.

**2.3.G.** EXERCISE (IF YOU HAVEN'T SEEN TENSOR PRODUCTS BEFORE). Calculate $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12)$. (This exercise is intended to give some hands-on practice with tensor products.)

**2.3.H.** IMPORTANT EXERCISE: RIGHT-EXACTNESS OF $\cdot \otimes_A N$. Show that $\cdot \otimes_A N$ gives a covariant functor $Mod_A \to Mod_A$. Show that $\cdot \otimes_A N$ is a **right-exact functor**, i.e. if

$$M' \to M \to M'' \to 0$$

is an exact sequence of $A$-modules (which means $f : M \to M''$ is surjective, and $M'$ surjects onto the kernel of $f$; see §2.6), then the induced sequence

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

is also exact. This exercise is repeated in Exercise 2.6.F, but you may get a lot out of doing it now. (You will be reminded of the definition of right-exactness in §2.6.4.)

The constructive definition $\otimes$ is a weird definition, and really the "wrong" definition. To motivate a better one: notice that there is a natural $A$-bilinear map $M \times N \to M \otimes_A N$. (If $M, N, P \in Mod_A$, a map $f : M \times N \to P$ is $A$**-bilinear** if $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$, $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$, and $f(am, n) = f(m, an) = af(m, n)$.) *Any* $A$-bilinear map $M \times N \to C$ factors through the tensor product uniquely: $M \times N \to M \otimes_A N \to C$. (Think this through!)

We can take this as the *definition* of the tensor product as follows. It is an $A$-module $T$ along with an $A$-bilinear map $t : M \times N \to T$, such that given any $A$-bilinear map $t' : M \times N \to T'$, there is a unique $A$-linear map $f : T \to T'$ such that $t' = f \circ t$.

$$\begin{tikzcd} M \times N \arrow[rr, "t"] \arrow[dr, "t'"'] & & T \arrow[dl, dotted, "\exists ! f"] \\ & T' & \end{tikzcd}$$

**2.3.I.** EXERCISE. Show that $(T, t : M \times N \to T)$ is unique up to unique isomorphism. Hint: first figure out what "unique up to unique isomorphism" means for such pairs. Then follow the analogous argument for the product.

In short: given $M$ and $N$, there is an $A$-bilinear map $t : M \times N \to M \otimes_A N$, unique up to unique isomorphism, defined by the following universal property: for any $A$-bilinear map $t' : M \times N \to T'$ there is a unique $A$-linear map $f : M \otimes_A N \to T'$ such that $t' = f \circ t$.

As with all universal property arguments, this argument shows uniqueness *assuming existence*. To show existence, we need an explicit construction.

**2.3.J.** EXERCISE. Show that the construction of §2.3.4 satisfies the universal property of tensor product.

The two exercises below are some useful facts about tensor products with which you should be familiar.

**2.3.K.** IMPORTANT EXERCISE. (a) If $M$ is an $A$-module and $A \to B$ is a morphism of rings, show that $B \otimes_A M$ naturally has the structure of a $B$-module. Show that this describes a functor $Mod_A \to Mod_B$.
(b) If further $A \to C$ is a morphism of rings, show that $B \otimes_A C$ has the structure of a ring. Hint: multiplication will be given by $(b_1 \otimes c_1)(b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2)$. (Exercise 2.3.T will interpret this construction as a coproduct.)

**2.3.L.** IMPORTANT EXERCISE. If $S$ is a multiplicative subset of $A$ and $M$ is an $A$-module, describe a natural isomorphism $(S^{-1}A) \otimes_A M \cong S^{-1}M$ (as $S^{-1}A$-modules *and* as $A$-modules).

**2.3.5. Important Example: Fibered products.** (This notion will be essential later.) Suppose we have morphisms $f : X \to Z$ and $g : Y \to Z$ (in *any* category). Then the **fibered product** is an object $X \times_Z Y$ along with morphisms $\pi_X : X \times_Z Y \to X$ and $\pi_Y : X \times_Z Y \to Y$, where the two compositions $f \circ \pi_X, g \circ \pi_Y : X \times_Z Y \to Z$ agree, such that given any object $W$ with maps to $X$ and $Y$ (whose compositions to $Z$ agree), these maps factor through some unique $W \to X \times_Z Y$:

$$\begin{tikzcd}
W \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

(Warning: the definition of the fibered product depends on $f$ and $g$, even though they are omitted from the notation $X \times_Z Y$.)

By the usual universal property argument, if it exists, it is unique up to unique isomorphism. (You should think this through until it is clear to you.) Thus the use of the phrase "the fibered product" (rather than "a fibered product") is reasonable, and we should reasonably be allowed to give it the name $X \times_Z Y$. We know what maps to it are: they are precisely maps to $X$ and maps to $Y$ that agree as maps to $Z$.

Depending on your religion, the diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

is called a **fibered/pullback/Cartesian diagram/square** (six possibilities).

The right way to interpret the notion of fibered product is first to think about what it means in the category of sets.

**2.3.M.** EXERCISE. Show that in *Sets*,

$$X \times_Z Y = \{(x \in X, y \in Y) \ : \ f(x) = g(y)\}.$$

More precisely, show that the right side, equipped with its evident maps to $X$ and $Y$, satisfies the universal property of the fibered product. (This will help you build intuition for fibered products.)

**2.3.N.** EXERCISE. If $X$ is a topological space, show that fibered products always exist in the category of open sets of $X$, by describing what a fibered product is. (Hint: it has a one-word description.)

**2.3.O.** EXERCISE. If $Z$ is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, show that "$X \times_Z Y = X \times Y$": "the" fibered product over $Z$ is uniquely isomorphic to "the" product. (This is an exercise about unwinding the definition.)

**2.3.P.** USEFUL EXERCISE: TOWERS OF FIBER DIAGRAMS ARE FIBER DIAGRAMS. If the two squares in the following commutative diagram are fiber diagrams, show that the "outside rectangle" (involving $U$, $V$, $Y$, and $Z$) is also a fiber diagram.

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

**2.3.Q.** EXERCISE. Given $X \to Y \to Z$, show that there is a natural morphism $X \times_Y X \to X \times_Z X$, assuming that both fibered products exist. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

**2.3.R.** USEFUL EXERCISE: THE MAGIC DIAGRAM. Suppose we are given morphisms $X_1, X_2 \to Y$ and $Y \to Z$. Describe the natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$. Show that the following diagram is a fibered square.

$$\boxed{\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}}$$

This diagram is surprisingly incredibly useful — so useful that we will call it the **magic diagram**.

**2.3.6.** *Coproducts.* Define **coproduct** in a category by reversing all the arrows in the definition of product. Define **fibered coproduct** in a category by reversing all the arrows in the definition of fibered product.

**2.3.S.** EXERCISE. Show that coproduct for *Sets* is disjoint union. (This is why we use the notation $\coprod$ for disjoint union.)

**2.3.T.** EXERCISE. Suppose $A \to B, C$ are two ring morphisms, so in particular $B$ and $C$ are $A$-modules. Recall (Exercise 2.3.K) that $B \otimes_A C$ has a ring structure. Show that there is a natural morphism $B \to B \otimes_A C$ given by $b \mapsto b \otimes 1$. (This is not necessarily an inclusion, see Exercise 2.3.G.) Similarly, there is a natural morphism

$C \to B \otimes_A C$. Show that this gives a fibered coproduct on rings, i.e. that

$$\begin{tikzcd} B \otimes_A C & C \arrow[l] \\ B \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

satisfies the universal property of fibered coproduct.

**2.3.7. Monomorphisms and epimorphisms.**

**2.3.8. Definition.** A morphism $f : X \to Y$ is a **monomorphism** if any two morphisms $g_1, g_2 : Z \to X$ such that $f \circ g_1 = f \circ g_2$ must satisfy $g_1 = g_2$. In other words, for any other object $Z$, the natural map $\mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ is an injection. This a generalization of an injection of sets. In other words, there is at most one way of filling in the dotted arrow so that the following diagram commutes.

$$\begin{tikzcd} Z \arrow[d, dotted, "\leq 1"'] \arrow[dr] & \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Intuitively, it is the categorical version of an injective map, and indeed this notion generalizes the familiar notion of injective maps of sets. (The reason we don't use the word "injective" is that in some contexts, "injective" will have an intuitive meaning which may not agree with "monomorphism". This is also the case with "epimorphism" vs. "surjective".)

**2.3.U.** EXERCISE. Show that the composition of two monomorphisms is a monomorphism.

**2.3.V.** EXERCISE. Prove that a morphism $X \to Y$ is a monomorphism if and only if the induced morphism $X \to X \times_Y X$ is an isomorphism. We may then take this as the definition of monomorphism. (Monomorphisms aren't central to future discussions, although they will come up again. This exercise is just good practice.)

**2.3.W.** EXERCISE. Suppose $Y \to Z$ is a monomorphism, and $X_1, X_2 \to Y$ are two morphisms. Show that $X_1 \times_Y X_2$ and $X_1 \times_Z X_2$ are canonically isomorphic. We will use this later when talking about fibered products. (Hint: for any object $V$, give a natural bijection between maps from $V$ to the first and maps from $V$ to the second. It is also possible to use the magic diagram, Exercise 2.3.R.)

The notion of an **epimorphism** is "dual" to the definition of monomorphism, where all the arrows are reversed. This concept will not be central for us, although it turns up in the definition of an abelian category. Intuitively, it is the categorical version of a surjective map.

**2.3.9. Representable functors and Yoneda's lemma.** Much of our discussion about universal properties can be cleanly expressed in terms of representable functors, under the rubric of "Yoneda's Lemma". Yoneda's lemma is an easy fact stated in a complicated way. Informally speaking, you can essentially recover an object in a category by knowing the maps into it. For example, we have seen that the data of maps to $X \times Y$ are naturally (canonically) the data of maps to $X$ and to $Y$. Indeed, we have now taken this as the *definition* of $X \times Y$.

Recall Example 2.2.20. Suppose $A$ is an object of category $\mathcal{C}$. For any object $C \in \mathcal{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

$$\mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A), \tag{2.3.9.1}$$

by composition: given a map from $C$ to $A$, we get a map from $B$ to $A$ by precomposing with $f : B \to C$. Hence this gives a contravariant functor $h_A : \mathcal{C} \to \mathit{Sets}$. Yoneda's Lemma states that the functor $h_A$ determines $A$ up to unique isomorphism. More precisely:

**2.3.X.** IMPORTANT EXERCISE THAT EVERYONE SHOULD DO ONCE IN THEIR LIFE (YONEDA'S LEMMA). Given two objects $A$ and $A'$ in a category $\mathcal{C}$, and bijections

$$i_C : \mathrm{Mor}(C, A) \to \mathrm{Mor}(C, A') \tag{2.3.9.2}$$

that commute with the maps (2.3.9.1). Prove $i_C$ is induced from a unique isomorphism $A \to A'$. (Hint: This sounds hard, but it really is not. This statement is so general that there are really only a couple of things that you could possibly try. For example, if you're hoping to find an isomorphism $A \to A'$, where will you find it? Well, you are looking for an element $\mathrm{Mor}(A, A')$. So just plug in $C = A$ to (2.3.9.2), and see where the identity goes. You will quickly find the desired morphism; show that it is an isomorphism, then show that it is unique.)

There is an analogous statement with the arrows reversed, where instead of maps into $A$, you think of maps *from* $A$. The role of the contravariant functor $h_A$ of Example 2.2.20 is played by the covariant functor $h^A$ of Example 2.2.14. Because the proof is the same (with the arrows reversed), you needn't think it through.

Yoneda's lemma properly refers to a more general statement. Although it looks more complicated, it is no harder to prove.

**2.3.Y.** ⋆ EXERCISE.
(a) Suppose $A$ and $B$ are objects in a category $\mathcal{C}$. Give a bijection between the natural transformations $h^A \to h^B$ of covariant functors $\mathcal{C} \to \mathit{Sets}$ (see Example 2.2.14 for the definition) and the morphisms $B \to A$.
(b) State and prove the corresponding fact for contravariant functors $h_A$ (see Exercise 2.2.20). Remark: A contravariant functor $F$ from $\mathcal{C}$ to *Sets* is said to be **representable** if there is a natural isomorphism

$$\xi : F \xrightarrow{\sim} h_A .$$

Thus the representing object $A$ is determined up to unique isomorphism by the pair $(F, \xi)$. There is a similar definition for covariant functors. (We will revisit this in §7.6, and this problem will appear again as Exercise 7.6.B.)
(c) **Yoneda's lemma.** Suppose $F$ is a covariant functor $\mathcal{C} \to \mathit{Sets}$, and $A \in \mathcal{C}$. Give a bijection between the natural transformations $h^A \to F$ and $F(A)$. State the corresponding fact for contravariant functors.

In fancy terms, Yoneda's lemma states the following. Given a category $\mathcal{C}$, we can produce a new category, called the *functor category* of $\mathcal{C}$, where the objects are contravariant functors $\mathcal{C} \to \mathit{Sets}$, and the morphisms are natural transformations of such functors. We have a functor (which we can usefully call $h$) from $\mathcal{C}$ to its functor category, which sends $A$ to $h_A$. Yoneda's Lemma states that this is a fully faithful functor, called the *Yoneda embedding*. (Fully faithful functors were defined in §2.2.15.)

## 2.4 Limits and colimits

Limits and colimits provide two important examples defined by universal properties. They generalize a number of familiar constructions. I'll give the definition first, and then show you why it is familiar. For example, fractions will be motivating examples of colimits (Exercise 2.4.B(a)), and the $p$-adic numbers (Example 2.4.3) will be motivating examples of limits.

**2.4.1. Limits.** We say that a category is a **small category** if the objects and the morphisms are sets. (This is a technical condition intended only for experts.) Suppose $\mathcal{I}$ is any small category, and $\mathcal{C}$ is any category. Then a functor $F : \mathcal{I} \to \mathcal{C}$ (i.e. with an object $A_i \in \mathcal{C}$ for each element $i \in \mathcal{I}$, and appropriate commuting morphisms dictated by $\mathcal{I}$) is said to be a **diagram indexed by** $\mathcal{I}$. We call $\mathcal{I}$ an **index category**. Our index categories will be partially ordered sets (Example 2.2.8), in which in particular there is at most one morphism between any two objects. (But other examples are sometimes useful.) For example, if $\square$ is the category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

and $\mathcal{A}$ is a category, then a functor $\square \to \mathcal{A}$ is precisely the data of a commuting square in $\mathcal{A}$.

Then the **limit** is an object $\varprojlim_{\mathcal{I}} A_i$ of $\mathcal{C}$ along with morphisms $f_j : \varprojlim_{\mathcal{I}} A_i \to A_j$ such that if $m : j \to k$ is a morphism in $\mathcal{I}$, then

$$\begin{tikzcd} \varprojlim_{\mathcal{I}} A_i \arrow[d, "f_j"'] \arrow[dr, "f_k"] & \\ A_j \arrow[r, "F(m)"] & A_k \end{tikzcd}$$

commutes, and this object and maps to each $A_i$ are universal (final) with respect to this property. More precisely, given any other object $W$ along with maps $g_i : W \to A_i$ commuting with the $F(m)$ (if $m : i \to j$ is a morphism in $\mathcal{I}$, then $g_j = F(m) \circ g_i$), then there is a unique map $g : W \to \varprojlim_{\mathcal{I}} A_i$ so that $g_i = f_i \circ g$ for all $i$. (In some cases, the limit is sometimes called the **inverse limit** or **projective limit**. We won't use this language.) By the usual universal property argument, if the limit exists, it is unique up to unique isomorphism.

**2.4.2.** *Examples: products.* For example, if $\mathcal{I}$ is the partially ordered set

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

we obtain the fibered product.

If $\mathcal{I}$ is

$$\bullet \qquad \bullet$$

we obtain the product.

If $\mathcal{I}$ is a set (i.e. the only morphisms are the identity maps), then the limit is called the **product** of the $A_i$, and is denoted $\prod_i A_i$. The special case where $\mathcal{I}$ has two elements is the example of the previous paragraph.

If $\mathcal{I}$ has an initial object $e$, then $A_e$ is the limit, and in particular the limit always exists.

**2.4.3.** *Example: the* $p$*-adic numbers.* The $p$-adic numbers, $\mathbb{Z}_p$, are often described informally (and somewhat unnaturally) as being of the form $\mathbb{Z}_p = ? + ?p + ?p^2 + ?p^3 + \cdots$. They are an example of a limit in the category of rings:

$$\begin{tikzcd} \mathbb{Z}_p \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\ \cdots \arrow[r] & \mathbb{Z}/p^3 \arrow[r] & \mathbb{Z}/p^2 \arrow[r] & \mathbb{Z}/p \end{tikzcd}$$

Limits do not always exist for any index category $\mathcal{I}$. However, you can often easily check that limits exist if the objects of your category can be interpreted as sets with additional structure, and arbitrary products exist (respecting the set-like structure).

**2.4.A.** Important Exercise. Show that in the category *Sets*,

$$\left\{ (a_i)_{i \in I} \in \prod_i A_i : F(m)(a_i) = a_j \text{ for all } m \in \mathrm{Mor}_{\mathcal{I}}(i,j) \in \mathrm{Mor}(\mathcal{I}) \right\},$$

along with the obvious projection maps to each $A_i$, is the limit $\varprojlim_{\mathcal{I}} A_i$.

This clearly also works in the category $\mathit{Mod}_A$ of $A$-modules, and its specializations such as $\mathit{Vec}_k$ and *Ab*.

From this point of view, $2 + 3p + 2p^2 + \cdots \in \mathbb{Z}_p$ can be understood as the sequence $(2, 2 + 3p, 2 + 3p + 2p^2, \dots)$.

**2.4.4. Colimits.** More immediately relevant for us will be the dual (arrow-reversed version) of the notion of limit (or inverse limit). We just flip all the arrows in that definition, and get the notion of a **colimit**. Again, if it exists, it is unique up to unique isomorphism. (In some cases, the colimit is sometimes called the **direct limit**, **inductive limit**, or **injective limit**. We won't use this language. I prefer using limit/colimit in analogy with kernel/cokernel and product/coproduct. This is more than analogy, as kernels and products may be interpreted as limits, and similarly with cokernels and coproducts. Also, I remember that kernels "map to", and cokernels are "mapped to", which reminds me that a limit maps *to* all the objects in the big commutative diagram indexed by $\mathcal{I}$; and a colimit has a map *from* all the objects.)

Even though we have just flipped the arrows, colimits behave quite differently from limits.

**2.4.5.** *Example.* The group $5^{-\infty}\mathbb{Z}$ of rational numbers whose denominators are powers of 5 is a colimit $\varinjlim 5^{-i}\mathbb{Z}$. More precisely, $5^{-\infty}\mathbb{Z}$ is the colimit of

$$\mathbb{Z} \longrightarrow 5^{-1}\mathbb{Z} \longrightarrow 5^{-2}\mathbb{Z} \longrightarrow \cdots$$

The colimit over an index *set* I is called the **coproduct**, denoted $\coprod_i A_i$, and is the dual (arrow-reversed) notion to the product.

**2.4.B.** EXERCISE. (a) Interpret the statement "$\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$". (b) Interpret the union of the some subsets of a given set as a colimit. (Dually, the intersection can be interpreted as a limit.) The objects of the category in question are the subsets of the given set.

Colimits don't always exist, but there are two useful large classes of examples for which they do.

**2.4.6. Definition.** A nonempty partially ordered set $(S, \geq)$ is **filtered** (or is said to be a **filtered set**) if for each $x, y \in S$, there is a $z$ such that $x \geq z$ and $y \geq z$. More generally, a nonempty category $\mathcal{I}$ is **filtered** if:

(i) for each $x, y \in \mathcal{I}$, there is a $z \in \mathcal{I}$ and arrows $x \to z$ and $y \to z$, and
(ii) for every two arrows $u, v : x \to y$, there is an arrow $w : y \to z$ such that $w \circ u = w \circ v$.

(Other terminologies are also commonly used, such as "directed partially ordered set" and "filtered index category", respectively.)

**2.4.C.** EXERCISE. Suppose $\mathcal{I}$ is filtered. (We will almost exclusively use the case where $\mathcal{I}$ is a filtered set.) Show that any diagram in *Sets* indexed by $\mathcal{I}$ has the following as a colimit:

$$\left\{ a \in \coprod_{i \in \mathcal{I}} A_i \right\} \Big/ \ (a_i \in A_i) \sim (f(a_i) \in A_j) \text{ for every } f : A_i \to A_j \text{ in the diagram.}$$

(Hint: Verify that $\sim$ is indeed an equivalence relation, by writing it as $(a_i \in A_i) \sim (a_j \in A_j)$ if there are $f : A_i \to A_k$ and $g : A_j \to A_k$ with $f(a_i) = g(a_j)$.)

This idea applies to many categories whose objects can be interpreted as sets with additional structure (such as abelian groups, $A$-modules, groups, etc.). For example, in Example 2.4.5, each element of the colimit is an element of something upstairs, but you can't say in advance what it is an element of. For example, 17/125 is an element of the $5^{-3}\mathbb{Z}$ (or $5^{-4}\mathbb{Z}$, or later ones), but not $5^{-2}\mathbb{Z}$. More generally, in the category of $A$-modules $Mod_A$, each element $a$ of the colimit $\varinjlim A_i$ can be interpreted as an element of *some* $a \in A_i$. The element $a \in \varinjlim A_i$ is 0 if there is some $m : i \to j$ such that $F(m)(a) = 0$ (i.e. if it becomes 0 "later in the diagram"). Furthermore, two elements interpreted as $a_i \in A_i$ and $a_j \in A_j$ are the same if there are some arrows $m : i \to k$ and $n : j \to k$ such that $F(m)(a_i) = F(n)(a_j)$, i.e. if they become the same "later in the diagram". To add two elements interpreted as $a_i \in A_i$ and $a_j \in A_j$, we choose arrows $m : i \to k$ and $n : j \to k$, and then interpret their sum as $F(m)(a_i) + F(n)(a_j)$.

**2.4.D.** EXERCISE. Verify that the $A$-module described above is indeed the colimit.

**2.4.E.** USEFUL EXERCISE (LOCALIZATION AS COLIMIT). Generalize Exercise 2.4.B(a) to interpret localization of an integral domain as a colimit over a filtered set: suppose $S$ is a multiplicative set of $A$, and interpret $S^{-1}A = \varinjlim \frac{1}{s}A$ where the limit is over $s \in S$. (Aside: Can you make some version of this work even if $A$ isn't an integral domain, e.g. $S^{-1}A = \varinjlim A_s$?)

A variant of this construction works without the filtered condition, if you have another means of "connecting elements in different objects of your diagram". For example:

**2.4.F.** EXERCISE: COLIMITS OF $A$-MODULES WITHOUT THE FILTERED CONDITION. Suppose you are given a diagram of $A$-modules indexed by $\mathcal{I}$: $F : \mathcal{I} \to Mod_A$, where we let $A_i := F(i)$. Show that the colimit is $\oplus_{i \in \mathcal{I}} A_i$ modulo the relations $a_j - F(m)(a_i)$ for every $m : i \to j$ in $\mathcal{I}$ (i.e. for every arrow in the diagram).

The following exercise shows that you have to be careful to remember which category you are working in.

**2.4.G.** UNIMPORTANT EXERCISE. Consider the filtered set of abelian groups $p^{-n}\mathbb{Z}_p/\mathbb{Z}_p$ (here $p$ is a fixed prime, and $n$ varies — you should be able to figure out the index set). Show that this system has colimit $\mathbb{Q}_p/\mathbb{Z}_p$ in the category of abelian groups, and the colimit $0$ in the category of finite abelian groups. Here $\mathbb{Q}_p$ is the fraction field of $\mathbb{Z}_p$, which can be interpreted as $\cup p^{-n}\mathbb{Z}_p$.

**2.4.7. Summary.** One useful thing to informally keep in mind is the following. In a category where the objects are "set-like", an element of a limit can be thought of as an element in each object in the diagram, that are "compatible" (Exercise 2.4.A). And an element of a colimit can be thought of ("has a representative that is") an element of a single object in the diagram (Exercise 2.4.C). Even though the definitions of limit and colimit are the same, just with arrows reversed, these interpretations are quite different.

## 2.5 Adjoints

We next come to a very useful construction closely related to universal properties. Just as a universal property "essentially" (up to unique isomorphism) determines an object in a category (assuming such an object exists), "adjoints" essentially determine a functor (again, assuming it exists). Two *covariant* functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are **adjoint** if there is a natural bijection for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\tau_{AB} : \mathrm{Mor}_{\mathcal{B}}(F(A), B) \to \mathrm{Mor}_{\mathcal{A}}(A, G(B)). \tag{2.5.0.1}$$

We say that $(F, G)$ form an **adjoint pair**, and that $F$ is **left-adjoint** to $G$ (and $G$ is **right-adjoint** to $F$). By "natural" we mean the following. For all $f : A \to A'$ in $\mathcal{A}$, we require

(2.5.0.2)
$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{B}}(F(A'),B) \arrow[r, "Ff^*"] \arrow[d, "\tau_{A'B}"] & \mathrm{Mor}_{\mathcal{B}}(F(A),B) \arrow[d, "\tau_{AB}"] \\ \mathrm{Mor}_{\mathcal{A}}(A',G(B)) \arrow[r, "f^*"] & \mathrm{Mor}_{\mathcal{A}}(A,G(B)) \end{tikzcd}$$

to commute, and for all $g : B \to B'$ in $\mathcal{B}$ we want a similar commutative diagram to commute. (Here $f^*$ is the map induced by $f : A \to A'$, and $Ff^*$ is the map induced by $Ff : F(A) \to F(A')$.)

**2.5.A.** EXERCISE. Write down what this diagram should be. (Hint: do it by extending diagram (2.5.0.2) above.)

**2.5.B.** EXERCISE. Show that the map $\tau_{AB}$ (2.5.0.1) is given as follows. For each $A$ there is a map $\eta_A : A \to GF(A)$ so that for any $g : F(A) \to B$, the corresponding $f : A \to G(B)$ is given by the composition

$$A \xrightarrow{\eta_A} GF(A) \xrightarrow{Gg} G(B).$$

Similarly, there is a map $\epsilon_B : FG(B) \to B$ for each $B$ so that for any $f : A \to G(B)$, the corresponding map $g : F(A) \to B$ is given by the composition

$$F(A) \xrightarrow{Ff} FG(B) \xrightarrow{\epsilon_B} B.$$

Here is an example of an adjoint pair.

**2.5.C.** EXERCISE. Suppose $M$, $N$, and $P$ are $A$-modules. Describe a bijection $\mathrm{Hom}_A(M \otimes_A N, P) \leftrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_A(N,P))$. (Hint: try to use the universal property.)

**2.5.D.** EXERCISE. Show that $\cdot \otimes_A N$ and $\mathrm{Hom}_A(N, \cdot)$ are adjoint functors.

**2.5.1.** ⋆ *Fancier remarks we won't use.* You can check that the left adjoint determines the right adjoint up to unique natural isomorphism, and vice versa, by a universal property argument. The maps $\eta_A$ and $\epsilon_B$ of Exercise 2.5.B are called the **unit** and **counit** of the adjunction. This leads to a different characterization of adjunction. Suppose functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are given, along with natural transformations $\epsilon : FG \to \mathrm{id}$ and $\eta : \mathrm{id} \to GF$ with the property that $G\epsilon \circ \eta G = \mathrm{id}_G$ (for each $B \in \mathcal{B}$, the composition of $\eta_{G(B)} : G(B) \to GFG(B)$ and $G(\epsilon_B) : GFG(B) \to G(B)$ is the identity) and $\eta F \circ F\epsilon = \mathrm{id}_F$. Then you can check that $F$ is left adjoint to $G$. These facts aren't hard to check, so if you want to use them, you should verify everything for yourself.

**2.5.2.** *Examples from other fields.* For those familiar with representation theory: Frobenius reciprocity may be understood in terms of adjoints. Suppose $V$ is a finite-dimensional representation of a finite group $G$, and $W$ is a representation of a subgroup $H < G$. Then induction and restriction are an adjoint pair $(\mathrm{Ind}_H^G, \mathrm{Res}_H^G)$ between the category of $G$-modules and the category of $H$-modules.

Topologists' favorite adjoint pair may be the suspension functor and the loop space functor.

**2.5.3. Example: groupification.** Here is another motivating example: getting an abelian group from an abelian semigroup. An abelian semigroup is just like an abelian group, except you don't require an inverse. One example is the non-negative integers $0, 1, 2, \ldots$ under addition. Another is the positive integers under multiplication $1, 2, \ldots$. From an abelian semigroup, you can create an abelian group. Here is a formalization of that notion. If $S$ is a semigroup, then its **groupification** is a map of semigroups $\pi : S \to G$ such that $G$ is a group, and any other map of semigroups from $S$ to a *group* $G'$ factors *uniquely* through $G$.

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "\pi"'] & G \arrow[d, dotted, "\exists!"] \\ & G' \end{tikzcd}$$

**2.5.E.** EXERCISE. Construct groupification $H$ from the category of abelian semigroups to the category of abelian groups. (One possibility of a construction: given an abelian semigroup $S$, the elements of its groupification $H(S)$ are $(a, b)$, which you may think of as $a - b$, with the equivalence that $(a, b) \sim (c, d)$ if $a + d + e = b + c + e$ for some $e \in S$. Describe addition in this group, and show that it satisfies the properties of an abelian group. Describe the semigroup map $S \to H(S)$.) Let $F$ be the forgetful morphism from the category of abelian groups *Ab* to the category of abelian semigroups. Show that $H$ is left-adjoint to $F$.

(Here is the general idea for experts: We have a full subcategory of a category. We want to "project" from the category to the subcategory. We have

$$\mathrm{Mor}_{\text{category}}(S, H) = \mathrm{Mor}_{\text{subcategory}}(G, H)$$

automatically; thus we are describing the left adjoint to the forgetful functor. How the argument worked: we constructed something which was in the smaller category, which automatically satisfies the universal property.)

**2.5.F.** EXERCISE. Show that if a semigroup is *already* a group then the identity morphism is the groupification ("the semigroup is groupified by itself"), by the universal property. (Perhaps better: the identity morphism is *a* groupification — but we don't want tie ourselves up in knots over categorical semantics.)

**2.5.G.** EXERCISE. The purpose of this exercise is to give you some practice with "adjoints of forgetful functors", the means by which we get groups from semigroups, and sheaves from presheaves. Suppose $A$ is a ring, and $S$ is a multiplicative subset. Then $S^{-1}A$-modules are a fully faithful subcategory of the category of $A$-modules (meaning: the objects of the first category are a subset of the objects of the second; and the morphisms between any two objects of the second that are secretly objects of the first are just the morphisms from the first). Then $Mod_A \to Mod_{S^{-1}A}$ can be interpreted as an adjoint to the forgetful functor $Mod_{S^{-1}A} \to Mod_A$. Figure out the correct statement, and prove that it holds.

(Here is the larger story. Every $S^{-1}A$-module is an $A$-module, and this is an injective map, so we have a covariant forgetful functor $F : Mod_{S^{-1}A} \to Mod_A$. In fact this is a fully faithful functor: it is injective on objects, and the morphisms between any two $S^{-1}A$-modules *as* $A$*-modules* are just the same when they are considered as $S^{-1}A$-modules. Then there is a functor $G : Mod_A \to Mod_{S^{-1}A}$, which might reasonably be called "localization with respect to $S$", which is left-adjoint to the forgetful functor. Translation: If $M$ is an $A$-module, and $N$ is an $S^{-1}A$-module, then $\mathrm{Mor}(GM, N)$ (morphisms as $S^{-1}A$-modules, which are the same as morphisms as $A$-modules) are in natural bijection with $\mathrm{Mor}(M, FN)$ (morphisms as $A$-modules).)

Here is a table of adjoints that will come up for us.

| situation | category $\mathcal{A}$ | category $\mathcal{B}$ | left-adjoint $F : \mathcal{A} \to \mathcal{B}$ | right-adjoint $G : \mathcal{B} \to \mathcal{A}$ |
|---|---|---|---|---|
| $A$-modules (Ex. 2.5.D) | | | $\cdot \otimes_A N$ | $\mathrm{Hom}_A(N, \cdot)$ |
| ring maps $A \to B$ | $Mod_A$ | $Mod_B$ | $\cdot \otimes_A B$ (extension of scalars) | forgetful (restriction of scalars) |
| (pre)sheaves on a topological space $X$ (Ex. 3.4.K) | presheaves on $X$ | sheaves on $X$ | sheafification | forgetful |
| (semi)groups (§2.5.3) | semigroups | groups | groupification | forgetful |
| sheaves, $f : X \to Y$ (Ex. 3.6.B) | sheaves on $Y$ | sheaves on $X$ | $f^{-1}$ | $f_*$ |
| sheaves of abelian groups or $\mathcal{O}$-modules, open immersions $f : U \hookrightarrow Y$ (Ex. 3.6.G) | sheaves on $U$ | sheaves on $Y$ | $f_!$ | $f^{-1}$ |
| quasicoherent sheaves, $f : X \to Y$ (Prop. 17.3.5) | quasicoherent sheaves on $Y$ | quasicoherent sheaves on $X$ | $f^*$ | $f_*$ |

Other examples will also come up, such as the adjoint pair $(\sim, \Gamma_\bullet)$ between graded modules over a graded ring, and quasicoherent sheaves on the corresponding projective scheme (§16.4).

**2.5.4.** *Useful comment for experts.* One last comment only for people who have seen adjoints before: If $(F, G)$ is an adjoint pair of functors, then $F$ commutes with colimits, and $G$ commutes with limits. Also, limits commute with limits and colimits commute with colimits. We will prove these facts (and a little more) in §2.6.10.

## 2.6 Kernels, cokernels, and exact sequences: A brief introduction to abelian categories

Since learning linear algebra, you have been familiar with the notions and behaviors of kernels, cokernels, etc. Later in your life you saw them in the category of abelian groups, and later still in the category of $A$-modules. Each of these notions generalizes the previous one.

We will soon define some new categories (certain sheaves) that will have familiar-looking behavior, reminiscent of that of modules over a ring. The notions of kernels, cokernels, images, and more will make sense, and they will behave "the way we expect" from our experience with modules. This can be made precise through the notion of an *abelian category*. Abelian categories are the right general setting in which one can do "homological algebra", in which notions of kernel, cokernel, and so on are used, and one can work with complexes and exact sequences.

We will see enough to motivate the definitions that we will see in general: monomorphism (and subobject), epimorphism, kernel, cokernel, and image. But in these notes we will avoid having to show that they behave "the way we expect" in a general abelian category because the examples we will see are directly interpretable in terms of modules over rings. In particular, it is not worth memorizing the definition of abelian category.

Two central examples of an abelian category are the category *Ab* of abelian groups, and the category $Mod_A$ of $A$-modules. The first is a special case of the second (just take $A = \mathbb{Z}$). As we give the definitions, you should verify that $Mod_A$ is an abelian category.

We first define the notion of *additive category*. We will use it only as a stepping stone to the notion of an abelian category.

**2.6.1. Definition.** A category $\mathcal{C}$ is said to be **additive** if it satisfies the following properties.

- Ad1. For each $A, B \in \mathcal{C}$, $\mathrm{Mor}(A, B)$ is an abelian group, such that composition of morphisms distributes over addition. (You should think about what this means — it translates to two distinct statements).
- Ad2. $\mathcal{C}$ has a zero object, denoted $0$. (This is an object that is simultaneously an initial object and a final object, Definition 2.3.2.)
- Ad3. It has products of two objects (a product $A \times B$ for any pair of objects), and hence by induction, products of any finite number of objects.

In an additive category, the morphisms are often called homomorphisms, and Mor is denoted by Hom. In fact, this notation Hom is a good indication that you're working in an additive category. A functor between additive categories preserving the additive structure of Hom, is called an **additive functor**.

**2.6.2.** *Remarks.* It is a consequence of the definition of additive category that finite direct products are also finite direct sums (coproducts) — the details don't matter to us. The symbol $\oplus$ is used for this notion. Also, it is quick to show that additive functors send zero objects to zero objects (show that $a$ is a $0$-object if and only if $\mathrm{id}_a = 0_a$; additive functors preserve both $\mathrm{id}$ and $0$), and preserves products.

One motivation for the name $0$-object is that the $0$-morphism in the abelian group $\mathrm{Hom}(A, B)$ is the composition $A \to 0 \to B$.

Real (or complex) Banach spaces are an example of an additive category. The category of free $A$-modules is another. The category of $A$-modules $Mod_A$ is also an example, but it has even more structure, which we now formalize as an example of an abelian category.

**2.6.3. Definition.** Let $\mathcal{C}$ be an additive category. A **kernel** of a morphism $f : B \to C$ is a map $i : A \to B$ such that $f \circ i = 0$, and that is universal with respect to this property. Diagramatically:

$$\begin{tikzcd} Z \arrow[d, dotted, "\exists!"'] \arrow[dr] \arrow[drr, "0"] & & \\ A \arrow[r, "i"] \arrow[rr, bend right, "0"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

(Note that the kernel is not just an object; it is a morphism of an object to $B$.) Hence it is unique up to unique isomorphism by universal property nonsense. A **cokernel** is defined dually by reversing the arrows — do this yourself. The kernel of $f : B \to C$ is the limit (§2.4) of the diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

and similarly the cokernel is a colimit.

If $i : A \to B$ is a monomorphism, then we say that $A$ is a **subobject** of $B$, where the map $i$ is implicit. Dually, there is the notion of **quotient object**, defined dually to subobject.

An **abelian category** is an additive category satisfying three additional properties.

(1) Every map has a kernel and cokernel.
(2) Every monomorphism is the kernel of its cokernel.
(3) Every epimorphism is the cokernel of its kernel.

It is a non-obvious (and imprecisely stated) fact that every property you want to be true about kernels, cokernels, etc. follows from these three. (Warning: in part of the literature, additional hypotheses are imposed as part of the definition.)

The **image** of a morphism $f : A \to B$ is defined as $\mathrm{im}(f) = \ker(\operatorname{coker} f)$. It is the unique factorization

$$A \xrightarrow{\text{epi.}} \mathrm{im}(f) \xrightarrow{\text{mono.}} B$$

It is the cokernel of the kernel, and the kernel of the cokernel. The reader may want to verify this as an exercise. It is unique up to unique isomorphism. The cokernel of a monomorphism is called the **quotient**.

We will leave the foundations of abelian categories untouched. The key thing to remember is that if you understand kernels, cokernels, images and so on in the category of modules over a ring $Mod_A$, you can manipulate objects in any abelian category. This is made precise by Freyd-Mitchell Embedding Theorem. (The Freyd-Mitchell Embedding Theorem: If $\mathcal{A}$ is an abelian category such that $\mathrm{Hom}(a, a')$ is a set for all $a, a' \in \mathcal{A}$, then there is a ring A and an exact, fully faithful functor from $\mathcal{A}$ into $Mod_A$, which embeds $\mathcal{A}$ as a full subcategory. A proof is sketched in [**W**, §1.6], and references to a complete proof are given there. The moral is that to prove something about a diagram in some abelian category, we may pretend that it is a diagram of modules over some ring, and we may then "diagram-chase" elements. Moreover, any fact about kernels, cokernels, and so on that holds in $Mod_A$ holds in any abelian category.) However, the abelian categories we will come across will obviously be related to modules, and our intuition will clearly carry over, so we needn't invoke a theorem whose proof we haven't read. For example, we'll show that sheaves of abelian groups on a topological space $X$ form an abelian category (§3.5), and the interpretation in terms of "compatible germs" will connect notions of kernels, cokernels etc. of sheaves of abelian groups to the corresponding notions of abelian groups.

**2.6.4. Complexes, exactness, and homology.**

We say a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{2.6.4.1}$$

is a **complex** if $g \circ f = 0$, and is **exact** if $\ker g = \operatorname{im} f$. An exact sequence with five terms, the first and last of which are $0$, is a **short exact sequence**. Note that $A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$ being exact is equivalent to describing $C$ as a cokernel of $f$ (with a similar statement for $0 \longrightarrow A \longrightarrow B \xrightarrow{g} C$ ).

If you would like practice in playing with these notions before thinking about homology, you can prove the Snake Lemma (stated in Example 2.7.5, with a stronger version in Exercise 2.7.B), or the Five Lemma (stated in Example 2.7.6, with a stronger version in Exercise 2.7.C).

If (2.6.4.1) is a complex, then its **homology** (often denoted $H$) is $\ker g \,/\, \operatorname{im} f$. We say that the $\ker g$ are the **cycles**, and $\operatorname{im} f$ are the **boundaries** (so homology is "cycles mod boundaries"). If the complex is indexed in decreasing order, the indices are often written as subscripts, and $H_i$ is the homology at $A_{i+1} \to A_i \to A_{i-1}$. If the complex is indexed in increasing order, the indices are often written as superscripts, and the homology $H^i$ at $A^{i-1} \to A^i \to A^{i+1}$ is often called **cohomology**.

An exact sequence

$$A^\bullet : \qquad \cdots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots \tag{2.6.4.2}$$

can be "factored" into short exact sequences

$$0 \longrightarrow \ker f^i \longrightarrow A^i \longrightarrow \ker f^{i+1} \longrightarrow 0$$

which is helpful in proving facts about long exact sequences by reducing them to facts about short exact sequences.

More generally, if (2.6.4.2) is assumed only to be a complex, then it can be "factored" into short exact sequences.

$$0 \longrightarrow \ker f^i \longrightarrow A^i \longrightarrow \operatorname{im} f^i \longrightarrow 0 \tag{2.6.4.3}$$

$$0 \longrightarrow \operatorname{im} f^{i-1} \longrightarrow \ker f^i \longrightarrow H^i(A^\bullet) \longrightarrow 0$$

**2.6.A.** EXERCISE. Describe exact sequences

$$0 \longrightarrow \operatorname{im} f^i \longrightarrow A^{i+1} \longrightarrow \operatorname{coker} f^i \longrightarrow 0 \tag{2.6.4.4}$$

$$0 \longrightarrow H^i(A^\bullet) \longrightarrow \operatorname{coker} f^{i-1} \longrightarrow \operatorname{im} f^i \longrightarrow 0$$

(These are somehow dual to (2.6.4.3). In fact in some mirror universe this might have been given as the standard definition of homology.)

**2.6.B.** EXERCISE. Suppose

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of finite-dimensional $k$-vector spaces (often called $A^\bullet$ for short). Show that $\sum(-1)^i \dim A^i = \sum(-1)^i h^i(A^\bullet)$. (Recall that $h^i(A^\bullet) = \dim \ker(d^i)/\operatorname{im}(d^{i-1})$.) In particular, if $A^\bullet$ is exact, then $\sum(-1)^i \dim A^i = 0$. (If you haven't dealt much with cohomology, this will give you some practice.)

**2.6.C.** IMPORTANT EXERCISE. Suppose $\mathcal{C}$ is an abelian category. Define the category $Com_{\mathcal{C}}$ as follows. The objects are infinite complexes

$$A^\bullet : \qquad \cdots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots$$

in $\mathcal{C}$, and the morphisms $A^\bullet \to B^\bullet$ are commuting diagrams

$$\begin{tikzcd} A^\bullet : \arrow[d] & \cdots \arrow[r] & A^{i-1} \arrow[r, "f^{i-1}"] \arrow[d] & A^i \arrow[r, "f^i"] \arrow[d] & A^{i+1} \arrow[r, "f^{i+1}"] \arrow[d] & \cdots \\ B^\bullet : & \cdots \arrow[r] & B^{i-1} \arrow[r, "g^{i-1}"] & B^i \arrow[r, "g^i"] & B^{i+1} \arrow[r, "g^{i+1}"] & \cdots \end{tikzcd} \tag{2.6.4.5}$$

Show that $Com_{\mathcal{C}}$ is an abelian category. (Feel free to deal with the special case $Mod_A$.)

**2.6.D.** IMPORTANT EXERCISE. Show that (2.6.4.5) induces a map of homology $H(A^i) \to H(B^i)$. (Again, feel free to deal with the special case $Mod_A$.)

We will later define when two maps of complexes are *homotopic* (§23.1), and show that homotopic maps induce isomorphisms on cohomology (Exercise 23.1.A), but we won't need that any time soon.

**2.6.5. Theorem (Long exact sequence). —** *A short exact sequence of complexes*

$$\begin{tikzcd} 0^\bullet : \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ A^\bullet : \arrow[d] & \cdots \arrow[r] & A^{i-1} \arrow[r, "f^{i-1}"] \arrow[d] & A^i \arrow[r, "f^i"] \arrow[d] & A^{i+1} \arrow[r, "f^{i+1}"] \arrow[d] & \cdots \\ B^\bullet : \arrow[d] & \cdots \arrow[r] & B^{i-1} \arrow[r, "g^{i-1}"] \arrow[d] & B^i \arrow[r, "g^i"] \arrow[d] & B^{i+1} \arrow[r, "g^{i+1}"] \arrow[d] & \cdots \\ C^\bullet : \arrow[d] & \cdots \arrow[r] & C^{i-1} \arrow[r, "h^{i-1}"] \arrow[d] & C^i \arrow[r, "h^i"] \arrow[d] & C^{i+1} \arrow[r, "h^{i+1}"] \arrow[d] & \cdots \\ 0^\bullet : & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

*induces a* **long exact sequence in cohomology**

$$\begin{tikzcd} & \cdots \arrow[r] & H^{i-1}(C^\bullet) \arrow[r] & {} \\ H^i(A^\bullet) \arrow[r] & H^i(B^\bullet) \arrow[r] & H^i(C^\bullet) \arrow[r] & {} \\ H^{i+1}(A^\bullet) \arrow[r] & \cdots & & \end{tikzcd}$$

(This requires a definition of the *connecting homomorphism* $H^{i-1}(C^\bullet) \to H^i(A^\bullet)$, which is natural in an appropriate sense.) For a concise proof in the case of complexes of modules, and a discussion of how to show this in general, see [**W**, §1.3]. It will also come out of our discussion of spectral sequences as well (again, in the category of modules over a ring), see Exercise 2.7.E, but this is a somewhat perverse way of proving it.

**2.6.6.** *Exactness of functors.* If $F : \mathcal{A} \to \mathcal{B}$ is a covariant additive functor from one abelian category to another, we say that $F$ is **right-exact** if the exactness of

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

in $\mathcal{A}$ implies that

$$F(A') \longrightarrow F(A) \longrightarrow F(A'') \longrightarrow 0$$

is also exact. Dually, we say that $F$ is **left-exact** if the exactness of

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \qquad \text{implies}$$

$$0 \longrightarrow F(A') \longrightarrow F(A) \longrightarrow F(A'') \qquad \text{is exact.}$$

A contravariant functor is **left-exact** if the exactness of

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow 0 \qquad \text{implies}$$

$$0 \longrightarrow F(A'') \longrightarrow F(A) \longrightarrow F(A') \qquad \text{is exact.}$$

The reader should be able to deduce what it means for a contravariant functor to be **right-exact**.

A covariant or contravariant functor is **exact** if it is both left-exact and right-exact.

**2.6.E.** EXERCISE. Suppose $F$ is an exact functor. Show that applying $F$ to an exact sequence preserves exactness. For example, if $F$ is covariant, and $A' \to A \to A''$ is exact, then $FA' \to FA \to FA''$ is exact. (This will be generalized in Exercise 2.6.H(c).)

**2.6.F.** EXERCISE. Suppose $A$ is a ring, $S \subset A$ is a multiplicative subset, and $M$ is an $A$-module.

(a) Show that localization of $A$-modules $Mod_A \to Mod_{S^{-1}A}$ is an exact covariant

functor.
(b) Show that $\cdot \otimes M$ is a right-exact covariant functor $Mod_A \to Mod_A$. (This is a repeat of Exercise 2.3.H.)
(c) Show that $\mathrm{Hom}(M, \cdot)$ is a left-exact covariant functor $Mod_A \to Mod_A$.
(d) Show that $\mathrm{Hom}(\cdot, M)$ is a left-exact contravariant functor $Mod_A \to Mod_A$.

**2.6.G.** EXERCISE. Suppose $M$ is a **finitely presented** $A$-module: $M$ has a finite number of generators, and with these generators it has a finite number of relations; or usefully equivalently, fits in an exact sequence

$$A^{\oplus q} \to A^{\oplus p} \to M \to 0 \tag{2.6.6.1}$$

Use (2.6.6.1) and the left-exactness of Hom to describe an isomorphism

$$S^{-1}\,\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

(You might be able to interpret this in light of a variant of Exercise 2.6.H below, for left-exact contravariant functors rather than right-exact covariant functors.)

**2.6.7.** $\star$ **Two useful facts in homological algebra.**

We now come to two (sets of) facts I wish I had learned as a child, as they would have saved me lots of grief. They encapsulate what is best and worst of abstract nonsense. The statements are so general as to be nonintuitive. The proofs are very short. They generalize some specific behavior it is easy to prove in an ad hoc basis. Once they are second nature to you, many subtle facts will become obvious to you as special cases. And you will see that they will get used (implicitly or explicitly) repeatedly.

**2.6.8.** $\star$ *Interaction of homology and (right/left-)exact functors.*

You might wait to prove this until you learn about cohomology in Chapter 20, when it will first be used in a serious way.

**2.6.H.** IMPORTANT EXERCISE (THE FHHF THEOREM). This result can take you far, and perhaps for that reason it has sometimes been called the fernbahnhof (FernbaHnHoF) theorem. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a covariant functor of abelian categories. Suppose $C^\bullet$ is a complex in $\mathcal{A}$.

(a) *($F$ right-exact yields $FH^\bullet \longrightarrow H^\bullet F$ )* If $F$ is right-exact, describe a natural morphism $FH^\bullet \to H^\bullet F$. (More precisely, for each $i$, the left side is $F$ applied to the cohomology at piece $i$ of $C^\bullet$, while the right side is the cohomology at piece $i$ of $FC^\bullet$.)
(b) *($F$ left-exact yields $FH^\bullet \longleftarrow H^\bullet F$ )* If $F$ is left-exact, describe a natural morphism $H^\bullet F \to FH^\bullet$.
(c) *($F$ exact yields $FH^\bullet \longleftrightarrow H^\bullet F$)* If $F$ is exact, show that the morphisms of (a) and (b) are inverses and thus isomorphisms.

Hint for (a): use $C^p \xrightarrow{d^p} C^{p+1} \longrightarrow \mathrm{coker}\, d^p \longrightarrow 0$ to give an isomorphism $F\,\mathrm{coker}\, d^p \cong \mathrm{coker}\, Fd^p$. Then use the first line of (2.6.4.4) to give a surjection $F\,\mathrm{im}\, d^p \twoheadrightarrow \mathrm{im}\, Fd^p$. Then use the second line of (2.6.4.4) to give the desired map $FH^pC^\bullet \longrightarrow H^pFC^\bullet$. While you are at it, you may as well describe a map for the fourth member of the quartet $\{\ker, \mathrm{coker}, \mathrm{im}, H, \}$: $F\ker d^p \longrightarrow \ker Fd^p$.

**2.6.9.** If this makes your head spin, you may prefer to think of it in the following specific case, where both $\mathcal{A}$ and $\mathcal{B}$ are the category of A-modules, and $F$ is $\cdot \otimes N$ for some fixed N-module. Your argument in this case will translate without change to yield a solution to Exercise 2.6.H(a) and (c) in general. If $\otimes N$ is exact, then $N$ is called a **flat** A-module. (The notion of flatness will turn out to be very important, and is discussed in detail in Chapter 24.)

For example, localization is exact, so $S^{-1}A$ is a *flat* A-algebra for all multiplicative sets $S$. Thus taking cohomology of a complex of A-modules commutes with localization — something you could verify directly.

**2.6.10.** $\star$ *Interaction of adjoints, (co)limits, and (left- and right-) exactness.*

A surprising number of arguments boil down to the statement:

*Limits commute with limits and right-adjoints. In particular, because kernels are limits, both right-adjoints and limits are left exact.*

as well as its dual:

*Colimits commute with colimits and left-adjoints. In particular, because cokernels are colimits, both left-adjoints and colimits are right exact.*

These statements were promised in §2.5.4. The latter has a useful extension:

*In an abelian category, colimits over filtered index categories are exact.*

("Filtered" was defined in §2.4.6.) If you want to use these statements (for example, later in these notes), you will have to prove them. Let's now make them precise.

**2.6.I.** EXERCISE (KERNELS COMMUTE WITH LIMITS). Suppose $\mathcal{C}$ is an abelian category, and $a : \mathcal{I} \to \mathcal{C}$ and $b : \mathcal{I} \to \mathcal{C}$ are two diagrams in $\mathcal{C}$ indexed by $\mathcal{I}$. For convenience, let $A_i = a(i)$ and $B_i = b(i)$ be the objects in those two diagrams. Let $h_i : A_i \to B_i$ be maps commuting with the maps in the diagram. (Translation: $h$ is a natural transformation of functors $a \to b$, see §2.2.21.) Then the $\ker h_i$ form another diagram in $\mathcal{I}$ indexed by $\mathcal{I}$. Describe a natural isomorphism $\varprojlim \ker h_i \cong \ker(\varprojlim A_i \to \varprojlim B_i)$.

**2.6.J.** EXERCISE. Make sense of the statement that "limits commute with limits" in a general category, and prove it. (Hint: recall that kernels are limits. The previous exercise should be a corollary of this one.)

**2.6.11. Proposition (right-adjoints commute with limits). —** *Suppose* $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ *is a pair of adjoint functors. If* $A = \varprojlim A_i$ *is a limit in* $\mathcal{D}$ *of a diagram indexed by* $I$*, then* $GA = \varprojlim GA_i$ *(with the corresponding maps* $GA \to GA_i$*) is a limit in* $\mathcal{C}$*.*

*Proof.* We must show that $GA \to GA_i$ satisfies the universal property of limits. Suppose we have maps $W \to GA_i$ commuting with the maps of $\mathcal{I}$. We wish to show that there exists a unique $W \to GA$ extending the $W \to GA_i$. By adjointness of $F$ and $G$, we can restate this as: Suppose we have maps $FW \to A_i$ commuting with the maps of $\mathcal{I}$. We wish to show that there exists a unique $FW \to A$ extending the $FW \to A_i$. But this is precisely the universal property of the limit. $\square$

Of course, the dual statements to Exercise 2.6.J and Proposition 2.6.11 hold by the dual arguments.

If $F$ and $G$ are additive functors between abelian categories, and $(F, G)$ is an adjoint pair, then (as kernels are limits and cokernels are colimits) $G$ is left-exact and $F$ is right-exact.

**2.6.K.** EXERCISE. Show that in $Mod_A$, colimits over filtered index categories are exact. (Your argument will apply without change to any abelian category whose objects can be interpreted as "sets with additional structure".) Right-exactness follows from the above discussion, so the issue is left-exactness. (Possible hint: After you show that localization is exact, Exercise 2.6.F(a), or sheafification is exact, Exercise 3.5.D, in a hands on way, you will be easily able to prove this. Conversely, if you do this exercise, those two will be easy.)

**2.6.L.** EXERCISE. Show that filtered colimits commute with homology. Hint: use the FHHF Theorem (Exercise 2.6.H), and the previous Exercise.

In light of Exercise 2.6.L, you may want to think about how limits (and colimits) commute with homology in general, and which way maps go; The statement of the FHHF Theorem should suggest the answer. (Are limits analogous to left-exact functors, or right-exact functors?) We won't directly use this insight.

**2.6.12.** $\star$ **Dreaming of derived functors.** When you see a left-exact functor, you should always dream that you are seeing the end of a long exact sequence. If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence in abelian category $\mathcal{A}$, and $F : \mathcal{A} \to \mathcal{B}$ is a left-exact functor, then

$$0 \to FM' \to FM \to FM''$$

is exact, and you should always dream that it should continue in some natural way. For example, the next term should depend only on $M'$, call it $R^1FM'$, and if it is zero, then $FM \to FM''$ is an epimorphism. This remark holds true for left-exact and contravariant functors too. In good cases, such a continuation exists, and is incredibly useful. We will discuss this in Chapter 23.

## 2.7 $\star$ Spectral sequences

Spectral sequences are a powerful book-keeping tool for proving things involving complicated commutative diagrams. They were introduced by Leray in the 1940's at the same time as he introduced sheaves. They have a reputation for being abstruse and difficult. It has been suggested that the name 'spectral' was given because, like spectres, spectral sequences are terrifying, evil, and dangerous. I have heard no one disagree with this interpretation, which is perhaps not surprising since I just made it up.

Nonetheless, the goal of this section is to tell you enough that you can use spectral sequences without hesitation or fear, and why you shouldn't be frightened when they come up in a seminar. What is perhaps different in this presentation is that we will use spectral sequence to prove things that you may have already seen, and that you can prove easily in other ways. This will allow you to get some hands-on experience for how to use them. We will also see them only in the special case of double complexes (which is the version by far the most often used in algebraic geometry), and not in the general form usually presented (filtered complexes, exact couples, etc.). See [**W**, Ch. 5] for more detailed information if you wish.

You should *not* read this section when you are reading the rest of Chapter 2. Instead, you should read it just before you need it for the first time. When you finally *do* read this section, you *must* do the exercises.

For concreteness, we work in the category $\mathit{Vec}_k$ of vector spaces over a field $k$. However, everything we say will apply in any abelian category, such as the category $\mathit{Mod}_A$ of $A$-modules.

**2.7.1. Double complexes.**

A **double complex** is a collection of vector spaces $E^{p,q}$ ($p, q \in \mathbb{Z}$), and "rightward" morphisms $d_{\rightarrow}^{p,q} : E^{p,q} \to E^{p,q+1}$ and "upward" morphisms $d_{\uparrow}^{p,q} : E^{p,q} \to E^{p+1,q}$. In the superscript, the first entry denotes the row number, and the second entry denotes the column number, in keeping with the convention for matrices, but opposite to how the $(x, y)$-plane is labeled. The subscript is meant to suggest the direction of the arrows. We will always write these as $d_{\rightarrow}$ and $d_{\uparrow}$ and ignore the superscripts. We require that $d_{\rightarrow}$ and $d_{\uparrow}$ satisfying (a) $d_{\rightarrow}^2 = 0$, (b) $d_{\uparrow}^2 = 0$, and one more condition: (c) either $d_{\rightarrow} d_{\uparrow} = d_{\uparrow} d_{\rightarrow}$ (all the squares commute) or $d_{\rightarrow} d_{\uparrow} + d_{\uparrow} d_{\rightarrow} = 0$ (they all anticommute). Both come up in nature, and you can switch from one to the other by replacing $d_{\uparrow}^{p,q}$ with $(-1)^q d_{\uparrow}^{p,q}$. So I will assume that all the squares anticommute, but that you know how to turn the commuting case into this one. (You will see that there is no difference in the recipe, basically because the image and kernel of a homomorphism $f$ equal the image and kernel respectively of $-f$.)

$$\begin{tikzcd}
E^{p+1,q} \arrow[r, "d_{\rightarrow}^{p+1,q}"] & E^{p+1,q+1} \\
E^{p,q} \arrow[u, "d_{\uparrow}^{p,q}"] \arrow[r, "d_{\rightarrow}^{p,q}"] \arrow[ur, phantom, "\text{anticommutes}"] & E^{p,q+1} \arrow[u, "d_{\uparrow}^{p,q+1}"']
\end{tikzcd}$$

There are variations on this definition, where for example the vertical arrows go downwards, or some different subset of the $E^{p,q}$ are required to be zero, but I will leave these straightforward variations to you.

From the double complex we construct a corresponding (single) complex $E^{\bullet}$ with $E^k = \oplus_i E^{i,k-i}$, with $d = d_{\rightarrow} + d_{\uparrow}$. In other words, when there is a *single* superscript $k$, we mean a sum of the $k$th antidiagonal of the double complex. The single complex is sometimes called the **total complex**. Note that $d^2 = (d_{\rightarrow} + d_{\uparrow})^2 = d_{\rightarrow}^2 + (d_{\rightarrow} d_{\uparrow} + d_{\uparrow} d_{\rightarrow}) + d_{\uparrow}^2 = 0$, so $E^{\bullet}$ is indeed a complex.

The **cohomology** of the single complex is sometimes called the **hypercohomology** of the double complex. We will instead use the phrase "cohomology of the double complex".

Our initial goal will be to find the cohomology of the double complex. You will see later that we secretly also have other goals.

A spectral sequence is a recipe for computing some information about the cohomology of the double complex. I won't yet give the full recipe. Surprisingly, this fragmentary bit of information is sufficent to prove lots of things.

**2.7.2. Approximate Definition.** A **spectral sequence** with **rightward orientation** is a sequence of tables or **pages** ${}_{\rightarrow}E_0^{p,q}, {}_{\rightarrow}E_1^{p,q}, {}_{\rightarrow}E_2^{p,q}, \dots$ ($p, q \in \mathbb{Z}$), where ${}_{\rightarrow}E_0^{p,q} = E^{p,q}$, along with a differential

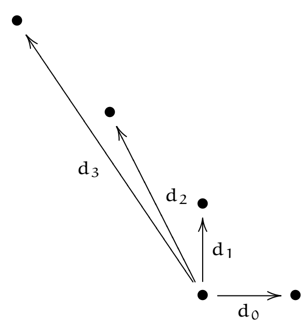
$${}_{\rightarrow}d_r^{p,q} : {}_{\rightarrow}E_r^{p,q} \rightarrow {}_{\rightarrow}E_r^{p+r,q-r+1}$$

with ${}_{\rightarrow}d_r^{p,q} \circ {}_{\rightarrow}d_r^{p,q} = 0$, and with an isomorphism of the cohomology of ${}_{\rightarrow}d_r$ at ${}_{\rightarrow}E_r^{p,q}$ (i.e. $\ker {}_{\rightarrow}d_r^{p,q} / \operatorname{im} {}_{\rightarrow}d_r^{p-r,q+r-1}$) with ${}_{\rightarrow}E_{r+1}^{p,q}$.

The orientation indicates that our 0th differential is the rightward one: $d_0 = d_{\rightarrow}$. The left subscript "$\rightarrow$" is usually omitted.

The order of the morphisms is best understood visually:

(2.7.2.1)




(the morphisms each apply to different pages). Notice that the map always is "degree 1" in the grading of the single complex $E^{\bullet}$.

The actual definition describes what $E_r^{\bullet,\bullet}$ and $d_r^{\bullet,\bullet}$ really are, in terms of $E^{\bullet,\bullet}$. We will describe $d_0$, $d_1$, and $d_2$ below, and you should for now take on faith that this sequence continues in some natural way.

Note that $E_r^{p,q}$ is always a subquotient of the corresponding term on the 0th page $E_0^{p,q} = E^{p,q}$. In particular, if $E^{p,q} = 0$, then $E_r^{p,q} = 0$ for all $r$, so $E_r^{p,q} = 0$ unless $p, q \in \mathbb{Z}^{\geq 0}$.

Suppose now that $E^{\bullet,\bullet}$ is a **first quadrant double complex**, i.e. $E^{p,q} = 0$ for $p < 0$ or $q < 0$. Then for any fixed $p, q$, once $r$ is sufficiently large, $E_{r+1}^{p,q}$ is computed from $(E_r^{\bullet,\bullet}, d_r)$ using the complex

$$0 \xrightarrow{d_r^{p+r,q-r-1}} E_r^{p,q} \xrightarrow{d_r^{p,q}} 0$$

and thus we have canonical isomorphisms

$$E_r^{p,q} \cong E_{r+1}^{p,q} \cong E_{r+2}^{p,q} \cong \cdots$$

We denote this module $E_\infty^{p,q}$. The same idea works in other circumstances, for example if the double complex is only nonzero in a finite number of rows — $E^{p,q} = 0$ unless $p_0 < p < p_q$. This will come up for example in the long exact sequence and mapping cone discussion (Exercises 2.7.E and 2.7.F below).

We now describe the first few pages of the spectral sequence explicitly. As stated above, the differential $d_0$ on $E_0^{\bullet,\bullet} = E^{\bullet,\bullet}$ is defined to be $d_\rightarrow$. The rows are complexes:

The 0th page $E_0$:
$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

and so $E_1$ is just the table of cohomologies of the rows. You should check that there are now vertical maps $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ of the row cohomology groups, induced by $d_\uparrow$, and that these make the columns into complexes. (This is essentially the fact that a map of complexes induces a map on homology.) We have "used up the horizontal morphisms", but "the vertical differentials live on".

The 1st page $E_1$:
$$\begin{tikzcd}
\bullet & \bullet & \bullet \\
\bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u]
\end{tikzcd}$$

We take cohomology of $d_1$ on $E_1$, giving us a new table, $E_2^{p,q}$. It turns out that there are natural morphisms from each entry to the entry two above and one to the left, and that the composition of these two is $0$. (It is a very worthwhile exercise to work out how this natural morphism $d_2$ should be defined. Your argument may be reminiscent of the connecting homomorphism in the Snake Lemma 2.7.5 or in the long exact sequence in cohomology arising from a short exact sequence of complexes, Exercise 2.6.C. This is no coincidence.)

The 2nd page $E_2$:
$$\begin{tikzcd}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet \arrow[uul] & \bullet \arrow[uul]
\end{tikzcd}$$

This is the beginning of a pattern.

Then it is a theorem that there is a filtration of $H^k(E^\bullet)$ by $E_\infty^{p,q}$ where $p+q=k$. (We can't yet state it as an official **Theorem** because we haven't precisely defined the pages and differentials in the spectral sequence.) More precisely, there is a filtration

$$E_\infty^{0,k} \xhookrightarrow{E_\infty^{1,k-1}} \; ? \; \xhookrightarrow{E_\infty^{2,k-2}} \cdots \xhookrightarrow{E^{0,k}} H^k(E^\bullet) \tag{2.7.2.2}$$

where the quotients are displayed above each inclusion. (I always forget which way the quotients are supposed to go, i.e. whether $E^{k,0}$ or $E^{0,k}$ is the subobject. One way of remembering it is by having some idea of how the result is proved.)

We say that the spectral sequence ${}_\rightarrow E_\bullet^{\bullet,\bullet}$ **converges** to $H^\bullet(E^\bullet)$. We often say that ${}_\rightarrow E_2^{\bullet,\bullet}$ (or any other page) abuts to $H^\bullet(E^\bullet)$.

Although the filtration gives only partial information about $H^\bullet(E^\bullet)$, sometimes one can find $H^\bullet(E^\bullet)$ precisely. One example is if all $E_\infty^{i,k-i}$ are zero, or if all but one of them are zero (e.g. if $E_r^{i,k-i}$ has precisely one non-zero row or column, in which case one says that the spectral sequence *collapses* at the $r$th step, although we will not use this term). Another example is in the category of vector spaces over a field, in which case we can find the dimension of $H^k(E^\bullet)$. Also, in lucky circumstances, $E_2$ (or some other small page) already equals $E_\infty$.

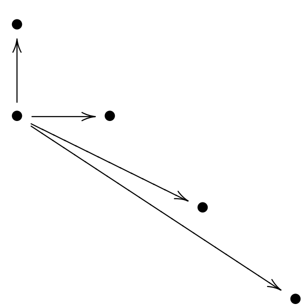
**2.7.A.** EXERCISE: INFORMATION FROM THE SECOND PAGE. Show that $H^0(E^\bullet) = E_\infty^{0,0} = E_2^{0,0}$ and

$$0 \longrightarrow E_2^{0,1} \longrightarrow H^1(E^\bullet) \longrightarrow E_2^{1,0} \xrightarrow{d_2^{1,0}} E_2^{0,2} \longrightarrow H^2(E^\bullet).$$

**2.7.3.** *The other orientation.*

You may have observed that we could as well have done everything in the opposite direction, i.e. reversing the roles of horizontal and vertical morphisms. Then the sequences of arrows giving the spectral sequence would look like this (compare to (2.7.2.1)).

(2.7.3.1)



This spectral sequence is denoted ${}_\uparrow E_\bullet^{\bullet,\bullet}$ ("with the upwards orientation"). Then we would again get pieces of a filtration of $H^\bullet(E^\bullet)$ (where we have to be a bit careful with the order with which ${}_\uparrow E_\infty^{p,q}$ corresponds to the subquotients — it in the opposite order to that of (2.7.2.2) for ${}_\rightarrow E_\infty^{p,q}$). Warning: in general there is no isomorphism between ${}_\rightarrow E_\infty^{p,q}$ and ${}_\uparrow E_\infty^{p,q}$.

In fact, this observation that we can start with either the horizontal or vertical maps was our secret goal all along. Both algorithms compute information about the same thing ($H^\bullet(E^\bullet)$), and usually we don't care about the final answer — we often care about the answer we get in one way, and we get at it by doing the spectral sequence in the *other* way.

**2.7.4. Examples.**

We are now ready to see how this is useful. The moral of these examples is the following. In the past, you may have proved various facts involving various sorts of diagrams, by chasing elements around. Now, you will just plug them into a spectral sequence, and let the spectral sequence machinery do your chasing for you.

**2.7.5.** *Example: Proving the Snake Lemma.* Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

where the rows are exact in the middle (at $B$, $C$, $D$, $G$, $H$, $I$) and the squares commute. (Normally the Snake Lemma is described with the vertical arrows pointing downwards, but I want to fit this into my spectral sequence conventions.) We wish to show that there is an exact sequence

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0. \tag{2.7.5.1}$$

We plug this into our spectral sequence machinery. We first compute the cohomology using the rightwards orientation, i.e. using the order (2.7.2.1). Then because the rows are exact, $E_1^{p,q} = 0$, so the spectral sequence has already converged: $E_\infty^{p,q} = 0$.

We next compute this "0" in another way, by computing the spectral sequence using the upwards orientation. Then ${}_{\uparrow}E_1^{\bullet,\bullet}$ (with its differentials) is:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma \arrow[r] & 0 \\
0 \arrow[r] & \ker\alpha \arrow[r] & \ker\beta \arrow[r] & \ker\gamma \arrow[r] & 0.
\end{tikzcd}$$

Then ${}_{\uparrow}E_2^{\bullet,\bullet}$ is of the form:

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & & & & & \\
& 0 \arrow[drr] & ?? \arrow[drr] & ? \arrow[drr] & ? & 0 & \\
& 0 & ? & ? \arrow[drr] & ?? \arrow[drr] & 0 & \\
& & & & & 0 & 0
\end{tikzcd}$$

We see that after ${}_{\uparrow}E_2$, all the terms will stabilize except for the double-question-marks — all maps to and from the single question marks are to and from 0-entries. And after ${}_{\uparrow}E_3$, even these two double-quesion-mark terms will stabilize. But in the end our complex must be the 0 complex. This means that in ${}_{\uparrow}E_2$, all the entries must be zero, except for the two double-question-marks, and these two must be isomorphic. This means that $0 \to \ker\alpha \to \ker\beta \to \ker\gamma$ and $\operatorname{coker}\alpha \to \operatorname{coker}\beta \to$

$\operatorname{coker}\gamma \to 0$ are both exact (that comes from the vanishing of the single-question-marks), and

$$\operatorname{coker}(\ker\beta \to \ker\gamma) \cong \ker(\operatorname{coker}\alpha \to \operatorname{coker}\beta)$$

is an isomorphism (that comes from the equality of the double-question-marks). Taken together, we have proved the exactness of (2.7.5.1), and hence the Snake Lemma! (Notice: in the end we didn't really care about the double complex. We just used it as a prop to prove the snake lemma.)

Spectral sequences make it easy to see how to generalize results further. For example, if $A \to B$ is no longer assumed to be injective, how would the conclusion change?

**2.7.B.** UNIMPORTANT EXERCISE (GRAFTING EXACT SEQUENCES, A WEAKER VERSION OF THE SNAKE LEMMA). Extend the snake lemma as follows. Suppose we have a commuting diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & A' \arrow[r] & \cdots \\
\cdots \arrow[r] & W \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "a"] & Y \arrow[r] \arrow[u, "b"] & Z \arrow[r] \arrow[u, "c"] & 0. \arrow[u] &
\end{tikzcd}$$

where the top and bottom rows are exact. Show that the top and bottom rows can be "grafted together" to an exact sequence

$$\cdots \longrightarrow W \longrightarrow \ker a \longrightarrow \ker b \longrightarrow \ker c$$

$$\longrightarrow \operatorname{coker} a \longrightarrow \operatorname{coker} b \longrightarrow \operatorname{coker} c \longrightarrow A' \longrightarrow \cdots.$$

**2.7.6.** *Example: the Five Lemma.* Suppose

(2.7.6.1)
$$\begin{tikzcd}
F \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & J \\
A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & D \arrow[r] \arrow[u, "\delta"] & E \arrow[u, "\epsilon"]
\end{tikzcd}$$

where the rows are exact and the squares commute.

Suppose $\alpha$, $\beta$, $\delta$, $\epsilon$ are isomorphisms. We will show that $\gamma$ is an isomorphism.

We first compute the cohomology of the total complex using the rightwards orientation (2.7.2.1). We choose this because we see that we will get lots of zeros. Then $_{\to}E_1^{\bullet,\bullet}$ looks like this:

$$\begin{tikzcd}
? & 0 & 0 & 0 & ? \\
? \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & ? \arrow[u]
\end{tikzcd}$$

Then $_{\to}E_2$ looks similar, and the sequence will converge by $E_2$, as we will never get any arrows between two non-zero entries in a table thereafter. We can't conclude that the cohomology of the total complex vanishes, but we can note that it vanishes in all but four degrees — and most important, it vanishes in the two degrees corresponding to the entries C and H (the source and target of $\gamma$).

We next compute this using the upwards orientation (2.7.3.1). Then ${}_{\uparrow}E_1$ looks like this:

$$0 \longrightarrow 0 \longrightarrow ? \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow ? \longrightarrow 0 \longrightarrow 0$$

and the spectral sequence converges at this step. We wish to show that those two question marks are zero. But they are precisely the cohomology groups of the total complex that we just showed *were* zero — so we're done!

The best way to become comfortable with this sort of argument is to try it out yourself several times, and realize that it really is easy. So you should do the following exercises!

**2.7.C.** EXERCISE: THE SUBTLE FIVE LEMMA. By looking at the spectral sequence proof of the Five Lemma above, prove a subtler version of the Five Lemma, where one of the isomorphisms can instead just be required to be an injection, and another can instead just be required to be a surjection. (I am deliberately not telling you which ones, so you can see how the spectral sequence is telling you how to improve the result.)

**2.7.D.** EXERCISE. If $\beta$ and $\delta$ (in (2.7.6.1)) are injective, and $\alpha$ is surjective, show that $\gamma$ is injective. Give the dual statement (whose proof is of course essentially the same).

**2.7.E.** EXERCISE. Use spectral sequences to show that a short exact sequence of complexes gives a long exact sequence in cohomology (Exercise 2.6.C).

**2.7.F.** EXERCISE (THE MAPPING CONE). Suppose $\mu : A^\bullet \to B^\bullet$ is a morphism of complexes. Suppose $C^\bullet$ is the single complex associated to the double complex $A^\bullet \to B^\bullet$. ($C^\bullet$ is called the *mapping cone* of $\mu$.) Show that there is a long exact sequence of complexes:

$$\cdots \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots.$$

(There is a slight notational ambiguity here; depending on how you index your double complex, your long exact sequence might look slightly different.) In particular, we will use the fact that $\mu$ induces an isomorphism on cohomology if and only if the mapping cone is exact. (We won't use it until the proof of Theorem 20.2.4.)

The Grothendieck (or composition of functor) spectral sequence (Exercise 23.3.D) will be an important example of a spectral sequence that specializes in a number of useful ways.

You are now ready to go out into the world and use spectral sequences to your heart's content!

**2.7.7.** $\star\star$ **Complete definition of the spectral sequence, and proof.**

You should most definitely not read this section any time soon after reading the introduction to spectral sequences above. Instead, flip quickly through it to convince yourself that nothing fancy is involved.

We consider the rightwards orientation. The upwards orientation is of course a trivial variation of this.

**2.7.8.** *Goals.* We wish to describe the pages and differentials of the spectral sequence explicitly, and prove that they behave the way we said they did. More precisely, we wish to:

(a) describe $E_r^{p,q}$,
(b) verify that $H^k(E^\bullet)$ is filtered by $E_\infty^{p,k-p}$ as in (2.7.2.2),
(c) describe $d_r$ and verify that $d_r^2 = 0$, and
(d) verify that $E_{r+1}^{p,q}$ is given by cohomology using $d_r$.

Before tacking these goals, you can impress your friends by giving this short description of the pages and differentials of the spectral sequence. We say that an element of $E^{\bullet,\bullet}$ is a $(p,q)$-*strip* if it is an element of $\oplus_{l \geq 0} E^{p+l,q-l}$ (see Fig. 2.1). Its non-zero entries lie on a semi-infinite antidiagonal starting with position $(p,q)$. We say that the $(p,q)$-entry (the projection to $E^{p,q}$) is the *leading term* of the $(p,q)$-strip. Let $\boxed{S^{p,q}} \subset E^{\bullet,\bullet}$ be the submodule of all the $(p,q)$-strips. Clearly $S^{p,q} \subset E^{p+q}$, and $S^{0,k} = E^k$.

$$\begin{array}{ccccc}
\ddots & 0 & 0 & 0 & 0 \\
0 & *^{p+2,q-2} & 0 & 0 & 0 \\
0 & 0 & *^{p+1,q-1} & 0 & 0 \\
0 & 0 & 0 & *^{p,q} & 0 \\
0 & 0 & 0 & 0 & 0^{p-1,q+1}
\end{array}$$

FIGURE 2.1. A $(p,q)$-strip (in $S^{p,q} \subset E^{p+q}$). Clearly $S^{0,k} = E^k$.

Note that the differential $d = d_\uparrow + d_\rightarrow$ sends a $(p,q)$-strip $x$ to a $(p,q+1)$-strip $dx$. If $dx$ is furthermore a $(p+r, q+r+1)$-strip ($r \in \mathbb{Z}^{\geq 0}$), we say that $x$ is an $r$-*closed* $(p,q)$-*strip*. We denote the set of such $\boxed{S_r^{p,q}}$ (so for example $S_0^{p,q} = S^{p,q}$, and $S_0^{0,k} = E^k$). An element of $S_r^{p,q}$ may be depicted as:

$$\begin{tikzcd}
\ddots \arrow[r] & ? & & & \\
& *^{p+2,q-2} \arrow[u] \arrow[r] & 0 & & \\
& & *^{p+1,q-1} \arrow[u] \arrow[r] & 0 & \\
& & & *^{p,q} \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

**2.7.9.** *Preliminary definition of* $E_r^{p,q}$. We are now ready to give a first definition of $E_r^{p,q}$, which by construction should be a subquotient of $E^{p,q} = E_0^{p,q}$. We describe it as such by describing two submodules $Y_r^{p,q} \subset X_r^{p,q} \subset E^{p,q}$, and defining $E_r^{p,q} = X_r^{p,q}/Y_r^{p,q}$. Let $X_r^{p,q}$ be those elements of $E^{p,q}$ that are the leading terms of $r$-closed $(p,q)$-strips. Note that by definition, $d$ sends $(r-1)$-closed $S^{p-(r-1),q+(r-1)-1}$-strips to $(p,q)$-strips. Let $Y_r^{p,q}$ be the leading $((p,q))$-terms of the differential $d$ of $(r-1)$-closed $(p-(r-1), q+(r-1)-1)$-strips (where the differential is considered as a $(p,q)$-strip).

We next give the definition of the differential $d_r$ of such an element $x \in X_r^{p,q}$. We take *any* $r$-closed $(p,q)$-strip with leading term $x$. Its differential $d$ is a $(p+r, q-r+1)$-strip, and we take its leading term. The choice of the $r$-closed $(p,q)$-strip means that this is not a well-defined element of $E^{p,q}$. But it is well-defined modulo the $(r-1)$-closed $(p+1, r+1)$-strips, and hence gives a map $E_r^{p,q} \to E_r^{p+r,q-r+1}$.

This definition is fairly short, but not much fun to work with, so we will forget it, and instead dive into a snakes' nest of subscripts and superscripts.

We begin with making some quick but important observations about $(p,q)$-strips.

**2.7.G.** EXERCISE. Verify the following.

(a) $S^{p,q} = S^{p+1,q-1} \oplus E^{p,q}$.

(b) (Any closed $(p,q)$-strip is $r$-closed for all $r$.) Any element $x$ of $S^{p,q} = S_0^{p,q}$ that is a cycle (i.e. $dx = 0$) is automatically in $S_r^{p,q}$ for all $r$. For example, this holds when $x$ is a boundary (i.e. of the form $dy$).

(c) Show that for fixed $p, q$,

$$S_0^{p,q} \supset S_1^{p,q} \supset \cdots \supset S_r^{p,q} \supset \cdots$$

stabilizes for $r \gg 0$ (i.e. $S_r^{p,q} = S_{r+1}^{p,q} = \cdots$). Denote the stabilized module $S_\infty^{p,q}$. Show $S_\infty^{p,q}$ is the set of closed $(p,q)$-strips (those $(p,q)$-strips annihilated by $d$, i.e. the cycles). In particular, $S_r^{0,k}$ is the set of cycles in $E^k$.

**2.7.10.** *Defining* $E_r^{p,q}$.

Define $X_r^{p,q} := S_r^{p,q}/S_{r-1}^{p+1,q-1}$ and $Y := dS_{r-1}^{p-(r-1),q+(r-1)-1}/S_{r-1}^{p+1,q-1}$.

Then $Y_r^{p,q} \subset X_r^{p,q}$ by Exercise 2.7.G(b). We define

$$\boxed{E_r^{p,q} = \frac{X_r^{p,q}}{Y_r^{p,q}} = \frac{S_r^{p,q}}{dS_{r-1}^{p-(r-1),q+(r-1)-1} + S_{r-1}^{p+1,q-1}}} \tag{2.7.10.1}$$

We have completed Goal 2.7.8(a).

You are welcome to verify that these definitions of $X_r^{p,q}$ and $Y_r^{p,q}$ and hence $E_r^{p,q}$ agree with the earlier ones of §2.7.9 (and in particular $X_r^{p,q}$ and $Y_r^{p,q}$ are both submodules of $E^{p,q}$), but we won't need this fact.

**2.7.H.** Exercise: $E_\infty^{p,k-p}$ gives subquotients of $H^k(E^\bullet)$. By Exercise 2.7.G(c), $E_r^{p,q}$ stabilizes as $r \to \infty$. For $r \gg 0$, interpret $S_r^{p,q}/dS_{r-1}^{p-(r-1),q+(r-1)-1}$ as the cycles in $S_\infty^{p,q} \subset E^{p+q}$ modulo those boundary elements of $dE^{p+q-1}$ contained in $S_\infty^{p,q}$. Finally, show that $H^k(E^\bullet)$ is indeed filtered as described in (2.7.2.2).

We have completed Goal 2.7.8(b).

**2.7.11.** *Definition of* $d_r$.

We shall see that the map $d_r : E_r^{p,q} \to E^{p+r,q-r+1}$ is just induced by our differential $d$. Notice that $d$ sends $r$-closed $(p,q)$-strips $S_r^{p,q}$ to $(p+r, q-r+1)$-strips $S^{p+r,q-r+1}$, by the definition "$r$-closed". By Exercise 2.7.G(b), the image lies in $S_r^{p+r,q-r+1}$.

**2.7.I.** Exercise. Verify that $d$ sends

$$dS_{r-1}^{p-(r-1),q+(r-1)-1} + S_{r-1}^{p+1,q-1} \to dS_{r-1}^{(p+r)-(r-1),(q-r+1)+(r-1)-1} + S_{r-1}^{(p+r)+1,(q-r+1)-1}.$$

(The first term on the left goes to $0$ from $d^2 = 0$, and the second term on the left goes to the first term on the right.)

Thus we may define

$$\boxed{d_r : E_r^{p,q} = \frac{S_r^{p,q}}{dS_{r-1}^{p-(r-1),q+(r-1)-1} + S_{r-1}^{p+1,q-1}} \to}$$

$$\boxed{\frac{S_r^{p+r,q-r+1}}{dS_{r-1}^{p+1,q-1} + S_{r-1}^{p+r+1,q-r}} = E_r^{p+r,q-r+1}}$$

and clearly $d_r^2 = 0$ (as we may interpret it as taking an element of $S_r^{p,q}$ and applying $d$ twice).

We have accomplished Goal 2.7.8(c).

**2.7.12.** *Verifying that the cohomology of* $d_r$ *at* $E_r^{p,q}$ *is* $E_{r+1}^{p,q}$. We are left with the unpleasant job of verifying that the cohomology of

$$\frac{S_r^{p-r,q+r-1}}{dS_{r-1}^{p-2r+1,q-3} + S_{r-1}^{p-r+1,q+r-2}} \xrightarrow{d_r} \frac{S_r^{p,q}}{dS_{r-1}^{p-r+1,q+r-2} + S_{r-1}^{p+1,q-1}} \tag{2.7.12.1}$$

$$\xrightarrow{d_r} \frac{S_r^{p+r,q-r+1}}{dS_{r-1}^{p+1,q-1} + S_{r-1}^{p+r+1,q-r}}$$

is naturally identified with

$$\frac{S^{p,q}_{r+1}}{dS^{p-r,q+r-1}_r + S^{p+1,q-1}_r}$$

and this will conclude our final Goal 2.7.8(d).

We begin by understanding the kernel of the right map of (2.7.12.1). Suppose $a \in S^{p,q}_r$ is mapped to $0$. This means that $da = db + c$, where $b \in S^{p+1,q-1}_{r-1}$. If $u = a - b$, then $u \in S^{p,q}$, while $du = c \in S^{p+r+1,q-r}_{r-1} \subset S^{p+r+1,q-r}$, from which $u$ is $r$-closed, i.e. $u \in S^{p,q}_{r+1}$. Hence $a = b + u + x$ where $dx = 0$, from which $a - x = b + c \in S^{p+1,q-1}_{r-1} + S^{p,q}_{r+1}$. However, $x \in S^{p,q}$, so $x \in S^{p,q}_{r+1}$ by Exercise 2.7.G(b). Thus $a \in S^{p+1,q-1}_{r-1} + S^{p,q}_{r+1}$. Conversely, any $a \in S^{p+1,q-1}_{r-1} + S^{p,q}_{r+1}$ satisfies

$$da \in dS^{p+r,q-r+1}_{r-1} + dS^{p,q}_{r+1} \subset dS^{p+r,q-r+1}_{r-1} + S^{p+r+1,q-r}_{r-1}$$

(using $dS^{p,q}_{r+1} \subset S^{p+r+1,q-r}_0$ and Exercise 2.7.G(b)) so any such $a$ is indeed in the kernel of

$$S^{p,q}_r \to \frac{S^{p+r,q-r+1}_r}{dS^{p+1,q-1}_{r-1} + S^{p+r+1,q-r}_{r-1}}.$$

Hence the kernel of the right map of (2.7.12.1) is

$$\ker = \frac{S^{p+1,q-1}_{r-1} + S^{p,q}_{r+1}}{dS^{p-r+1,q+r-2}_{r-1} + S^{p+1,q-1}_{r-1}}.$$

Next, the image of the left map of (2.7.12.1) is immediately

$$\mathrm{im} = \frac{dS^{p-r,q+r-1}_r + dS^{p-r+1,q+r-2}_{r-1} + S^{p+1,q-1}_{r-1}}{dS^{p-r+1,q+r-2}_{r-1} + S^{p+1,q-1}_{r-1}} = \frac{dS^{p-r,q+r-1}_r + S^{p+1,q-1}_{r-1}}{dS^{p-r+1,q+r-2}_{r-1} + S^{p+1,q-1}_{r-1}}$$

(as $S^{p-r,q-r+1}_r$ contains $S^{p-r+1,q+r-1}_{r-1}$).

Thus the cohomology of (2.7.12.1) is

$$\ker/\mathrm{im} = \frac{S^{p+1,q-1}_{r-1} + S^{p,q}_{r+1}}{dS^{p-r,q+r-1}_r + S^{p+1,q-1}_{r-1}} = \frac{S^{p,q}_{r+1}}{S^{p,q}_{r+1} \cap (dS^{p-r,q+r-1}_r + S^{p+1,q-1}_{r-1})}$$

where the equality on the right uses the fact that $dS^{p-r,q+r+1}_r \subset S^{p,q}_{r+1}$ and an isomorphism theorem. We thus must show

$$S^{p,q}_{r+1} \cap (dS^{p-r,q+r-1}_r + S^{p+1,q-1}_{r-1}) = dS^{p-r,q+r-1}_r + S^{p+1,q-1}_r.$$

However,

$$S^{p,q}_{r+1} \cap (dS^{p-r,q+r-1}_r + S^{p+1,q-1}_{r-1}) = dS^{p-r,q+r-1}_r + S^{p,q}_{r+1} \cap S^{p+1,q-1}_{r-1}$$

and $S^{p,q}_{r+1} \cap S^{p+1,q-1}_{r-1}$ consists of $(p,q)$-strips whose differential vanishes up to row $p+r$, from which $S^{p,q}_{r+1} \cap S^{p+1,q-1}_{r-1} = S^{p,q}_r$ as desired.

This completes the explanation of how spectral sequences work for a first-quadrant double complex. The argument applies without significant change to more general situations, including filtered complexes.

CHAPTER 3

# Sheaves

It is perhaps surprising that geometric spaces are often best understood in terms of (nice) functions on them. For example, a differentiable manifold that is a subset of $\mathbb{R}^n$ can be studied in terms of its differentiable functions. Because "geometric spaces" can have few (everywhere-defined) functions, a more precise version of this insight is that the structure of the space can be well understood by considering all functions on all open subsets of the space. This information is encoded in something called a *sheaf*. Sheaves were introduced by Leray in the 1940's, and Serre introduced them to algebraic geometry. (The reason for the name will be somewhat explained in Remark 3.4.3.) We will define sheaves and describe useful facts about them. We will begin with a motivating example to convince you that the notion is not so foreign.

One reason sheaves are slippery to work with is that they keep track of a huge amount of information, and there are some subtle local-to-global issues. There are also three different ways of getting a hold of them.

- in terms of open sets (the definition §3.2) — intuitive but in some ways the least helpful
- in terms of stalks (see §3.4.1)
- in terms of a base of a topology (§3.7).

Knowing which to use requires experience, so it is essential to do a number of exercises on different aspects of sheaves in order to truly understand the concept.

## 3.1 Motivating example: The sheaf of differentiable functions.

Consider differentiable functions on the topological space $X = \mathbb{R}^n$ (or more generally on a smooth manifold $X$). The sheaf of differentiable functions on $X$ is the data of all differentiable functions on all open subsets on $X$. We will see how to manage this data, and observe some of its properties. On each open set $U \subset X$, we have a ring of differentiable functions. We denote this ring of functions $\mathcal{O}(U)$.

Given a differentiable function on an open set, you can restrict it to a smaller open set, obtaining a differentiable function there. In other words, if $U \subset V$ is an inclusion of open sets, we have a "restriction map" $\mathrm{res}_{V,U} : \mathcal{O}(V) \to \mathcal{O}(U)$.

Take a differentiable function on a big open set, and restrict it to a medium open set, and then restrict that to a small open set. The result is the same as if you restrict the differentiable function on the big open set directly to the small open set.

In other words, if $U \hookrightarrow V \hookrightarrow W$, then the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}(W) \arrow[rr, "\mathrm{res}_{W,V}"] \arrow[dr, "\mathrm{res}_{W,U}"'] & & \mathcal{O}(V) \arrow[dl, "\mathrm{res}_{V,U}"] \\ & \mathcal{O}(U) & \end{tikzcd}$$

Next take two differentiable functions $f_1$ and $f_2$ on a big open set $U$, and an open cover of $U$ by some $\{U_i\}$. Suppose that $f_1$ and $f_2$ agree on each of these $U_i$. Then they must have been the same function to begin with. In other words, if $\{U_i\}_{i \in I}$ is a cover of $U$, and $f_1, f_2 \in \mathcal{O}(U)$, and $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$, then $f_1 = f_2$. Thus we can *identify* functions on an open set by looking at them on a covering by small open sets.

Finally, given the same $U$ and cover $\{U_i\}$, take a differentiable function on each of the $U_i$ — a function $f_1$ on $U_1$, a function $f_2$ on $U_2$, and so on — and they agree on the pairwise overlaps. Then they can be "glued together" to make one differentiable function on all of $U$. In other words, given $f_i \in \mathcal{O}(U_i)$ for all $i$, such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all $i$ and $j$, then there is some $f \in \mathcal{O}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all $i$.

The entire example above would have worked just as well with continuous function, or smooth functions, or just plain functions. Thus all of these classes of "nice" functions share some common properties. We will soon formalize these properties in the notion of a sheaf.

**3.1.1. The germ of a differentiable function.** Before we do, we first give another definition, that of the germ of a differentiable function at a point $p \in X$. Intuitively, it is a "shred" of a differentiable function at $p$. Germs are objects of the form $\{(f, \text{open } U) \; : \; p \in U, f \in \mathcal{O}(U)\}$ modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ containing $p$ where $f|_W = g|_W$ (i.e., $\mathrm{res}_{U,W} f = \mathrm{res}_{V,W} g$). In other words, two functions that are the same in a neighborhood of $p$ (but may differ elsewhere) have the same germ. We call this set of germs the stalk at $p$, and denote it $\mathcal{O}_p$. Notice that the stalk is a ring: you can add two germs, and get another germ: if you have a function $f$ defined on $U$, and a function $g$ defined on $V$, then $f + g$ is defined on $U \cap V$. Moreover, $f + g$ is well-defined: if $f'$ has the same germ as $f$, meaning that there is some open set $W$ containing $p$ on which they agree, and $g'$ has the same germ as $g$, meaning they agree on some open $W'$ containing $p$, then $f' + g'$ is the same function as $f + g$ on $U \cap V \cap W \cap W'$.

Notice also that if $p \in U$, you get a map $\mathcal{O}(U) \to \mathcal{O}_p$. Experts may already see that we are talking about germs as colimits.

We can see that $\mathcal{O}_p$ is a local ring as follows. Consider those germs vanishing at $p$, which we denote $\mathfrak{m}_p \subset \mathcal{O}_p$. They certainly form an ideal: $\mathfrak{m}_p$ is closed under addition, and when you multiply something vanishing at $p$ by any other function, the result also vanishes at $p$. We check that this ideal is maximal by showing that the quotient map is a field:

$$0 \longrightarrow \mathfrak{m}_p := \text{ideal of germs vanishing at } p \longrightarrow \mathcal{O}_p \xrightarrow{f \mapsto f(p)} \mathbb{R} \longrightarrow 0 \tag{3.1.1.1}$$

**3.1.A.** EXERCISE. Show that this is the only maximal ideal of $\mathcal{O}_p$. (Hint: show that every element of $\mathcal{O}_p \setminus \mathfrak{m}$ is invertible.)

Note that we can interpret the value of a function at a point, or the value of a germ at a point, as an element of the local ring modulo the maximal ideal. (We will see that this doesn't work for more general sheaves, but *does* work for things behaving like sheaves of functions. This will be formalized in the notion of a *local-ringed space*, which we will see, briefly, in §7.3.

**3.1.2.** *Aside.* Notice that $\mathfrak{m}/\mathfrak{m}^2$ is a module over $\mathcal{O}_p/\mathfrak{m} \cong \mathbb{R}$, i.e. it is a real vector space. It turns out to be naturally (whatever that means) the cotangent space to the manifold at $p$. This insight will prove handy later, when we define tangent and cotangent spaces of schemes.

**3.1.B.** EXERCISE FOR THOSE WITH DIFFERENTIAL GEOMETRIC BACKGROUND. Prove this.

## 3.2 Definition of sheaf and presheaf

We now formalize these notions, by defining presheaves and sheaves. Presheaves are simpler to define, and notions such as kernel and cokernel are straightforward. Sheaves are more complicated to define, and some notions such as cokernel require more thought. But sheaves are more useful because they are in some vague sense more geometric; you can get information about a sheaf locally.

**3.2.1. Definition of sheaf and presheaf on a topological space $X$.**

To be concrete, we will define sheaves of sets. However, in the definition the category *Sets* can be replaced by any category, and other important examples are abelian groups *Ab*, $k$-vector spaces $Vec_k$, rings *Rings*, modules over a ring $Mod_A$, and more. (You may have to think more when dealing with a category of objects that aren't "sets with additional structure", but there aren't any new complications. In any case, this won't be relevant for us.) Sheaves (and presheaves) are often written in calligraphic font. The fact that $\mathcal{F}$ is a sheaf on a topological space $X$ is often written as

$$\begin{array}{c}\mathcal{F}\\ \Big| \\ X\end{array}$$

**3.2.2. Definition: Presheaf.** A **presheaf** $\mathcal{F}$ on a topological space $X$ is the following data.

• To each open set $U \subset X$, we have a set $\mathcal{F}(U)$ (e.g. the set of differentiable functions in our motivating example). (Notational warning: Several notations are in use, for various good reasons: $\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F})$. We will use them all.) The elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over** $U$.

• For each inclusion $U \hookrightarrow V$ of open sets, we have a **restriction map** $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ (just as we did for differentiable functions).

The data is required to satisfy the following two conditions.

• The map $\mathrm{res}_{U,U}$ is the identity: $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$.

• If $U \hookrightarrow V \hookrightarrow W$ are inclusions of open sets, then the restriction maps commute, i.e.

$$\begin{tikzcd} \mathcal{F}(W) \arrow[rr, "\mathrm{res}_{W,V}"] \arrow[dr, "\mathrm{res}_{W,U}"'] & & \mathcal{F}(V) \arrow[dl, "\mathrm{res}_{V,U}"] \\ & \mathcal{F}(U) & \end{tikzcd}$$

commutes.

**3.2.A.** EXERCISE FOR CATEGORY-LOVERS: "A PRESHEAF IS THE SAME AS A CONTRAVARIANT FUNCTOR". Given any topological space $X$, we have a "category of open sets" (Example 2.2.9), where the objects are the open sets and the morphisms are inclusions. Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of $X$ to the category of sets. (This interpretation is surprisingly useful.)

**3.2.3. Definition: Stalks and germs.** We define the stalk of a presheaf at a point in two equivalent ways. One will be hands-on, and the other will be as a colimit.

**3.2.4.** Define the **stalk** of a presheaf $\mathcal{F}$ at a point $p$ to be the set of **germs** of $\mathcal{F}$ at $p$, denoted $\mathcal{F}_p$, as in the example of §3.1.1. Germs correspond to sections over some open set containing $p$, and two of these sections are considered the same if they agree on some smaller open set. More precisely: the stalk is

$$\{(f, \text{open } U) \ : \ p \in U, f \in \mathcal{F}(U)\}$$

modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ where $\mathrm{res}_{U,W} f = \mathrm{res}_{V,W} g$.

**3.2.5.** A useful (and better) equivalent definition of a stalk is as a colimit of all $\mathcal{F}(U)$ over all open sets $U$ containing $p$:

$$\mathcal{F}_p = \varinjlim \mathcal{F}(U).$$

The index category is a directed set (given any two such open sets, there is a third such set contained in both), so these two definitions are the same by Exercise 2.4.C. Hence we can define stalks for sheaves of sets, groups, rings, and other things for which colimits exist for directed sets.

If $p \in U$, and $f \in \mathcal{F}(U)$, then the image of $f$ in $\mathcal{F}_p$ is called the **germ of** $f$ **at** $p$. (Warning: unlike the example of §3.1.1, in general, the value of a section at a point doesn't make sense.)

**3.2.6. Definition: Sheaf.** A presheaf is a **sheaf** if it satisfies two more axioms. Notice that these axioms use the additional information of when some open sets cover another.

**Identity axiom.** If $\{U_i\}_{i \in I}$ is an open cover of $U$, and $f_1, f_2 \in \mathcal{F}(U)$, and $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$ for all $i$, then $f_1 = f_2$.

(A presheaf satisfying the identity axiom is called a **separated presheaf**, but we will not use that notation in any essential way.)

**Gluability axiom.** If $\{U_i\}_{i \in I}$ is a open cover of $U$, then given $f_i \in \mathcal{F}(U_i)$ for all $i$, such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all $i, j$, then there is some $f \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all $i$.

In mathematics, definitions often come paired: "at most one" and "at least one". In this case, identity means there is at most one way to glue, and gluability means that there is at least one way to glue.

(For experts and scholars of the empty set only: an additional axiom sometimes included is that $F(\varnothing)$ is a one-element set, and in general, for a sheaf with values in a category, $F(\varnothing)$ is required to be the final object in the category. This actually follows from the above definitions, assuming that the empty product is appropriately defined as the final object.)

*Example.* If $U$ and $V$ are disjoint, then $\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V)$. Here we use the fact that $F(\varnothing)$ is the final object.

The **stalk of a sheaf** at a point is just its stalk as a presheaf — the same definition applies — and similarly for the **germs** of a section of a sheaf.

**3.2.B.** UNIMPORTANT EXERCISE: PRESHEAVES THAT ARE NOT SHEAVES. Show that the following are presheaves on $\mathbb{C}$ (with the classical topology), but not sheaves: (a) bounded functions, (b) holomorphic functions admitting a holomorphic square root.

Both of the presheaves in the previous Exercise satisfy the identity axiom. A "natural" example failing even the identity axiom will be given in Remark 3.7.2.

We now make a couple of points intended only for category-lovers.

**3.2.7.** *Interpretation in terms of the equalizer exact sequence.* The two axioms for a presheaf to be a sheaf can be interpreted as "exactness" of the "equalizer exact sequence": $\cdot \longrightarrow \mathcal{F}(U) \longrightarrow \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)$. Identity is exactness at $\mathcal{F}(U)$, and gluability is exactness at $\prod \mathcal{F}(U_i)$. I won't make this precise, or even explain what the double right arrow means. But you may be able to figure it out from the context.

**3.2.C.** EXERCISE. The gluability axiom may be interpreted as saying that $\mathcal{F}(\cup_{i\in I} U_i)$ is a certain limit. What is that limit?

We now give a number of examples of sheaves.

**3.2.D.** EXERCISE. (a) Verify that the examples of §3.1 are indeed sheaves (of differentiable functions, or continuous functions, or smooth functions, or functions on a manifold or $\mathbb{R}^n$).
(b) Show that real-valued continuous functions on (open sets of) a topological space $X$ form a sheaf.

**3.2.8.** *Important Example: Restriction of a sheaf.* Suppose $\mathcal{F}$ is a sheaf on $X$, and $U \subset$ is an open set. Define the **restriction of $\mathcal{F}$ to** $U$, denoted $\mathcal{F}|_U$, to be the collection $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for all $V \subset U$. Clearly this is a sheaf on $U$.

**3.2.9.** *Important Example: skyscraper sheaf.* Suppose $X$ is a topological space, with $p \in X$, and $S$ is a set. Then $S_p$ defined by

$$S_p(U) = \begin{cases} S & \text{if } p \in U, \text{ and} \\ \{e\} & \text{if } p \notin U \end{cases}$$

forms a sheaf. Here $\{e\}$ is any one-element set. (Check this if it isn't clear to you.) This is called a **skyscraper sheaf**, because the informal picture of it looks like a skyscraper at $p$. There is an analogous definition for sheaves of abelian groups, except $S_p(U) = \{0\}$ if $p \notin U$; and for sheaves with values in a category more generally, $S_p(U)$ should be a final object. (Warning: the notation $S_p$ is imperfect, as the subscript $p$ also denotes the stalk at $p$.)

**3.2.10.** *Constant presheaves and constant sheaves.* Let $X$ be a topological space, and $S$ a set. Define $\underline{S}^{pre}(U) = S$ for all open sets $U$. You will readily verify that $\underline{S}^{pre}$ forms a presheaf (with restriction maps the identity). This is called the **constant presheaf associated to** $S$. This isn't (in general) a sheaf. (It may be distracting to say why. Lovers of the empty set will note that the sheaf axioms force the sections over the empty set to be the final object in the category, i.e. a one-element set. But even if we patch the definition by setting $\underline{S}^{pre}(\emptyset) = \{e\}$, if $S$ has more than one element, and $X$ is the two-point space with the discrete topology, you can check that $\underline{S}^{pre}$ fails gluability.)

**3.2.E.** EXERCISE (CONSTANT SHEAVES). Now let $\mathcal{F}(U)$ be the maps to $S$ that are *locally constant*, i.e. for any point $x$ in $U$, there is a neighborhood of $x$ where the function is constant. Show that this is a *sheaf*. (A better description is this: endow $S$ with the discrete topology, and let $\mathcal{F}(U)$ be the continuous maps $U \to S$.) This is called the **constant sheaf** (associated to $S$); do not confuse it with the constant presheaf. We denote this sheaf $\underline{S}$.

**3.2.F.** EXERCISE ("MORPHISMS GLUE"). Suppose $Y$ is a topological space. Show that "continuous maps to $Y$" form a sheaf of sets on $X$. More precisely, to each open set $U$ of $X$, we associate the set of continuous maps of $U$ to $Y$. Show that this forms a sheaf. (Example 3.2.D(b), with $Y = \mathbb{R}$, and Exercise 3.2.E(b), with $Y = S$ with the discrete topology, are both special cases.)

**3.2.G.** EXERCISE. This is a fancier example of the previous exercise.
(a) (sheaf of sections of a map) Suppose we are given a continuous map $f : Y \to X$. Show that "sections of $f$" form a sheaf. More precisely, to each open set $U$ of $X$, associate the set of continuous maps $s : U \to Y$ such that $f \circ s = id|_U$. Show that this forms a sheaf. (For those who have heard of vector bundles, these are a good example.) This is motivation for the phrase "section of a sheaf".
(b) (This exercise is for those who know what a topological group is. If you don't know what a topological group is, you might be able to guess.) Suppose that $Y$ is a topological group. Show that continuous maps to $Y$ form a sheaf of *groups*. (Example 3.2.D(b), with $Y = \mathbb{R}$, is a special case.)

**3.2.11.** $\star$ *The espace étalé of a (pre)sheaf.* Depending on your background, you may prefer the following perspective on sheaves, which we will not discuss further. Suppose $\mathcal{F}$ is a presheaf (e.g. a sheaf) on a topological space $X$. Construct a topological space $Y$ along with a continuous map to $X$ as follows: as a set, $Y$ is the disjoint union of all the stalks of $X$. This also describes a natural set map $Y \to X$. We topologize $Y$ as follows. Each section $s$ of $\mathcal{F}$ over an open set $U$ determines a section of $Y \to X$ over $U$, sending $s$ to each of its germs for each $x \in U$. The topology on $Y$ is the weakest topology such that these sections are continuous. This is called the **espace étalé** of $\mathcal{F}$. Then the reader may wish to show that (a) if $\mathcal{F}$ is a sheaf, then the sheaf of sections of $Y \to X$ (see the previous exercise 3.2.G(a)) can be naturally identified with the sheaf $\mathcal{F}$ itself. (b) Moreover, if $\mathcal{F}$ is a presheaf, the sheaf of sections of $Y \to X$ is the *sheafification* of $\mathcal{F}$, to be defined in Definition 3.4.5 (see Remark 3.4.7). Example 3.2.E may be interpreted as an example of this construction.

**3.2.H.** IMPORTANT EXERCISE: THE PUSHFORWARD SHEAF OR DIRECT IMAGE SHEAF. Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on $X$. Then define $f_*\mathcal{F}$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, where $V$ is an open subset of $Y$. Show that $f_*\mathcal{F}$ is a sheaf. This is called a **direct image sheaf** or **pushforward sheaf**. More precisely, $f_*\mathcal{F}$ is called the **pushforward of $\mathcal{F}$ by $f$**.

The skyscraper sheaf (Example 3.2.9) can be interpreted as the pushforward of the constant sheaf $\underline{S}$ on a one-point space $p$, under the morphism $f : \{p\} \to X$.

Once we realize that sheaves form a category, we will see that the pushforward is a functor from sheaves on $X$ to sheaves on $Y$ (Exercise 3.3.A).

**3.2.I.** EXERCISE (PUSHFORWARD INDUCES MAPS OF STALKS). Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf of sets (or rings or $A$-modules) on $X$. If $f(x) = y$, describe the natural morphism of stalks $(f_*\mathcal{F})_y \to \mathcal{F}_x$. (You can use the explicit definition of stalk using representatives, §3.2.4, or the universal property, §3.2.5. If you prefer one way, you should try the other.) Once we define the category of sheaves of sets on a topological space in §3.3.1, you will see that your construction will make the following diagram commute:

$$\begin{tikzcd} Sets_X \arrow[r, "f_*"] \arrow[d] & Sets_Y \arrow[d] \\ Sets \arrow[r] & Sets \end{tikzcd}$$

**3.2.12. Important Example: Ringed spaces, and $\mathcal{O}_X$-modules.** Suppose $\mathcal{O}_X$ is a sheaf of rings on a topological space $X$ (i.e. a sheaf on $X$ with values in the category of *Rings*). Then $(X, \mathcal{O}_X)$ is called a **ringed space**. The sheaf of rings is often denoted by $\mathcal{O}_X$, pronounced "oh-of-X". This sheaf is called the **structure sheaf** of the ringed space. We now define the notion of an **$\mathcal{O}_X$-module**. The notion is analogous to one we've seen before: just as we have modules over a ring, we have $\mathcal{O}_X$-modules over the structure sheaf (of rings) $\mathcal{O}_X$.

There is only one possible definition that could go with this name. An $\mathcal{O}_X$-module is a sheaf of abelian groups $\mathcal{F}$ with the following additional structure. For each $U$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module. Furthermore, this structure should behave well with respect to restriction maps: if $U \subset V$, then

(3.2.12.1)
$$\begin{tikzcd} \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r, "\text{action}"] \arrow[d, "\text{res}_{V,U} \times \text{res}_{V,U}"'] & \mathcal{F}(V) \arrow[d, "\text{res}_{V,U}"] \\ \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\text{action}"] & \mathcal{F}(U) \end{tikzcd}$$

commutes. (You should convince yourself that I haven't forgotten anything.)

Recall that the notion of A-module generalizes the notion of abelian group, because an abelian group is the same thing as a $\mathbb{Z}$-module. Similarly, the notion of $\mathcal{O}_X$-module generalizes the notion of sheaf of abelian groups, because the latter is the same thing as a $\underline{\mathbb{Z}}$-module, where $\underline{\mathbb{Z}}$ is the constant sheaf associated to $\mathbb{Z}$. Hence when we are proving things about $\mathcal{O}_X$-modules, we are also proving things about sheaves of abelian groups.

**3.2.13.** *For those who know about vector bundles.* The motivating example of $\mathcal{O}_X$-modules is the sheaf of sections of a vector bundle. If $(X, \mathcal{O}_X)$ is a differentiable manifold (so $\mathcal{O}_X$ is the sheaf of differentiable functions), and $\pi : V \to X$ is a vector bundle over X, then the sheaf of differentiable sections $\phi : X \to V$ is an $\mathcal{O}_X$-module. Indeed, given a section $s$ of $\pi$ over an open subset $U \subset X$, and a function $f$ on $U$, we can multiply $s$ by $f$ to get a new section $fs$ of $\pi$ over $U$. Moreover, if $V$ is a smaller subset, then we could multiply $f$ by $s$ and then restrict to $V$, or we could restrict both $f$ and $s$ to $V$ and then multiply, and we would get the same answer. That is precisely the commutativity of (3.2.12.1).

## 3.3 Morphisms of presheaves and sheaves

**3.3.1.** Whenever one defines a new mathematical object, category theory teaches to try to understand maps between them. We now define morphisms of presheaves, and similarly for sheaves. In other words, we will descibe the *category of presheaves* (of sets, abelian groups, etc.) and the *category of sheaves*.

A **morphism of presheaves** of sets (or indeed of sheaves with values in any category) on X, $f : \mathcal{F} \to \mathcal{G}$, is the data of maps $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all U behaving well with respect to restriction: if $U \hookrightarrow V$ then

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "f(V)"] \arrow[d, "\mathrm{res}_{V,U}"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ \mathcal{F}(U) \arrow[r, "f(U)"] & \mathcal{G}(U) \end{tikzcd}$$

commutes. (Notice: the underlying space of both $\mathcal{F}$ and $\mathcal{G}$ is X.)

**Morphisms of sheaves** are defined identically: the morphisms from a sheaf $\mathcal{F}$ to a sheaf $\mathcal{G}$ are precisely the morphisms from $\mathcal{F}$ to $\mathcal{G}$ as presheaves. (Translation: The category of sheaves on X is a full subcategory of the category of presheaves on X.)

An example of a morphism of sheaves is the map from the sheaf of differentiable functions on $\mathbb{R}$ to the sheaf of continuous functions. This is a "forgetful map": we are forgetting that these functions are differentiable, and remembering only that they are continuous.

We may as well set some notation: let $Sets_X$, $Ab_X$, etc. denote the category of sheaves of sets, abelian groups, etc. on a topological space X. Let $Mod_{\mathcal{O}_X}$ denote the category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$. Let $Sets_X^{\mathrm{pre}}$, etc. denote the category of presheaves of sets, etc. on X.

**3.3.2.** *Side-remark for category-lovers.* If you interpret a presheaf on $X$ as a contravariant functor (from the category of open sets), a morphism of presheaves on $X$ is a natural transformation of functors (§2.2.21).

**3.3.A.** EXERCISE. Suppose $f : X \to Y$ is a continuous map of topological spaces (i.e. a morphism in the category of topological spaces). Show that pushforward gives a functor $\mathit{Sets}_X \to \mathit{Sets}_Y$. Here *Sets* can be replaced by many other categories. (Watch out for some possible confusion: a presheaf is a functor, and presheaves form a category. It may be best to forget that presheaves are functors for now.)

**3.3.B.** IMPORTANT EXERCISE AND DEFINITION: "SHEAF Hom". Suppose $\mathcal{F}$ and $\mathcal{G}$ are two sheaves of sets on $X$. (In fact, it will suffice that $\mathcal{F}$ is a presheaf.) Let $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U).$$

(Recall the notation $\mathcal{F}|_U$, the restriction of the sheaf to the open set $U$, Example 3.2.8.) Show that this is a sheaf of sets on $X$. This is called the "sheaf $\mathcal{H}om$". (Strictly speaking, we should reserve Hom for when we are in additive category, so this should possibly be called "sheaf Mor". But the terminology sheaf $\mathcal{H}om$ is too established to uproot.) Show that if $\mathcal{G}$ is a sheaf of abelian groups, then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups. Implicit in this fact is that $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is an abelian group. (This exercise is somewhat tedious, but in the end very rewarding.) The same construction will "obviously" work for sheaves with values in any category.

Warning: $\mathcal{H}om$ does not commute with taking stalks. More precisely: it is not true that $\mathcal{H}om(\mathcal{F}, \mathcal{G})_p$ is isomorphic to $\mathrm{Hom}(\mathcal{F}_p, \mathcal{G}_p)$. (Can you think of a counterexample? Does there at least exist a map from one of these to the other?)

We will use many variants of the definition of $\mathcal{H}om$. For example, if $\mathcal{F}$ and $\mathcal{G}$ are sheaves of abelian groups on $X$, then $\mathcal{H}om_{Ab_X}(\mathcal{F}, \mathcal{G})$ is defined by taking $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$ to be the maps *as sheaves of abelian groups* $\mathcal{F}|_U \to \mathcal{G}|_U$. Similarly, if $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, we define $\mathcal{H}om_{Mod_{\mathcal{O}_X}}(\mathcal{F}, \mathcal{G})$ in the analogous way. Obnoxiously, the subscripts $Ab_X$ and $Mod_{\mathcal{O}_X}$ are essentially always dropped (here and in the literature), so be careful which category you are working in! We call $\mathcal{H}om_{Mod_{\mathcal{O}_X}}(\mathcal{F}, \mathcal{O}_X)$ the *dual* of the $\mathcal{O}_X$-module $\mathcal{F}$, and denoted it $\mathcal{F}^\vee$.

**3.3.C.** UNIMPORTANT EXERCISE (REALITY CHECK).

(a) If $\mathcal{F}$ is a sheaf of sets on $X$, then show that $\mathcal{H}om(\underline{\{p\}}, \mathcal{F}) \cong \mathcal{F}$, where $\underline{\{p\}}$ is the constant sheaf associated to the one element set $\{p\}$.
(b) If $\mathcal{F}$ is a sheaf of abelian groups on $X$, then show that $\mathcal{H}om_{Ab_X}(\underline{\mathbb{Z}}, \mathcal{F}) \cong \mathcal{F}$.
(c) If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then show that $\mathcal{H}om_{Mod_{\mathcal{O}_X}}(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}$.

A key idea in (b) and (c) is that 1 "generates" (in some sense) $\underline{\mathbb{Z}}$ (in (b)) and $\mathcal{O}_X$ (in (c)).

**3.3.3. Presheaves of abelian groups (and even "presheaf $\mathcal{O}_X$-modules") form an abelian category.**

We can make module-like constructions using presheaves of abelian groups on a topological space $X$. (In this section, all (pre)sheaves are of abelian groups.) For example, we can clearly add maps of presheaves and get another map of presheaves: if $f, g : \mathcal{F} \to \mathcal{G}$, then we define the map $f + g$ by $(f + g)(V) =$

$f(V) + g(V)$. (There is something small to check here: that the result is indeed a map of presheaves.) In this way, presheaves of abelian groups form an additive category (Definition 2.6.1). For exactly the same reasons, sheaves of abelian groups also form an additive category.

If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, define the **presheaf kernel** $\ker_{pre} f$ by $(\ker_{pre} f)(U) = \ker f(U)$.

**3.3.D.** EXERCISE. Show that $\ker_{pre} f$ is a presheaf. (Hint: if $U \hookrightarrow V$, define the restriction map by chasing the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \ker_{pre} f(V) \arrow[r] \arrow[d, dotted, "\exists!"] & \mathcal{F}(V) \arrow[r] \arrow[d, "\mathrm{res}_{V,U}"] & \mathcal{G}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ 0 \arrow[r] & \ker_{pre} f(U) \arrow[r] & \mathcal{F}(U) \arrow[r] & \mathcal{G}(U) \end{tikzcd}$$

You should check that the restriction maps compose as desired.)

Define the **presheaf cokernel** $\operatorname{coker}_{pre} f$ similarly. It is a presheaf by essentially the same argument.

**3.3.E.** EXERCISE: THE COKERNEL DESERVES ITS NAME. Show that the presheaf cokernel satisfies the universal property of cokernels (Definition 2.6.3) in the category of presheaves.

Similarly, $\ker_{pre} f \to \mathcal{F}$ satisfies the universal property for kernels in the category of presheaves.

It is not too tedious to verify that presheaves of abelian groups form an abelian category, and the reader is free to do so. The key idea is that all abelian-categorical notions may be defined and verified "open set by open set". We needn't worry about restriction maps — they "come along for the ride". Hence we can define terms such as **subpresheaf**, **image presheaf**, **quotient presheaf**, **cokernel presheaf**, and they behave the way one expect. You construct kernels, quotients, cokernels, and images open set by open set. Homological algebra (exact sequences and so forth) works, and also "works open set by open set". In particular:

**3.3.F.** EASY EXERCISE. Show (or observe) that for a topological space $X$ with open set $U$, $\mathcal{F} \mapsto \mathcal{F}(U)$ gives a functor from presheaves of abelian groups on $X$, $Ab_X^{pre}$, to abelian groups, $Ab$. Then show that this functor is exact.

**3.3.G.** EXERCISE. Show that $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is exact if and only if $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is exact for all $U$.

The above discussion essentially carries over without change to presheaves with values in any abelian category. (Think this through if you wish.)

However, we are interested in more geometric objects, sheaves, where things can be understood in terms of their local behavior, thanks to the identity and gluing axioms. We will soon see that sheaves of abelian groups also form an abelian category, but a complication will arise that will force the notion of *sheafification* on us. Sheafification will be the answer to many of our prayers. We just don't realize it yet.

To begin with, sheaves $Ab_X$ may be easily seen to form an additive category (essentially because presheaves $Ab_X^{\text{pre}}$ already do, and sheaves form a full subcategory).

Kernels work just as with presheaves:

**3.3.H.** IMPORTANT EXERCISE. Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of *sheaves*. Show that the presheaf kernel $\ker_{\text{pre}} f$ is in fact a sheaf. Show that it satisfies the universal property of kernels (Definition 2.6.3). (Hint: the second question follows immediately from the fact that $\ker_{\text{pre}} f$ satisfies the universal property in the category of *presheaves*.)

Thus if $f$ is a morphism of sheaves, we define

$$\ker f := \ker_{\text{pre}} f.$$

The problem arises with the cokernel.

**3.3.I.** IMPORTANT EXERCISE. Let $X$ be $\mathbb{C}$ with the classical topology, let $\underline{\mathbb{Z}}$ be the constant sheaf on $X$ associated to $\mathbb{Z}$, $\mathcal{O}_X$ the sheaf of holomorphic functions, and $\mathcal{F}$ the *presheaf* of functions admitting a holomorphic logarithm. Describe an exact sequence of presheaves on $X$:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \longrightarrow \qquad \mathcal{F} \longrightarrow 0$$

where $\underline{\mathbb{Z}} \to \mathcal{O}_X$ is the natural inclusion and $\mathcal{O}_X \to \mathcal{F}$ is given by $f \mapsto \exp 2\pi i f$. (Be sure to verify exactness.) Show that $\mathcal{F}$ is *not* a sheaf. (Hint: $\mathcal{F}$ does not satisfy the gluability axiom. The problem is that there are functions that don't have a logarithm but locally have a logarithm.) This will come up again in Example 3.4.9.

We will have to put our hopes for understanding cokernels of sheaves on hold for a while. We will first learn to understand sheaves using stalks.

## 3.4 Properties determined at the level of stalks, and sheafification

**3.4.1. Properties determined by stalks.** In this section, we will see that lots of facts about sheaves can be checked "at the level of stalks". This isn't true for presheaves, and reflects the local nature of sheaves. We will see that sections and morphisms are determined "by their stalks", and the property of a morphism being an isomorphism may be checked at stalks. (The last one is the trickiest.)

**3.4.A.** IMPORTANT EXERCISE **(sections are determined by germs)**. Prove that a section of a sheaf of sets is determined by its germs, i.e. the natural map

$$\mathcal{F}(U) \to \prod_{p \in U} \mathcal{F}_p \tag{3.4.1.1}$$

is injective. Hint 1: you won't use the gluability axiom, so this is true for separated presheaves. Hint 2: it is false for presheaves in general, see Exercise 3.4.F, so you *will* use the identity axiom. (Your proof will also apply to sheaves of groups, rings, etc.)

This exercise suggests an important question: which elements of the right side of (3.4.1.1) are in the image of the left side?

**3.4.2. Important definition.** We say that an element $\prod_{p \in U} s_p$ of the right side $\prod_{p \in U} \mathcal{F}_p$ of (3.4.1.1) consists of **compatible germs** if for all $p \in U$, there is some representative $(U_p, s'_p \in \mathcal{F}(U_p))$ for $s_p$ (where $p \in U_p \subset U$) such that the germ of $s'_p$ at all $y \in U_p$ is $s_y$. You will have to think about this a little. Clearly any section $s$ of $\mathcal{F}$ over $U$ gives a choice of compatible germs for $U$ — take $(U_p, s'_p) = (U, s)$.

**3.4.B.** IMPORTANT EXERCISE. Prove that any choice of compatible germs for a sheaf $\mathcal{F}$ over $U$ is the image of a section of $\mathcal{F}$ over $U$. (Hint: you will use gluability.)

We have thus completely described the image of (3.4.1.1), in a way that we will find useful.

**3.4.3.** *Remark.* This perspective is part of the motivation for the agricultural terminology "sheaf": it is (the data of) a bunch of stalks, bundled together appropriately.

Now we throw morphisms into the mix.

**3.4.C.** EXERCISE. Show a morphism of (pre)sheaves (of sets, or rings, or abelian groups, or $\mathcal{O}_X$-modules) induces a morphism of stalks. More precisely, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of (pre)sheaves on $X$, and $p \in X$, describe a natural map $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$. (You may wish to state this in the language of functors.)

**3.4.D.** EXERCISE **(morphisms are determined by stalks)**. If $\phi_1$ and $\phi_2$ are morphisms from $\mathcal{F}$ to $\mathcal{G}$ that induce the same maps on each stalk, show that $\phi_1 = \phi_2$. Hint: consider the following diagram.

(3.4.3.1)
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, hook] & \mathcal{G}(U) \arrow[d, hook] \\ \prod_{p \in U} \mathcal{F}_p \arrow[r] & \prod_{p \in U} \mathcal{G}_p \end{tikzcd}$$

**3.4.E.** TRICKY EXERCISE **(isomorphisms are determined by stalks)**. Show that a morphism of sheaves of sets is an isomorphism if and only if it induces an isomorphism of all stalks. Hint: Use (3.4.3.1). Injectivity of maps of stalks uses the previous exercise 3.4.D. Once you have injectivity, show surjectivity using gluability; this is more subtle.

**3.4.F.** EXERCISE. (a) Show that Exercise 3.4.A is false for general presheaves.
(b) Show that Exercise 3.4.D is false for general presheaves.
(c) Show that Exercise 3.4.E is false for general presheaves.
(General hint for finding counterexamples of this sort: consider a 2-point space with the discrete topology, i.e. every subset is open.)

**3.4.4. Sheafification.**

Every sheaf is a presheaf (and indeed by definition sheaves on $X$ form a full subcategory of the category of presheaves on $X$). Just as groupification (§2.5.3) gives a group that best approximates a semigroup, sheafification gives the sheaf that best approximates a presheaf, with an analogous universal property. (One possible example to keep in mind is the sheafification of the presheaf of holomorphic functions admitting a square root on $\mathbb{C}$ with the classical topology.)

**3.4.5.** *Definition.* If $\mathcal{F}$ is a presheaf on $X$, then a morphism of presheaves $sh : \mathcal{F} \to \mathcal{F}^{sh}$ on $X$ is a **sheafification of** $\mathcal{F}$ if $\mathcal{F}^{sh}$ is a sheaf, and for any other sheaf $\mathcal{G}$, and any presheaf morphism $g : \mathcal{F} \to \mathcal{G}$, there *exists* a *unique* morphism of sheaves $f : \mathcal{F}^{sh} \to \mathcal{G}$ making the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "sh"] \arrow[dr, "g"'] & \mathcal{F}^{sh} \arrow[d, "f"] \\ & \mathcal{G} \end{tikzcd}$$

commute.

**3.4.G.** EXERCISE. Show that sheafification is unique up to unique isomorphism. Show that if $\mathcal{F}$ is a sheaf, then the sheafification is $\mathcal{F} \xrightarrow{id} \mathcal{F}$. (This should be second nature by now.)

**3.4.6. Construction.** We next show that any presheaf of sets (or groups, rings, etc.) has a sheafification. Suppose $\mathcal{F}$ is a *presheaf*. Define $\mathcal{F}^{sh}$ by defining $\mathcal{F}^{sh}(U)$ as the set of compatible germs of the presheaf $\mathcal{F}$ over $U$. Explicitly:

$$\begin{aligned} \mathcal{F}^{sh}(U) &:= \{(f_x \in \mathcal{F}_x)_{x \in U} : \text{for all } x \in U, \text{ there exists } x \in V \subset U \text{ and } s \in \mathcal{F}(V) \\ &\qquad \text{with } s_y = f_y \text{ for all } y \in V\}. \end{aligned}$$

(Those who want to worry about the empty set are welcome to.)

**3.4.H.** EASY EXERCISE. Show that $\mathcal{F}^{sh}$ (using the tautological restriction maps) forms a sheaf.

**3.4.I.** EASY EXERCISE. Describe a natural map of presheaves $sh : \mathcal{F} \to \mathcal{F}^{sh}$.

**3.4.J.** EXERCISE. Show that the map $sh$ satisfies the universal property of sheafification (Definition 3.4.5). (This is easier than you might fear.)

**3.4.K.** USEFUL EXERCISE, NOT JUST FOR CATEGORY-LOVERS. Show that the sheafification functor is left-adjoint to the forgetful functor from sheaves on $X$ to presheaves on $X$. This is not difficult — it is largely a restatement of the universal property. But it lets you use results from §2.6.10, and can "explain" why you don't need to sheafify when taking kernel, and why you need to sheafify when taking cokernel and (soon, in Exercise 3.5.H) $\otimes$.

**3.4.L.** EASY EXERCISE. Use the universal property to show that for any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$. Show that sheafification is a functor from presheaves on $X$ to sheaves on $X$.

**3.4.M.** EXERCISE. Show $\mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ induces an isomorphism of stalks. (Possible hint: Use the concrete description of the stalks. Another possibility once you read Remark 3.6.3: judicious use of adjoints.)

**3.4.7.** ⋆ *Remark.* The *espace étalé* construction (§3.2.11) yields a different-sounding description of sheafification which may be preferred by some readers. The fundamental idea is identical. This is essentially the same construction as the one given here. Another construction is described in [**EH**].

**3.4.8. Subsheaves and quotient sheaves.**

**3.4.N.** EXERCISE. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (of sets) on a topological space $X$. Show that the following are equivalent.

(a) $\phi$ is a monomorphism in the category of sheaves.
(b) $\phi$ is injective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ injective for all $x \in X$.
(c) $\phi$ is injective on the level of open sets: $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open $U \subset X$.

(Possible hints: for (b) implies (a), recall that morphisms are determined by stalks, Exercise 3.4.D. For (a) implies (c), use the "indicator sheaf" with one section over every open set contained in $U$, and no section over any other open set.)

If these conditions hold, we say that $\mathcal{F}$ is a **subsheaf** of $\mathcal{G}$ (where the "inclusion" $\phi$ is sometimes left implicit).

**3.4.O.** EXERCISE. Continuing the notation of the previous exercise, show that the following are equivalent.

(a) $\phi$ is an epimorphism in the category of sheaves.
(b) $\phi$ is surjective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ surjective for all $x \in X$.

If these conditions hold, we say that $\mathcal{G}$ is a **quotient sheaf** of $\mathcal{F}$.

Thus *monomorphisms and epimorphisms — subsheafiness and quotient sheafiness — can be checked at the level of stalks.*

Both exercises generalize readily to sheaves with values in any reasonable category, where "injective" is replaced by "monomorphism" and "surjective" is replaced by "epimorphism".

Notice that there was no part (c) to Exercise 3.4.O, and Example 3.4.9 shows why. (But there is a version of (c) that *implies* (a) and (b): surjectivity on all open sets in the base of a topology implies surjectivity of the map of sheaves, Exercise 3.7.E.)

**3.4.9.** *Example (cf. Exercise 3.3.I).* Let $X = \mathbb{C}$ with the classical topology, and define $\mathcal{O}_X$ to be the sheaf of holomorphic functions, and $\mathcal{O}_X^*$ to be the sheaf of invertible (nowhere zero) holomorphic functions. This is a sheaf of abelian groups under multiplication. We have maps of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1 \tag{3.4.9.1}$$

where $\underline{\mathbb{Z}}$ is the constant sheaf associated to $\mathbb{Z}$. (You can figure out what the sheaves $0$ and $1$ mean; they are isomorphic, and are written in this way for reasons that may be clear.) We will soon interpret this as an exact sequence of sheaves of abelian groups (the *exponential exact sequence*), although we don't yet have the language to do so.

**3.4.P.** ENLIGHTENING EXERCISE. Show that $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$ describes $\mathcal{O}_X^*$ as a quotient sheaf of $\mathcal{O}_X$. Show that it is not surjective on all open sets.

This is a great example to get a sense of what "surjectivity" means for sheaves: nowhere vanishing holomorphic functions have logarithms locally, but they need not globally.

## 3.5 Sheaves of abelian groups, and $\mathcal{O}_X$-modules, form abelian categories

We are now ready to see that sheaves of abelian groups, and their cousins, $\mathcal{O}_X$-modules, form abelian categories. In other words, we may treat them similarly to vector spaces, and modules over a ring. In the process of doing this, we will see that this is much stronger than an analogy; kernels, cokernels, exactness, and so forth can be understood at the level of germs (which are just abelian groups), and the compatibility of the germs will come for free.

The category of sheaves of abelian groups is clearly an additive category (Definition 2.6.1). In order to show that it is an abelian category, we must show that any morphism $\phi : \mathcal{F} \to \mathcal{G}$ has a kernel and a cokernel. We have already seen that $\phi$ has a kernel (Exercise 3.3.H): the presheaf kernel is a sheaf, and is a kernel in the category of sheaves.

**3.5.A.** EXERCISE. Show that the stalk of the kernel is the kernel of the stalks: there is a natural isomorphism

$$(\ker(\mathcal{F} \to \mathcal{G}))_x \cong \ker(\mathcal{F}_x \to \mathcal{G}_x).$$

We next address the issue of the cokernel. Now $\phi : \mathcal{F} \to \mathcal{G}$ has a cokernel in the category of presheaves; call it $\mathcal{H}^{\text{pre}}$ (where the superscript is meant to remind us that this is a presheaf). Let $\mathcal{H}^{\text{pre}} \xrightarrow{sh} \mathcal{H}$ be its sheafification. Recall that the cokernel is defined using a universal property: it is the colimit of the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[d] & \mathcal{G} \\ 0 & \end{tikzcd}$$

in the category of presheaves. We claim that $\mathcal{H}$ is the cokernel of $\phi$ in the category of sheaves, and show this by proving the universal property. Given any sheaf $\mathcal{E}$ and a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[d] & \mathcal{G} \arrow[d] \\ 0 \arrow[r] & \mathcal{E} \end{tikzcd}$$

We construct

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\phi"] \arrow[d] & \mathcal{G} \arrow[d] \arrow[ddrr, bend left] & & \\
0 \arrow[r] \arrow[drrr] & \mathcal{H}^{\text{pre}} \arrow[r, "\text{sh}"] & \mathcal{H} & \\
& & & \mathcal{E}
\end{tikzcd}$$

We show that there is a unique morphism $\mathcal{H} \to \mathcal{E}$ making the diagram commute. As $\mathcal{H}^{\text{pre}}$ is the cokernel in the category of presheaves, there is a unique morphism of presheaves $\mathcal{H}^{\text{pre}} \to \mathcal{E}$ making the diagram commute. But then by the universal property of sheafification (Definition 3.4.5), there is a unique morphism of *sheaves* $\mathcal{H} \to \mathcal{E}$ making the diagram commute.

**3.5.B.** EXERCISE. Show that the stalk of the cokernel is naturally isomorphic to the cokernel of the stalk.

We have now defined the notions of kernel and cokernel, and verified that they may be checked at the level of stalks. We have also verified that the properties of a morphism being a monomorphism or epimorphism are also determined at the level of stalks (Exercises 3.4.N and 3.4.O). Hence sheaves of abelian groups on $X$ form an abelian category.

We see more: all structures coming from the abelian nature of this category may be checked at the level of stalks. For example:

**3.5.C.** EXERCISE. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups. Show that the image sheaf $\operatorname{im} \phi$ is the sheafification of the image presheaf. (You must use the definition of image in an abelian category. In fact, this gives the accepted definition of image sheaf for a morphism of sheaves of sets.) Show that the stalk of the image is the image of the stalk.

As a consequence, **exactness of a sequence of sheaves may be checked at the level of stalks**. In particular:

**3.5.D.** IMPORTANT EXERCISE. Show that taking the stalk of a sheaf of abelian groups is an exact functor. More precisely, if $X$ is a topological space and $p \in X$ is a point, show that taking the stalk at $p$ defines an exact functor $\mathcal{A}b_X \to \mathcal{A}b$.

**3.5.E.** EXERCISE (LEFT-EXACTNESS OF THE SECTION FUNCTOR). Suppose $U \subset X$ is an open set, and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups. Show that

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. (You can do this "by hand", or use the fact that sheafification is an adjoint.) Show that the global section functor need not be exact: show that if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of sheaves of abelian groups, then

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

need not be exact. (Hint: the exponential exact sequence (3.4.9.1).)

**3.5.F.** EXERCISE: LEFT-EXACTNESS OF PUSHFORWARD. Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups on $X$. If $f : X \to Y$ is a continuous map, show that

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H}$$

is exact. (The previous exercise, dealing with the left-exactness of the global section functor can be interpreted as a special case of this, in the case where $Y$ is a point.)

**3.5.G.** EXERCISE. Show that if $(X, \mathcal{O}_X)$ is a ringed space, then $\mathcal{O}_X$-modules form an abelian category. (There isn't much more to check!)

We end with a useful construction using some of the ideas in this section.

**3.5.H.** IMPORTANT EXERCISE: TENSOR PRODUCTS OF $\mathcal{O}_X$-MODULES. (a) Suppose $\mathcal{O}_X$ is a sheaf of rings on $X$. Define (categorically) what we should mean by **tensor product of two $\mathcal{O}_X$-modules**. Give an explicit construction, and show that it satisfies your categorical definition. *Hint:* take the "presheaf tensor product" — which needs to be defined — and sheafify. Note: $\otimes_{\mathcal{O}_X}$ is often written $\otimes$ when the subscript is clear from the context. (An example showing sheafification is necessary will arise in Example 15.1.1.) )
(b) Show that the tensor product of stalks is the stalk of tensor product.

**3.5.1.** *Conclusion.* Just as presheaves are abelian categories because all abelian-categorical notions make sense open set by open set, sheaves are abelian categories because all abelian-categorical notions make sense stalk by stalk.

## 3.6 The inverse image sheaf

We next describe a notion that is fundamental, but rather intricate. We will not need it for some time, so this may be best left for a second reading. Suppose we have a continuous map $f : X \to Y$. If $\mathcal{F}$ is a sheaf on $X$, we have defined the pushforward or direct image sheaf $f_*\mathcal{F}$, which is a sheaf on $Y$. There is also a notion of inverse image sheaf. (We will not call it the pullback sheaf, reserving that name for a later construction for quasicoherent sheaves, §17.3.) This is a covariant functor $f^{-1}$ from sheaves on $Y$ to sheaves on $X$. If the sheaves on $Y$ have some additional structure (e.g. group or ring), then this structure is respected by $f^{-1}$.

**3.6.1.** *Definition by adjoint: elegant but abstract.* We define $f^{-1}$ as the left-adjoint to $f_*$.

This isn't really a definition; we need a construction to show that the adjoint exists. Note that we then get canonical maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ (associated to the identity in $\mathrm{Mor}_Y(f_*\mathcal{F}, f_*\mathcal{F})$) and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ (associated to the identity in $\mathrm{Mor}_X(f^{-1}\mathcal{G}, f^{-1}\mathcal{G})$).

**3.6.2.** *Construction: concrete but ugly.* Define the temporary notation $f^{-1}\mathcal{G}^{\mathrm{pre}}(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V)$. (Recall the explicit description of colimit: sections are sections on open sets containing $f(U)$, with an equivalence relation. Note that $f(U)$ won't be an open set in general.)

**3.6.A.** EXERCISE. Show that this defines a presheaf on $X$.

Now define the **inverse image of** $\mathcal{G}$ by $f^{-1}\mathcal{G} := (f^{-1}\mathcal{G}^{\text{pre}})^{\text{sh}}$. The next exercise shows that this satisfies the universal property. But you may wish to try the later exercises first, and come back to Exercise 3.6.B later. (For the later exercises, try to give two proofs, one using the universal property, and the other using the explicit description.)

**3.6.B.** IMPORTANT TRICKY EXERCISE. If $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on $X$ and $\mathcal{G}$ is a sheaf on $Y$, describe a bijection

$$\mathrm{Mor}_X(f^{-1}\mathcal{G}, \mathcal{F}) \leftrightarrow \mathrm{Mor}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Observe that your bijection is "natural" in the sense of the definition of adjoints (i.e. functorial in both $\mathcal{F}$ and $\mathcal{G}$).

**3.6.3.** *Remark.* As a special case, if $X$ is a point $p \in Y$, we see that $f^{-1}\mathcal{G}$ is the stalk $\mathcal{G}_p$ of $\mathcal{G}$, and maps from the stalk $\mathcal{G}_p$ to a set $S$ are the same as maps of sheaves on $Y$ from $\mathcal{G}$ to the skyscraper sheaf with set $S$ supported at $p$. You may prefer to prove this special case by hand directly before solving Exercise 3.6.B, as it is enlightening. (It can also be useful — can you use it to solve Exercises 3.4.M and 3.4.O?)

**3.6.C.** EXERCISE. Show that the stalks of $f^{-1}\mathcal{G}$ are the same as the stalks of $\mathcal{G}$. More precisely, if $f(p) = q$, describe a natural isomorphism $\mathcal{G}_q \cong (f^{-1}\mathcal{G})_p$. (Possible hint: use the concrete description of the stalk, as a colimit. Recall that stalks are preserved by sheafification, Exercise 3.4.M. Alternatively, use adjointness.) This, along with the notion of compatible stalks, may give you a way of thinking about inverse image sheaves.

**3.6.D.** EXERCISE (EASY BUT USEFUL). If $U$ is an open subset of $Y$, $i : U \to Y$ is the inclusion, and $\mathcal{G}$ is a sheaf on $Y$, show that $i^{-1}\mathcal{G}$ is naturally isomorphic to $\mathcal{G}|_U$.

**3.6.E.** EXERCISE. Show that $f^{-1}$ is an exact functor from sheaves of abelian groups on $Y$ to sheaves of abelian groups on $X$ (cf. Exercise 3.5.D). (Hint: exactness can be checked on stalks, and by Exercise 3.6.C, the stalks are the same.) The identical argument will show that $f^{-1}$ is an exact functor from $\mathcal{O}_Y$-modules (on $Y$) to $f^{-1}\mathcal{O}_Y$-modules (on $X$), but don't bother writing that down. (Remark for experts: $f^{-1}$ is a left-adjoint, hence right-exact by abstract nonsense, as discussed in §2.6.10. Left-exactness holds because colimits over directed systems are exact.)

**3.6.F.** EXERCISE. (a) Suppose $Z \subset Y$ is a closed subset, and $i : Z \hookrightarrow Y$ is the inclusion. If $\mathcal{F}$ is a sheaf on $Z$, then show that the stalk $(i_*\mathcal{F})_y$ is a one element-set if $y \notin Z$, and $\mathcal{F}_y$ if $y \in Z$.
(b) *Definition:* Define the **support** of a sheaf $\mathcal{F}$ of sets, denoted $\mathrm{Supp}\,\mathcal{F}$, as the locus where the stalks are not the one-element set:

$$\mathrm{Supp}\,\mathcal{F} := \{x \in X \ : \ |\mathcal{F}_x| \neq 1\}.$$

(More generally, if the sheaf has value in some category, the support consists of points where the stalk is not the final object. For sheaves of abelian groups, the support consists of points with non-zero stalks.) Suppose $\mathrm{Supp}\,\mathcal{F} \subset Z$ where $Z$ is closed. Show that the natural map $\mathcal{F} \to i_*i^{-1}\mathcal{F}$ is an isomorphism. Thus a sheaf supported on a closed subset can be considered a sheaf on that closed subset. ("Support" is a useful notion, and will arise again in §14.7.E.)

**3.6.G.** EXTENSION BY ZERO $f_!$: AN OCCASIONAL LEFT-ADJOINT TO $f^{-1}$. In addition to always being a left-adjoint, $f^{-1}$ can sometimes be a right-adjoint. Suppose $i : U \hookrightarrow Y$ is an open immersion of ringed spaces Define **extension by zero** $i_! : \mathit{Mod}_{\mathcal{O}_U} \to \mathit{Mod}_{\mathcal{O}_Y}$ as follows. Suppose $\mathcal{F}$ is an $\mathcal{O}_U$-module. For open $W \subset Y$, $i_!\mathcal{F}(W) = \mathcal{F}(W)$ if $W \subset U$, and 0 otherwise (with the obvious restriction maps). Note that $i_!\mathcal{F}$ is an $\mathcal{O}_Y$-module, and that this clearly defines a functor. (The symbol "!" is read as "shriek". I have no idea why. Thus $i_!$ is read as "$i$-lower-shriek".)
(a) For $y \in Y$, show that $(i_!\mathcal{F})_y = \mathcal{F}_y$ if $y \in U$, and 0 otherwise.
(b) Show that $i_!$ is an exact functor.
(c) Describe an inclusion $i_!i^{-1}\mathcal{F} \hookrightarrow \mathcal{F}$.
(d) Show that $(i_!, i^{-1})$ is an adjoint pair, so there is a natural bijection $\mathrm{Hom}_{\mathcal{O}_Y}(i_!\mathcal{F}, \mathcal{G}) \leftrightarrow \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}\mathcal{G}|_U)$ for any $\mathcal{O}_Y$-module $\mathcal{G}$. (In particular, the sections of $\mathcal{G}$ over $U$ can be identified with $\mathrm{Hom}_{\mathcal{O}_Y}(i_!\mathcal{O}_U, \mathcal{G})$.)

## 3.7 Recovering sheaves from a "sheaf on a base"

Sheaves are natural things to want to think about, but hard to get our hands on. We like the identity and gluability axioms, but they make proving things trickier than for presheaves. We have discussed how we can understand sheaves using stalks. We now introduce a second way of getting a hold of sheaves, by introducing the notion of a *sheaf on a base*. Warning: this way of understanding an entire sheaf from limited information is confusing. It may help to keep sight of the central insight that this limited information lets you understand germs, and the notion of when they are compatible (with nearby germs).

First, we define the notion of a **base of a topology**. Suppose we have a topological space $X$, i.e. we know which subsets $U_i$ of $X$ are open. Then a base of a topology is a subcollection of the open sets $\{B_j\} \subset \{U_i\}$, such that each $U_i$ is a union of the $B_j$. Here is one example that you have seen early in your mathematical life. Suppose $X = \mathbb{R}^n$. Then the way the usual topology is often first defined is by defining *open balls* $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, and declaring that any union of open balls is open. So the balls form a base of the classical topology — we say they *generate* the classical topology. As an application of how we use them, to check continuity of some map $f : X \to \mathbb{R}^n$, you need only think about the pullback of balls on $\mathbb{R}^n$.

Now suppose we have a sheaf $\mathcal{F}$ on $X$, and a base $\{B_i\}$ on $X$. Then consider the information $(\{\mathcal{F}(B_i)\}, \{\mathrm{res}_{B_i,B_j} : \mathcal{F}(B_i) \to \mathcal{F}(B_j)\})$, which is a subset of the information contained in the sheaf — we are only paying attention to the information involving elements of the base, not all open sets.

We can recover the entire sheaf from this information. This is because we can determine the stalks from this information, and we can determine when germs are compatible.

**3.7.A.** EXERCISE. Make this precise.

This suggests a notion, called a **sheaf on a base**. A sheaf of sets (rings etc.) on a base $\{B_i\}$ is the following. For each $B_i$ in the base, we have a set $F(B_i)$. If $B_i \subset B_j$, we have maps $\mathrm{res}_{B_j,B_i} : F(B_j) \to F(B_i)$. (Things called $B$ are always assumed to be in the base.) If $B_i \subset B_j \subset B_k$, then $\mathrm{res}_{B_k,B_i} = \mathrm{res}_{B_j,B_i} \circ \mathrm{res}_{B_k,B_j}$. So far we have defined a **presheaf on a base** $\{B_i\}$.

We also require the **base identity** axiom: If $B = \cup B_i$, then if $f, g \in F(B)$ such that $\mathrm{res}_{B,B_i} f = \mathrm{res}_{B,B_i} g$ for all $i$, then $f = g$.

We require the **base gluability** axiom too: If $B = \cup B_i$, and we have $f_i \in F(B_i)$ such that $f_i$ agrees with $f_j$ on any basic open set contained in $B_i \cap B_j$ (i.e. $\mathrm{res}_{B_i,B_k} f_i = \mathrm{res}_{B_j,B_k} f_j$ for all $B_k \subset B_i \cap B_j$) then there exists $f \in F(B)$ such that $\mathrm{res}_{B,B_i} f = f_i$ for all $i$.

**3.7.1. Theorem. —** *Suppose $\{B_i\}$ is a base on $X$, and $F$ is a sheaf of sets on this base. Then there is a sheaf $\mathcal{F}$ extending $F$ (with isomorphisms $\mathcal{F}(B_i) \cong F(B_i)$ agreeing with the restriction maps). This sheaf $\mathcal{F}$ is unique up to unique isomorphism*

*Proof.* We will define $\mathcal{F}$ as the sheaf of compatible germs of $F$.

Define the **stalk** of a base presheaf $F$ at $p \in X$ by

$$F_p = \varinjlim F(B_i)$$

where the colimit is over all $B_i$ (in the base) containing $p$.

We will say a family of germs in an open set $U$ is compatible near $p$ if there is a section $s$ of $F$ over some $B_i$ containing $p$ such that the germs over $B_i$ are precisely the germs of $s$. More formally, define

$$\mathcal{F}(U) := \{(f_p \in F_p)_{p\in U} : \text{ for all } p \in U, \text{ there exists } B \text{ with } p \subset B \subset U, s \in F(B),$$

$$\text{with } s_q = f_q \text{ for all } q \in B\}$$

where each $B$ is in our base.

This is a sheaf (for the same reasons as the sheaf of compatible germs was earlier, cf. Exercise 3.4.H).

I next claim that if $B$ is in our base, the natural map $F(B) \to \mathcal{F}(B)$ is an isomorphism.

**3.7.B.** TRICKY EXERCISE. Describe the inverse map $\mathcal{F}(B) \to F(B)$, and verify that it is indeed inverse. Possible hint: elements of $\mathcal{F}(U)$ are determined by stalks, as are elements of $F(U)$. □

Thus sheaves on $X$ can be recovered from their "restriction to a base". This is a statement about *objects* in a category, so we should hope for a similar statement about *morphisms*.

**3.7.C.** IMPORTANT EXERCISE: MORPHISMS OF SHEAVES CORRESPOND TO MORPHISMS OF SHEAVES ON A BASE. Suppose $\{B_i\}$ is a base for the topology of $X$.
(a) Verify that a morphism of sheaves is determined by the induced morphism of sheaves on the base.

(b) Show that a morphism of sheaves on the base (i.e. such that the diagram

$$\begin{tikzcd} F(B_i) \arrow[r] \arrow[d] & G(B_i) \arrow[d] \\ F(B_j) \arrow[r] & G(B_j) \end{tikzcd}$$

commutes for all $B_j \hookrightarrow B_i$) gives a morphism of the induced sheaves. (Possible hint: compatible stalks.)

**3.7.D.** IMPORTANT EXERCISE. Suppose $X = \cup U_i$ is an open cover of $X$, and we have sheaves $\mathcal{F}_i$ on $U_i$ along with isomorphisms $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ (with $\phi_{ii}$ the identity) that agree on triple overlaps (i.e. $\phi_{jk} \circ \phi_{ij} = \phi_{ij}$ on $U_i \cap U_j \cap U_k$). Show that these sheaves can be glued together into a sheaf $\mathcal{F}$ on $X$ (unique up to unique isomorphism), such that $\mathcal{F}_i = \mathcal{F}|_{U_i}$, and the isomorphisms over $U_i \cap U_j$ are the obvious ones. (Thus we can "glue sheaves together", using limited patching information.) Warning: we are not assuming this is a finite cover, so you cannot use induction. For this reason this exercise can be perplexing. (You can use the ideas of this section to solve this problem, but you don't necessarily need to. Hint: As the base, take those open sets contained in *some* $U_i$. Small observation: the hypothesis that $\phi_{ii}$ is extraneous, as it follows from the cocycle condition.)

**3.7.2.** *Remark for experts.* Exercise 3.7.D almost says that the "set" of sheaves forms a sheaf itself, but not quite. Making this precise leads one to the notion of a *stack*.

**3.7.E.** UNIMPORTANT EXERCISE. Suppose a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ on a base $B_i$ is surjective for all $B_i$ (i.e. $\mathcal{F}(B_i) \to \mathcal{G}(B_i)$ is surjective for all $i$). Show that the morphism of sheaves (*not* on the base) is surjective. The converse is not true, unlike the case for injectivity. This gives a useful criterion for surjectivity ("surjectivity on small enough open sets").

# Part II

# Schemes

CHAPTER 4

# Toward affine schemes: the underlying set, and the underlying topological space

*The very idea of scheme is of infantile simplicity — so simple, so humble, that no one before me thought of stooping so low. So childish, in short, that for years, despite all the evidence, for many of my erudite colleagues, it was really "not serious"! — Grothendieck*

## 4.1 Toward schemes

We are now ready to consider the notion of a *scheme*, which is the type of geometric space central to algebraic geometry. We should first think through what we mean by "geometric space". You have likely seen the notion of a manifold, and we wish to abstract this notion so that it can be generalized to other settings, notably so that we can deal with non-smooth and arithmetic objects.

The key insight behind this generalization is the following: we can understand a geometric space (such as a manifold) well by understanding the functions on this space. More precisely, we will understand it through the sheaf of functions on the space. If we are interested in differentiable manifolds, we will consider differentiable functions; if we are interested in smooth manifolds, we will consider smooth functions; and so on.

Thus we will define a scheme to be the following data

- *The set:* the points of the scheme
- *The topology:* the open sets of the scheme
- *The structure sheaf:* the sheaf of "algebraic functions" (a sheaf of rings) on the scheme.

Recall that a topological space with a sheaf of rings is called a *ringed space* (§3.2.12).

We will try to draw pictures throughout. Pictures can help develop geometric intuition, which can guide the algebraic development (and, eventually, vice versa). Some people find pictures very helpful, while others are repulsed or nonplussed or confused.

We will try to make all three notions as intuitive as possible. For the set, in the key example of complex (affine) varieties (roughly, things cut out in $\mathbb{C}^n$ by polynomials), we will see that the points are the "traditional points" ($n$-tuples of complex numbers), plus some extra points that will be handy to have around. For the topology, we will require that "algebraic functions vanish on closed sets", and require nothing else. For the sheaf of algebraic functions (the structure sheaf), we will expect that in the complex plane, $(3x^2 + y^2)/(2x + 4xy + 1)$ should be an algebraic function on the open set consisting of points where the denominator doesn't vanish, and this will largely motivate our definition.

**4.1.1. Example: Differentiable manifolds.** As motivation, we return to our example of differentiable manifolds, reinterpreting them in this light. We will be quite informal in this discussion. Suppose $X$ is a manifold. It is a topological space, and has a *sheaf of differentiable functions* $\mathcal{O}_X$ (see §3.1). This gives $X$ the structure of a ringed space. We have observed that evaluation at a point $p \in X$ gives a surjective map from the stalk to $\mathbb{R}$

$$\mathcal{O}_{X,p} \twoheadrightarrow \mathbb{R},$$

so the kernel, the (germs of) functions vanishing at $p$, is a maximal ideal $\mathfrak{m}_X$ (see §3.1.1).

We could *define* a differentiable real manifold as a topological space $X$ with a sheaf of rings. We would require that there is a cover of $X$ by open sets such that on each open set the ringed space is isomorphic to a ball around the origin in $\mathbb{R}^n$ (with the sheaf of differentiable functions on that ball). With this definition, the ball is the basic patch, and a general manifold is obtained by gluing these patches together. (Admittedly, a great deal of geometry comes from how one chooses to patch the balls together!) In the algebraic setting, the basic patch is the notion of an *affine scheme*, which we will discuss soon. (In the definition of manifold, there is an additional requirement that the topological space be Hausdorff, to avoid pathologies. Schemes are often required to be "separated" to avoid essentially the same pathologies. Separatedness will be discussed in Chapter 11.)

*Functions are determined by their values at points.* This is an obvious statement, but won't be true for schemes in general. We will see an example in Exercise 4.2.A(a), and discuss this behavior further in §4.2.9.

*Morphisms of manifolds.* How can we describe differentiable maps of manifolds $X \to Y$? They are certainly continuous maps — but which ones? We can pull back functions along continuous maps. Differentiable functions pull back to differentiable functions. More formally, we have a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. (The inverse image sheaf $f^{-1}$ was defined in §3.6.) Inverse image is left-adjoint to pushforward, so we also get a map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Certainly given a differentiable map of manifolds, differentiable functions pull back to differentiable functions. It is less obvious that *this is a sufficient condition for a continuous function to be differentiable*.

**4.1.A.** IMPORTANT EXERCISE FOR THOSE WITH A LITTLE EXPERIENCE WITH MANIFOLDS. Prove that a continuous function of differentiable manifolds $f : X \to Y$ is differentiable if differentiable functions pull back to differentiable functions, i.e. if pullback by $f$ gives a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. (Hint: check this on small patches. Once you figure out what you are trying to show, you'll realize that the result is immediate.)

**4.1.B.** EXERCISE. Show that a morphism of differentiable manifolds $f : X \to Y$ with $f(p) = q$ induces a morphism of stalks $f^\# : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$. Show that $f^\#(\mathfrak{m}_{Y,q}) \subset \mathfrak{m}_{X,p}$. In other words, if you pull back a function that vanishes at $q$, you get a function that vanishes at $p$ — not a huge surprise. (In §7.3, we formalize this by saying that maps of differentiable manifolds are maps of local-ringed spaces.)

**4.1.2.** *Aside.* Here is a little more for experts: Notice that this induces a map on tangent spaces (see Aside 3.1.2)

$$(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2)^\vee \to (\mathfrak{m}_{Y,q}/\mathfrak{m}_{Y,q}^2)^\vee.$$

This is the tangent map you would geometrically expect. Again, it is interesting that the cotangent map $\mathfrak{m}_{Y,q}/\mathfrak{m}_{Y,q}^2 \to \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$ is algebraically more natural than the tangent map (there are no "duals").

Experts are now free to try to interpret other differential-geometric information using only the map of topological spaces and map of sheaves. For example: how can one check if $f$ is a submersion? How can one check if $f$ is an immersion? (We will see that the algebro-geometric version of these notions are *smooth morphism* and *locally closed immersion*, see Chapter 25 and §9.1.2 respectively.)

**4.1.3.** *Side Remark.* Manifolds are covered by disks that are all isomorphic. This isn't true for schemes (even for "smooth complex varieties"). There are examples of two "smooth complex curves" (the algebraic version of Riemann surfaces) $X$ and $Y$ so that no non-empty open subset of $X$ is isomorphic to a non-empty open subset of $Y$. And there is an example of a Riemann surface $X$ such that no two open subsets of $X$ are isomorphic. Informally, this is because in the Zariski topology on schemes, all non-empty open sets are "huge" and have more "structure".

**4.1.4. Other examples.** If you are interested in differential geometry, you will be interested in differentiable manifolds, on which the functions under consideration are differentiable functions. Similarly, if you are interested in topology, you will be interested in topological spaces, on which you will consider the continuous function. If you are interested in complex geometry, you will be interested in complex manifolds (or possibly "complex analytic varieties"), on which the functions are holomorphic functions. In each of these cases of interesting "geometric spaces", the topological space and sheaf of functions is clear. The notion of scheme fits naturally into this family.

## 4.2 The underlying set of affine schemes

For any ring $A$, we are going to define something called $\operatorname{Spec} A$, the **spectrum of** $A$. In this section, we will define it as a set, but we will soon endow it with a topology, and later we will define a sheaf of rings on it (the structure sheaf). Such an object is called an *affine scheme*. Later $\operatorname{Spec} A$ will denote the set along with the topology, and a sheaf of functions. But for now, as there is no possibility of confusion, $\operatorname{Spec} A$ will just be the set.

**4.2.1.** The set $\operatorname{Spec} A$ is the set of prime ideals of $A$. The point of $\operatorname{Spec} A$ corresponding to the prime ideal $\mathfrak{p}$ will be denoted $[\mathfrak{p}]$. Elements $a \in A$ will be called **functions on** $\operatorname{Spec} A$, and their **value** at the point $[\mathfrak{p}]$ will be $a \pmod{\mathfrak{p}}$. *This is weird: a function can take values in different rings at different points — the function* $5$ *on* $\operatorname{Spec} \mathbb{Z}$ *takes the value* $1 \pmod 2$ *at* $[(2)]$ *and* $2 \pmod 3$ *at* $[(3)]$. "An element $a$ of the ring lying in a prime ideal $\mathfrak{p}$" translates to "a function $a$ that is $0$ at the point $[\mathfrak{p}]$" or "a function $a$ vanishing at the point $[\mathfrak{p}]$", and we will use these phrases interchangeably. Notice that if you add or multiply two functions, you add or multiply their values at all points; this is a translation of the fact that $A \to A/\mathfrak{p}$ is a ring homomorphism. These translations are important — make sure you are very comfortable with them! They should become second nature.

We now give some examples.

**Example 1 (the complex affine line): $\mathbb{A}^1_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x]$.** Let's find the prime ideals of $\mathbb{C}[x]$. As $\mathbb{C}[x]$ is an integral domain, $0$ is prime. Also, $(x-a)$ is prime, for any $a \in \mathbb{C}$: it is even a maximal ideal, as the quotient by this ideal is field:

$$0 \longrightarrow (x-a) \longrightarrow \mathbb{C}[x] \xrightarrow{f \mapsto f(a)} \mathbb{C} \longrightarrow 0$$

(This exact sequence may remind you of (3.1.1.1) in our motivating example of manifolds.)

We now show that there are no other prime ideals. We use the fact that $\mathbb{C}[x]$ has a division algorithm, and is a unique factorization domain. Suppose $\mathfrak{p}$ is a prime ideal. If $\mathfrak{p} \neq (0)$, then suppose $f(x) \in \mathfrak{p}$ is a non-zero element of smallest degree. It is not constant, as prime ideals can't contain 1. If $f(x)$ is not linear, then factor $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have positive degree. (Here we use that $\mathbb{C}$ is algebraically closed.) Then $g(x) \in \mathfrak{p}$ or $h(x) \in \mathfrak{p}$, contradicting the minimality of the degree of $f$. Hence there is a linear element $x - a$ of $\mathfrak{p}$. Then I claim that $\mathfrak{p} = (x-a)$. Suppose $f(x) \in \mathfrak{p}$. Then the division algorithm would give $f(x) = g(x)(x-a) + m$ where $m \in \mathbb{C}$. Then $m = f(x) - g(x)(x-a) \in \mathfrak{p}$. If $m \neq 0$, then $1 \in \mathfrak{p}$, giving a contradiction.

Thus we have a picture of $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$ (see Figure 4.1). There is one point for each complex number, plus one extra point $[(0)]$. We can mostly picture $\mathbb{A}^1_{\mathbb{C}}$ as $\mathbb{C}$: the point $[(x-a)]$ we will reasonably associate to $a \in \mathbb{C}$. Where should we picture the point $[(0)]$? The best way of thinking about it is somewhat zen. It is somewhere on the complex line, but nowhere in particular. Because $(0)$ is contained in all of these primes, we will somehow associate it with this line passing through all the other points. $[(0)]$ is called the "generic point" of the line; it is "generically on the line" but you can't pin it down any further than that. (We will formally define "generic point" in §4.6.) We will place it far to the right for lack of anywhere better to put it. You will notice that we sketch $\mathbb{A}^1_{\mathbb{C}}$ as one-(real-)dimensional (even though we picture it as an enhanced version of $\mathbb{C}$); this is to later remind ourselves that this will be a one-dimensional space, where dimensions are defined in an algebraic (or complex-geometric) sense. (Dimension will be defined in Chapter 12.)

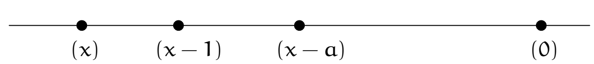



FIGURE 4.1. A picture of $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$

To give you some feeling for this space, we make some statements that are currently undefined, but suggestive. The functions on $\mathbb{A}^1_{\mathbb{C}}$ are the polynomials. So $f(x) = x^2 - 3x + 1$ is a function. What is its value at $[(x-1)]$, which we think of as the point $1 \in \mathbb{C}$? Answer: $f(1)$! Or equivalently, we can evalute $f(x)$ modulo $x - 1$ — this is the same thing by the division algorithm. (What is its value at $(0)$? It is $f(x) \pmod 0$, which is just $f(x)$.)

Here is a more complicated example: $g(x) = (x-3)^3/(x-2)$ is a "rational function". It is defined everywhere but $x = 2$. (When we know what the structure sheaf is, we will be able to say that it is an element of the structure sheaf on the open set $\mathbb{A}^1_{\mathbb{C}} - \{2\}$.) We want to say that $g(x)$ has a triple zero at 3, and a single pole at 2, and we will be able to after §13.3.

**Example 2 (the affine line over $k = \overline{k}$): $\mathbb{A}^1_k := \operatorname{Spec} k[x]$ where $k$ is an algebraically closed field.** This is called the affine line over $k$. All of our discussion in the previous example carries over without change. We will use the same picture, which is after all intended to just be a metaphor.

**Example 3: $\operatorname{Spec} \mathbb{Z}$.** An amazing fact is that from our perspective, this will look a lot like the affine line $\mathbb{A}^1_{\overline{k}}$. The integers, like $\overline{k}[x]$, form a unique factorization domain, with a division algorithm. The prime ideals are: $(0)$, and $(p)$ where $p$ is prime. Thus everything from Example 1 carries over without change, even the picture. Our picture of $\operatorname{Spec} \mathbb{Z}$ is shown in Figure 4.2.

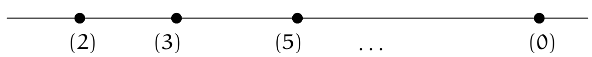



FIGURE 4.2. A "picture" of $\operatorname{Spec} \mathbb{Z}$, which looks suspiciously like Figure 4.1

Let's blithely carry over our discussion of functions to this space. $100$ is a function on $\operatorname{Spec} \mathbb{Z}$. It's value at (3) is "$1 \pmod 3$". It's value at (2) is "$0 \pmod 2$", and in fact it has a double zero. 27/4 is a rational function on $\operatorname{Spec} \mathbb{Z}$, defined away from (2). We want to say that it has a double pole at (2), and a triple zero at (3). Its value at (5) is

$$27 \times 4^{-1} \equiv 2 \times (-1) \equiv 3 \pmod 5.$$

**Example 4: silly but important examples.** The set $\operatorname{Spec} k$ where $k$ is any field is boring: only one point. $\operatorname{Spec} 0$, where $0$ is the zero-ring, is the empty set, as $0$ has no prime ideals.

**4.2.A.** A SMALL EXERCISE ABOUT SMALL SCHEMES. (a) Describe the set $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$. The ring $k[\epsilon]/(\epsilon^2)$ is called the ring of **dual numbers**, and will turn out to be quite useful. You should think of $\epsilon$ as a very small number, so small that its square is $0$ (although it itself is not $0$). It is a non-zero function whose value at all points is zero, thus giving our first example of functions not being determined by their values at points. We will discuss this phenomenon further in §4.2.9.
(b) Describe the set $\operatorname{Spec} k[x]_{(x)}$. We will see this scheme again repeatedly, starting with §4.2.6 and Exercise 4.4.J.

In Example 2, we restricted to the case of algebraically closed fields for a reason: things are more subtle if the field is not algebraically closed.

**Example 5 (the affine line over $\mathbb{R}$): $\mathbb{R}[x]$.** Using the fact that $\mathbb{R}[x]$ is a unique factorization domain, similar arguments to those of Examples 1–3 show that the primes are $(0)$, $(x-a)$ where $a \in \mathbb{R}$, and $(x^2+ax+b)$ where $x^2+ax+b$ is an irreducible quadratic. The latter two are maximal ideals, i.e. their quotients are fields. For example: $\mathbb{R}[x]/(x-3) \cong \mathbb{R}$, $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.

**4.2.B.** UNIMPORTANT EXERCISE. Show that for the last type of prime, of the form $(x^2 + ax + b)$, the quotient is *always* isomorphic to $\mathbb{C}$.

So we have the points that we would normally expect to see on the real line, corresponding to real numbers; the generic point $0$; and new points which we may interpret as *conjugate pairs* of complex numbers (the roots of the quadratic). This last type of point should be seen as more akin to the real numbers than to the generic point. You can picture $\mathbb{A}^1_{\mathbb{R}}$ as the complex plane, folded along the real axis. But the key point is that Galois-conjugate points (such as $i$ and $-i$) are considered glued.

Let's explore functions on this space. Consider the function $f(x) = x^3 - 1$. Its value at the point $[(x-2)]$ is $f(x) = 7$, or perhaps better, $7 \pmod{x-2}$. How about at $(x^2+1)$? We get

$$x^3 - 1 \equiv -x - 1 \pmod{x^2+1},$$

which may be profitably interpreted as $-i - 1$.

One moral of this example is that we can work over a non-algebraically closed field if we wish. It is more complicated, but we can recover much of the information we care about.

**4.2.C.** IMPORTANT EXERCISE. Describe the set $\mathbb{A}^1_{\mathbb{Q}}$. (This is harder to picture in a way analogous to $\mathbb{A}^1_{\mathbb{R}}$. But the rough cartoon of points on a line, as in Figure 4.1, remains a reasonable sketch.)

**Example 6 (the affine line over $\mathbb{F}_p$): $\mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[x]$.** As in the previous examples, $\mathbb{F}_p[x]$ is a Euclidean domain, so the prime ideals are of the form $(0)$ or $(f(x))$ where $f(x) \in \mathbb{F}_p[x]$ is an irreducible polynomial, which can be of any degree. Irreducible polynomials correspond to sets of Galois conjugates in $\overline{\mathbb{F}}_p$.

Note that $\operatorname{Spec} \mathbb{F}_p[x]$ has $p$ points corresponding to the elements of $\mathbb{F}_p$, but also (infinitely) many more. This makes this space much richer than simply $p$ points. For example, a polynomial $f(x)$ is not determined by its values at the $p$ elements of $\mathbb{F}_p$, but it *is* determined by its values at the points of $\operatorname{Spec} \mathbb{F}_p[x]$. (As we have mentioned before, this is not true for all schemes.)

You should think about this, even if you are a geometric person — this intuition will later turn up in geometric situations. Even if you think you are interested only in working over an algebraically closed field (such as $\mathbb{C}$), you will have non-algebraically closed fields (such as $\mathbb{C}(x)$) forced upon you.

**Example 7 (the complex affine plane): $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$.** (As with Examples 1 and 2, our discussion will apply with $\mathbb{C}$ replaced by *any* algebraically closed field.) Sadly, $\mathbb{C}[x, y]$ is not a principal ideal domain: $(x, y)$ is not a principal ideal. We can quickly name *some* prime ideals. One is $(0)$, which has the same flavor as the $(0)$ ideals in the previous examples. $(x-2, y-3)$ is prime, and indeed maximal, because $\mathbb{C}[x, y]/(x-2, y-3) \cong \mathbb{C}$, where this isomorphism is via $f(x, y) \mapsto f(2, 3)$. More generally, $(x-a, y-b)$ is prime for any $(a, b) \in \mathbb{C}^2$. Also, if $f(x, y)$ is an irreducible polynomial (e.g. $y - x^2$ or $y^2 - x^3$) then $(f(x, y))$ is prime.

**4.2.D.** EXERCISE. (We will see a different proof of this in §12.2.3.) Show that we have identified all the prime ideals of $\mathbb{C}[x, y]$. Hint: Suppose $\mathfrak{p}$ is a prime ideal that is not principal. Show you can find $f(x, y), g(x, y) \in \mathfrak{p}$ with no common factor. By considering the Euclidean algorithm in the Euclidean domain $k(y)[x]$, show that you can find a nonzero $h(x) \in (f(x,y), g(x,y)) \subset \mathfrak{p}$. Using primality, show that you can one of the linear factors of $h(x)$, say $(x-a)$, is in $\mathfrak{p}$. Similarly show there is some $(y-b) \in \mathfrak{p}$.

We now attempt to draw a picture of $\mathbb{A}^2_{\mathbb{C}}$. The maximal primes of $\mathbb{C}[x,y]$ correspond to the traditional points in $\mathbb{C}^2$: $[(x-a, y-b)]$ corresponds to $(a,b) \in \mathbb{C}^2$. We now have to visualize the "bonus points". $[(0)]$ somehow lives behind all of the traditional points; it is somewhere on the plane, but nowhere in particular. So for example, it does not lie on the parabola $y = x^2$. The point $[(y-x^2)]$ lies on the parabola $y = x^2$, but nowhere in particular on it. You can see from this picture that we already are implicitly thinking about "dimension". The primes $(x-a, y-b)$ are somehow of dimension 0, the primes $(f(x,y))$ are of dimension 1, and $(0)$ is of dimension 2. (All of our dimensions here are *complex* or *algebraic* dimensions. The complex plane $\mathbb{C}^2$ has real dimension 4, but complex dimension 2. Complex dimensions are in general half of real dimensions.) We won't define dimension precisely until Chapter 12, but you should feel free to keep it in mind before then.

Note too that maximal ideals correspond to the "smallest" points. Smaller ideals correspond to "bigger" points. "One prime ideal contains another" means that the points "have the opposite containment." All of this will be made precise once we have a topology. This order-reversal is a little confusing, and will remain so even once we have made the notions precise.

We now come to the obvious generalization of Example 7.

**Example 8 (complex affine $n$-space): $\mathbb{A}^n_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$.** (More generally, $\mathbb{A}^n_A$ is defined to be $\operatorname{Spec} A[x_1, \dots, x_n]$, where $A$ is an arbitrary ring.) For concreteness, let's consider $n = 3$. We now have an interesting question in what at first appears to be pure algebra: What are the prime ideals of $\mathbb{C}[x,y,z]$?

Analogously to before, $(x-a, y-b, z-c)$ is a prime ideal. This is a maximal ideal, with residue field $\mathbb{C}$; we think of these as "0-dimensional points". We will often write $(a,b,c)$ for $[(x-a, y-b, z-c)]$ because of our geometric interpretation of these ideals. There are no more maximal ideals, by Hilbert's Weak Nullstellensatz.

**4.2.2. Hilbert's Weak Nullstellensatz. —** *If $k$ is an algebraically closed field, then the maximal ideals $k[x_1, \dots, x_n]$, are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$.*

We may as well state a slightly stronger version now.

**4.2.3. Hilbert's Nullstellensatz. —** *If $k$ is any field, the maximal ideals of $k[x_1, \dots, x_n]$ are precisely those with residue field a finite extension of $k$.*

The Nullstellensatz 4.2.3 clearly implies the Weak Nullstellensatz 4.2.2. You will prove the Nullstellensatz in Exercise 12.2.B.

There are other prime ideals of $\mathbb{C}[x,y,z]$ too. We have $(0)$, which is corresponds to a "3-dimensional point". We have $(f(x,y,z))$, where $f$ is irreducible. To this we associate the hypersurface $f = 0$, so this is "2-dimensional" in nature. But we have not found them all! One clue: we have prime ideals of "dimension" 0, 2, and 3 — we are missing "dimension 1". Here is one such prime ideal: $(x,y)$. We picture this as the locus where $x = y = 0$, which is the $z$-axis. This is a prime ideal, as the corresponding quotient $\mathbb{C}[x,y,z]/(x,y) \cong \mathbb{C}[z]$ is an integral domain (and should be interpreted as the functions on the $z$-axis). There are lots of one-dimensional primes, and it is not possible to classify them in a reasonable way. It will turn out that they correspond to things that we think of as irreducible curves. Thus remarkably the answer to the purely algebraic question ("what are the primes of $\mathbb{C}[x, y, z]$") is fundamentally geometric!

The fact that the closed points $\mathbb{A}^1_{\mathbb{Q}}$ can be interpreted as points of $\overline{\mathbb{Q}}$ where Galois-conjugates are glued together (Exercise 4.2.C) extends to $\mathbb{A}^n_{\mathbb{Q}}$. For example, in $\mathbb{A}^2_{\mathbb{Q}}$, $(\sqrt{2}, \sqrt{2})$ is glued to $(-\sqrt{2}, -\sqrt{2})$ but not to $(\sqrt{2}, -\sqrt{2})$. The following exercise will give you some idea of how this works.

**4.2.E.** EXERCISE. Describe the maximal ideal of $\mathbb{Q}[x, y]$ corresponding $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Describe the maximal ideal of $\mathbb{Q}[x, y]$ corresponding $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$. What are the residue fields in both cases?

**4.2.4. Quotients and localizations.** Two natural ways of getting new rings from old — quotients and localizations — have interpretations in terms of spectra.

**4.2.5.** *Quotients:* $\operatorname{Spec} A/I$ *as a subset of* $\operatorname{Spec} A$. It is an important fact that the primes of $A/I$ are in bijection with the primes of $A$ containing $I$.

**4.2.F.** ESSENTIAL ALGEBRA EXERCISE (MANDATORY IF YOU HAVEN'T SEEN IT BEFORE). Suppose $A$ is a ring, and $I$ an ideal of $A$. Let $\phi : A \to A/I$. Show that $\phi^{-1}$ gives an inclusion-preserving bijection between primes of $A/I$ and primes of $A$ containing $I$. Thus we can picture $\operatorname{Spec} A/I$ as a subset of $\operatorname{Spec} A$.

As an important motivational special case, you now have a picture of *complex affine varieties*. Suppose $A$ is a finitely generated $\mathbb{C}$-algebra, generated by $x_1, \dots, x_n$, with relations $f_1(x_1, \dots, x_n) = \cdots = f_r(x_1, \dots, x_n) = 0$. Then this description in terms of generators and relations naturally gives us an interpretation of $\operatorname{Spec} A$ as a subset of $\mathbb{A}^n_{\mathbb{C}}$, which we think of as "traditional points" ($n$-tuples of complex numbers) along with some "bonus" points we haven't yet fully described. To see which of the traditional points are in $\operatorname{Spec} A$, we simply solve the equations $f_1 = \cdots = f_r = 0$. For example, $\operatorname{Spec} \mathbb{C}[x, y, z]/(x^2+y^2-z^2)$ may be pictured as shown in Figure 4.3. (Admittedly this is just a "sketch of the $\mathbb{R}$-points", but we will still find it helpful later.) This entire picture carries over (along with the Nullstellensatz) with $\mathbb{C}$ replaced by any algebraically closed field. Indeed, the picture of Figure 4.3 can be said to depict $k[x, y, z]/(x^2 + y^2 - z^2)$ for most algebraically closed fields $k$ (although it is misleading in characteristic 2, because of the coincidence $x^2 + y^2 - z^2 = (x + y + z)^2$).

**4.2.6.** *Localizations:* $\operatorname{Spec} S^{-1}A$ *as a subset of* $\operatorname{Spec} A$. The following exercise shows how prime behave under localization.

**4.2.G.** ESSENTIAL ALGEBRA EXERCISE (MANDATORY IF YOU HAVEN'T SEEN IT BEFORE). Suppose $S$ is a multiplicative subset of $A$. The map $\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ gives an order-preserving bijection of the primes of $S^{-1}A$ with the primes of $A$ that *don't meet* the multiplicative set $S$.

Recall from §2.3.3 that there are two important flavors of localization. The first is $A_f = \{1, f, f^2, \dots\}^{-1}A$ where $f \in A$. A motivating example is $A = \mathbb{C}[x, y]$,

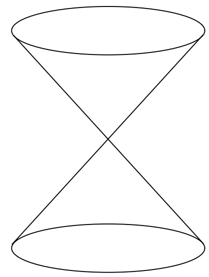


FIGURE 4.3. A "picture" of $\operatorname{Spec} \mathbb{C}[x, y, z]/(x^2 + y^2 - z^2)$

$f = y - x^2$. The second is $A_{\mathfrak{p}} = (A - \mathfrak{p})^{-1}A$, where $\mathfrak{p}$ is a prime ideal. A motivating example is $A = \mathbb{C}[x, y]$, $S = A - (x, y)$.

If $S = \{1, f, f^2, \dots\}$, the primes of $S^{-1}A$ are just those primes not containing $f$ — the points where "$f$ doesn't vanish". (In §4.5, we will call this a *distinguished open set*, once we know what open sets are.) So to picture $\operatorname{Spec} \mathbb{C}[x, y]_{y-x^2}$, we picture the affine plane, and throw out those points on the parabola $y - x^2$ — the points $(a, a^2)$ for $a \in \mathbb{C}$ (by which we mean $[(x - a, y - a^2)]$), as well as the "new kind of point" $[(y - x^2)]$.

It can be initially confusing to think about localization in the case where zero-divisors are inverted, because localization $A \to S^{-1}A$ is not injective (Exercise 2.3.C). Geometric intuition can help. Consider the case $A = \mathbb{C}[x, y]/(xy)$ and $f = x$. What is the localization $A_f$? The space $\operatorname{Spec} \mathbb{C}[x, y]/(xy)$ "is" the union of the two axes in the plane. Localizing means throwing out the locus where $x$ vanishes. So we are left with the $x$-axis, minus the origin, so we expect $\operatorname{Spec} \mathbb{C}[x]_x$. So there should be some natural isomorphism $(\mathbb{C}[x, y]/(xy))_x \cong \mathbb{C}[x]_x$.

**4.2.H.** EXERCISE. Show that these two rings are isomorphic. (You will see that $y$ on the left goes to $0$ on the right.)

If $S = A - \mathfrak{p}$, the primes of $S^{-1}A$ are just the primes of $A$ contained in $\mathfrak{p}$. In our example $A = \mathbb{C}[x, y]$, $\mathfrak{p} = (x, y)$, we keep all those points corresponding to "things through the origin", i.e. the $0$-dimensional point $(x, y)$, the 2-dimensional point $(0)$, and those 1-dimensional points $(f(x, y))$ where $f(0, 0) = 0$, i.e. those "irreducible curves through the origin". You can think of this being a shred of the plane near the origin; anything not actually "visible" at the origin is discarded (see Figure 4.4).

Another example is when $A = \operatorname{Spec} k[x]$, and $\mathfrak{p} = (x)$ (or more generally when $\mathfrak{p}$ is any maximal ideal). Then $A_{\mathfrak{p}}$ has only two prime ideals (Exercise 4.2.A(b)). You should see this as the germ of a "smooth curve", where one point is the "classical point", and the other is the "generic point of the curve". This is an example of a discrete valuation ring, and indeed all discrete valuation rings should be visualized in such a way. We will discuss discrete valuation rings in §13.3. By then we will have justified the use of the words "smooth" and "curve". (Reality check: try to picture Spec of $\mathbb{Z}$ localized at $(2)$ and at $(0)$. How do the two pictures differ?)

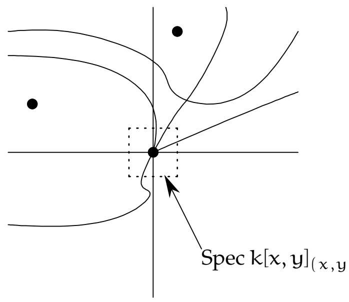



FIGURE 4.4. Picturing $\operatorname{Spec} \mathbb{C}[x,y]_{(x,y)}$ as a "shred of $\mathbb{A}^2_{\mathbb{C}}$". Only those points near the origin remain.

**4.2.7. Important fact: Maps of rings induce maps of spectra (as sets).** We now make an observation that will later grow up to be morphisms of schemes.

**4.2.I.** IMPORTANT EASY EXERCISE. If $\phi : B \to A$ is a map of rings, and $\mathfrak{p}$ is a prime ideal of $A$, show that $\phi^{-1}(\mathfrak{p})$ is a prime ideal of $B$.

Hence a map of rings $\phi : B \to A$ induces a map of sets $\operatorname{Spec} A \to \operatorname{Spec} B$ "in the opposite direction". This gives a contravariant functor from the category of rings to the category of sets: the composition of two maps of rings induces the composition of the corresponding maps of spectra.

**4.2.J.** EASY EXERCISE. Let $B$ be a ring.
(a) Suppose $I \subset B$ is an ideal. Show that the map $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is the inclusion of §4.2.5.
(b) Suppose $S \subset B$ is a multiplicative set. Show that the map $\operatorname{Spec} S^{-1}B \to \operatorname{Spec} B$ is the inclusion of §4.2.6.

**4.2.8.** *An explicit example.* In the case of affine complex varieties (or indeed affine varieties over any algebraically closed field), the translation between maps given by explicit formulas and maps of rings is quite direct. For example, consider a map from the parabola $\mathbb{C}^2$ (with coordinates $a$ and $b$) given by $b = a^2$, to the "curve" in $\mathbb{C}^3$ (with coordinates $x$, $y$, and $z$) cut out by the equations $y = x^2$ and $z = y^2$. Suppose the map sends the point $(a, b) \in \mathbb{C}^2$ to the point $(a, b, b^2) \in \mathbb{C}^3$. In our new language, we have map

$$\operatorname{Spec} \mathbb{C}[a,b]/(b-a^2) \longrightarrow \operatorname{Spec} \mathbb{C}[x,y,z]/(y-x^2, z-y^2)$$

given by

$$\mathbb{C}[a,b]/(b-a^2) \longleftarrow \mathbb{C}[x,y,z]/(y-x^2, z-y^2)$$

$$(a, b, b^2) \longleftarrow\!\shortmid (x, y, z),$$

i.e. $x \mapsto a$, $y \mapsto b$, and $z \mapsto b^2$. If the idea is not yet clear, the following two exercises may help.

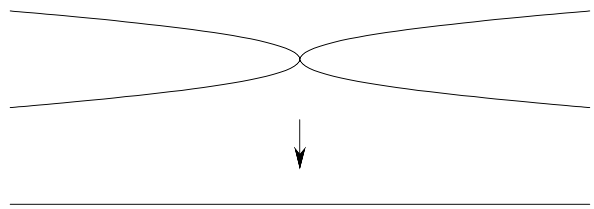


FIGURE 4.5. The map $\mathbb{C} \to \mathbb{C}$ given by $y \mapsto y^2$

**4.2.K.** EXERCISE (SPECIAL CASE). Consider the map of complex manifolds sending $\mathbb{C} \to \mathbb{C}$ via $y \mapsto y^2$; you can picture it as the projection of the parabola $x = y^2$ in the plane to the $x$-axis (see Figure 4.5). Interpret the corresponding map of rings as given by $\mathbb{C}[x] \mapsto \mathbb{C}[y]$ by $x \mapsto y^2$. Verify that the preimage (the fiber) above the point $a \in \mathbb{C}$ is the point(s) $\pm\sqrt{a} \in \mathbb{C}$, using the definition given above. (A more sophisticated version of this example appears in Example 10.3.3.)

**4.2.L.** EXERCISE (GENERAL CASE). (a) Show that the map

$$\phi : (y_1, y_2, \dots, y_n) \mapsto (f_1(x_1, \dots, x_m), f_2(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m))$$

determines a map

$$\operatorname{Spec} \mathbb{C}[x_1, \dots, x_m]/I \to \operatorname{Spec} \mathbb{C}[y_1, \dots, y_n]/J$$

if $\phi(J) \subset I$.
(b) Via the identification of the Nullstellensatz, interpret the map of (a) as a map $\mathbb{C}^m \to \mathbb{C}^n$ given by

$$(x_1, \dots, x_m) \mapsto (f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)).$$

The converse to (a) isn't quite true. Once you have more experience and intuition, you can figure out when it is true, and when it can be false. The failure of the converse to hold has to do with nilpotents, which we come to very shortly (§4.2.9).

**4.2.M.** IMPORTANT EXERCISE. Consider the map of sets $f : \mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$, given by the ring map $\mathbb{Z} \to \mathbb{Z}[x_1, \dots, x_n]$. If $p$ is prime, describe a bijection between the fiber $f^{-1}([(p)])$ and $\mathbb{A}^n_{\mathbb{F}_p}$. This exercise may give you a sense of how to picture maps (see Figure 4.6), and in particular why you can think of $\mathbb{A}^n_{\mathbb{Z}}$ as an "$\mathbb{A}^n$-bundle" over $\operatorname{Spec} \mathbb{Z}$. (Can you interpret the fiber over $[(0)]$ as $\mathbb{A}^n_k$ for some field $k$?)

**4.2.9. Functions are not determined by their values at points: the fault of nilpotents.** We conclude this section by describing some strange behavior. We are developing machinery that will let us bring our geometric intuition to algebra. There is one serious serious point where your intuition will be false, so you should know now, and adjust your intuition appropriately. As noted by Mumford ([**M-CAS**, p. 12]), "it is this aspect of schemes which was most scandalous when Grothendieck defined them."

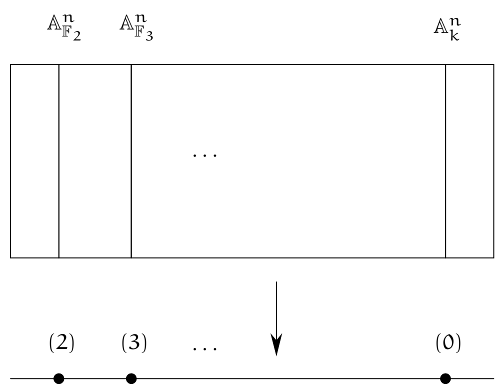



FIGURE 4.6. A picture of $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ as a "family of $\mathbb{A}^n$'s", or an "$\mathbb{A}^n$-bundle over $\operatorname{Spec} \mathbb{Z}$". What is $k$?

Suppose we have a function (ring element) vanishing at all points. Then it is not necessarily the zero function! The translation of this question is: is the intersection of all prime ideals necessarily just $0$? The answer is no, as is shown by the example of the ring of dual numbers $k[\epsilon]/(\epsilon^2)$: $\epsilon \neq 0$, but $\epsilon^2 = 0$. (We saw this scheme in Exercise 4.2.A(a).) Any function whose power is zero certainly lies in the intersection of all prime ideals.

**4.2.N.** EXERCISE. Ring elements that have a power that is $0$ are called **nilpotents**. (a) If $I$ is an ideal of nilpotents, show that the inclusion $\operatorname{Spec} B/I \to \operatorname{Spec} B$ of Exercise 4.2.F is a bijection. Thus nilpotents don't affect the underlying set. (We will soon see in §4.4.5 that they won't affect the topology either — the difference will be in the structure sheaf.) (b) (easy) Show that the nilpotents of a ring $B$ form an ideal. This ideal is called the **nilradical**, and is denoted $\mathfrak{N}$.

Thus the nilradical is contained in the intersection of all the prime ideals. The converse is also true:

**4.2.10. Theorem. —** *The nilradical $\mathfrak{N}(A)$ is the intersection of all the primes of $A$.*

**4.2.O.** EXERCISE. If you don't know this theorem, then look it up, or even better, prove it yourself. (Hint: Use the fact that any proper ideal of $A$ is contained in a maximal ideal, which requires the axiom of choice. Possible further hint: Suppose $x \notin \mathfrak{N}(A)$. We wish to show that there is a prime ideal not containing $x$. Show that $A_x$ is not the $0$-ring, by showing that $1 \neq 0$.)

**4.2.11.** In particular, although it is upsetting that functions are not determined by their values at points, we have precisely specified what the failure of this intuition is: two functions have the same values at points if and only if they differ by a nilpotent. You should think of this geometrically: a function vanishes at every point of the spectrum of a ring if and only if it has a power that is zero. And if there are no non-zero nilpotents — if $\mathfrak{N} = (0)$ — then functions *are* determined by their values at points. If a ring has no non-zero nilpotents, we say that it is **reduced**.

**4.2.P.** FUN UNIMPORTANT EXERCISE: DERIVATIVES WITHOUT DELTAS AND EPSILONS. Suppose we have a polynomial $f(x) \in k[x]$. Instead, we work in $k[x, \epsilon]/\epsilon^2$. What then is $f(x + \epsilon)$? (Do a couple of examples, then prove the pattern you observe.) This is a hint that nilpotents will be important in defining differential information (Chapter 22).

## 4.3 Visualizing schemes I: generic points

For years, you have been able to picture $x^2 + y^2 = 1$ in the plane, and you now have an idea of how to picture $\operatorname{Spec} \mathbb{Z}$. If we are claiming to understand rings as geometric objects (through the Spec functor), then we should wish to develop geometric insight into them. To develop geometric intuition about schemes, it is helpful to have pictures in your mind, extending your intuition about geometric spaces you are already familiar with. As we go along, we will empirically develop some idea of what schemes should look like. This section summarizes what we have gleaned so far.

Some mathematicians prefer to think completely algebraically, and never think in terms of pictures. Others will be disturbed by the fact that this is an art, not a science. And finally, this hand-waving will necessarily never be used in the rigorous development of the theory. For these reasons, you may wish to skip these sections. However, having the right picture in your mind can greatly help understanding what facts should be true, and how to prove them.

Our starting point is the example of "affine complex varieties" (things cut out by equations involving a finite number variables over $\mathbb{C}$), and more generally similar examples over arbitrary algebraically closed fields. We begin with notions that are intuitive ("traditional" points behaving the way you expect them to), and then add in the two features which are new and disturbing, generic points and non-reduced behavior. You can then extend this notion to seemingly different spaces, such as $\operatorname{Spec} \mathbb{Z}$.

Hilbert's Weak Nullstellensatz 4.2.2 shows that the "traditional points" are present as points of the scheme, and this carries over to any algebraically closed field. If the field is not algebraically closed, the traditional points are glued together into clumps by Galois conjugation, as in Examples 5 (the real affine line) and 6 (the affine line over $\mathbb{F}_p$) in §4.2 above. This is a geometric interpretation of Hilbert's Nullstellensatz 4.2.3.

But we have some additional points to add to the picture. You should remember that they "correspond" to "irreducible" "closed" (algebraic) subsets. As motivation, consider the case of the complex affine plane (Example 7): we had one for each irreducible polynomial, plus one corresponding to the entire plane. We will make "closed" precise when we define the Zariski topology (in the next section). You may already have an idea of what "irreducible" should mean; we make that precise at the start of §4.6. By "correspond" we mean that each closed irreducible subset has a corresponding point sitting on it, called its *generic point* (defined in §4.6). It is a new point, distinct from all the other points in the subset.

The correspondence is described in Exercise 4.7.E for $\operatorname{Spec} A$, and in Exercise 6.1.B for schemes in general. We don't know precisely where to draw the generic point, so we may stick it arbitrarily anywhere, but you should think of it as being "almost everywhere", and in particular, near every other point in the subset.

In §4.2.5, we saw how the points of $\operatorname{Spec} A/I$ should be interpreted as a subset of $\operatorname{Spec} A$. So for example, when you see $\operatorname{Spec} \mathbb{C}[x, y]/(x + y)$, you should picture this not just as a line, but as a line in the $xy$-plane; the choice of generators $x$ and $y$ of the algebra $\mathbb{C}[x, y]$ implies an inclusion into affine space.

In §4.2.6, we saw how the points of $\operatorname{Spec} S^{-1}A$ should be interpreted as subsets of $\operatorname{Spec} A$. The two most important cases were discussed. The points of $\operatorname{Spec} A_f$ correspond to the points of $\operatorname{Spec} A$ where $f$ doesn't vanish; we will later (§4.5) interpret this as a distinguished open set.

If $\mathfrak{p}$ is a prime ideal, then $\operatorname{Spec} A_{\mathfrak{p}}$ should be seen as a "shred of the space $\operatorname{Spec} A$ near the subset corresponding to $\mathfrak{p}$". The simplest nontrivial case of this is $(x) \subset \operatorname{Spec} k[x]$ (see Exercise 4.2.A, which we discuss again in Exercise 4.4.J).

## 4.4 The Zariski topology: The underlying topological space of an affine scheme

We next introduce the *Zariski topology* on the spectrum of a ring. For example, consider $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, the complex plane (with a few extra points). In algebraic geometry, we will only be allowed to consider algebraic functions, i.e. polynomials in $x$ and $y$. The locus where a polynomial vanishes should reasonably be a closed set, and the Zariski topology is defined by saying that the only sets we should consider closed should be these sets, and other sets forced to be closed by these. In other words, it is the coarsest topology where these sets are closed.

In particular, although topologies are often described using open subsets, it will more convenient for us to define this topology in terms of closed subsets. If $S$ is a subset of a ring $A$, define the V**anishing set** of $S$ by

$$V(S) := \{[\mathfrak{p}] \in \operatorname{Spec} A \ : \ S \subset \mathfrak{p}\}.$$

It is the set of points on which all elements of $S$ are zero. (It should now be second nature to equate "vanishing at a point" with "contained in a prime".) We declare that these — and no other — are the closed subsets.

For example, consider $V(xy, yz) \subset \mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$. Which points are contained in this locus? We think of this as solving $xy = yz = 0$. Of the "traditional" points (interpreted as ordered triples of complex numbers, thanks to the Hilbert's Nullstellensatz 4.2.2), we have the points where $y = 0$ or $x = z = 0$: the $xz$-plane and the $y$-axis respectively. Of the "new" points, we have the generic point of the $xz$-plane (also known as the point $[(y)]$), and the generic point of the $y$-axis (also known as the point $[(x, z)]$). You might imagine that we also have a number of "one-dimensional" points contained in the $xz$-plane.

**4.4.A.** Easy Exercise. Check that the $x$-axis is contained in $V(xy, yz)$.

Let's return to the general situation. The following exercise lets us restrict attention to vanishing sets of *ideals*.

**4.4.B.** EASY EXERCISE. Show that if $(S)$ is the ideal generated by $S$, then $V(S) = V((S))$.

We define the **Zariski topology** by declaring that $V(S)$ is closed for all $S$. Let's check that this is a topology:

**4.4.C.** EXERCISE. (a) Show that $\varnothing$ and $\operatorname{Spec} A$ are both open.
(b) If $I_i$ is a collection of ideals (as $i$ runs over some index set), show that $\cap_i V(I_i) = V(\sum_i I_i)$. Hence the union of any collection of open sets is open.
(c) Show that $V(I_1) \cup V(I_2) = V(I_1 I_2)$. Hence the intersection of any finite number of open sets is open.

**4.4.1. Properties of the "vanishing set" function $V(\cdot)$.** The function $V(\cdot)$ is obviously inclusion-reversing: If $S_1 \subset S_2$, then $V(S_2) \subset V(S_1)$. Warning: We could have equality in the second inclusion without equality in the first, as the next exercise shows.

**4.4.D.** EXERCISE/DEFINITION. If $I \subset R$ is an ideal, then define its **radical** by

$$\sqrt{I} := \{r \in R \ : \ r^n \in I \text{ for some } n \in \mathbb{Z}^{\geq 0}\}.$$

For example, the nilradical $\mathfrak{N}$ (§4.2.N) is $\sqrt{(0)}$. Show that $V(\sqrt{I}) = V(I)$. We say an **ideal is radical** if it equals its own radical.

Here are two useful consequences. As $(I \cap J)^2 \subset IJ \subset I \cap J$, we have that $V(IJ) = V(I \cap J)$ ($= V(I) \cup V(J)$ by Exercise 4.4.C(b)). Also, combining this with Exercise 4.4.B, we see $V(S) = V((S)) = V(\sqrt{(S)})$.

**4.4.E.** EXERCISE (RADICALS COMMUTE WITH FINITE INTERSECTION). If $I_1, \dots, I_n$ are ideals of a ring $A$, show that $\sqrt{\cap_{i=1}^n I_i} = \cap_{i=1}^n \sqrt{I_i}$. We will use this property without referring back to this exercise.

**4.4.F.** EXERCISE FOR LATER USE. Show that $\sqrt{I}$ is the intersection of all the prime ideals containing $I$. (Hint: Use Theorem 4.2.10 on an appropriate ring.)

**4.4.2. Examples.** Let's see how this meshes with our examples from the previous section.

Recall that $\mathbb{A}^1_{\mathbb{C}}$, as a set, was just the "traditional" points (corresponding to maximal ideals, in bijection with $a \in \mathbb{C}$), and one "new" point $(0)$. The Zariski topology on $\mathbb{A}^1_{\mathbb{C}}$ is not that exciting: the open sets are the empty set, and $\mathbb{A}^1_{\mathbb{C}}$ minus a finite number of maximal ideals. (It "almost" has the cofinite topology. Notice that the open sets are determined by their intersections with the "traditional points". The "new" point $(0)$ comes along for the ride, which is a good sign that it is harmless. Ignoring the "new" point, observe that the topology on $\mathbb{A}^1_{\mathbb{C}}$ is a coarser topology than the classical topology on $\mathbb{C}$.)

The case $\operatorname{Spec} \mathbb{Z}$ is similar. The topology is "almost" the cofinite topology in the same way. The open sets are the empty set, and $\operatorname{Spec} \mathbb{Z}$ minus a finite number of "ordinary" ($(p)$ where $p$ is prime) primes.

**4.4.3.** *Closed subsets of $\mathbb{A}^2_{\mathbb{C}}$.* The case $\mathbb{A}^2_{\mathbb{C}}$ is more interesting. You should think through where the "one-dimensional primes" fit into the picture. In Exercise 4.2.D, we identified all the primes of $\mathbb{C}[x, y]$ (i.e. the points of $\mathbb{A}^2_{\mathbb{C}}$) as the maximal ideals

$(x-a, y-b)$ (where $a, b \in \mathbb{C}$), the "one-dimensional points" $(f(x,y))$ (where $f(x,y)$ is irreducible), and the "two-dimensional point" $(0)$.

Then the closed subsets are of the following form:

(a) the entire space, and

(b) a finite number (possibly zero) of "curves" (each of which is the closure of a "one-dimensional point") and a finite number (possibly zero) of closed points.

**4.4.4. Important fact: Maps of rings induce continuous maps of topological spaces.** We saw in §4.2.7 that a map of rings $\phi : B \to A$ induces a map of sets $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$.

**4.4.G.** IMPORTANT EXERCISE. By showing that closed sets pull back to closed sets, show that $\pi$ is a *continuous map*.

Not all continuous maps arise in this way. Consider for example the continuous map on $\mathbb{A}^1_{\mathbb{C}}$ that is the identity except $0$ and $1$ (i.e. $[(x)]$ and $[(x-1)]$) are swapped; no polynomial can manage this marvellous feat.

In §4.2.7, we saw that $\operatorname{Spec} B/I$ and $\operatorname{Spec} S^{-1}B$ are naturally *subsets* of $\operatorname{Spec} B$. It is natural to ask if the Zariski topology behaves well with respect to these inclusions, and indeed it does.

**4.4.H.** IMPORTANT EXERCISE (CF. EXERCISE 4.2.J). Suppose that $I, S \subset B$ are an ideal and multiplicative subset respectively. Show that $\operatorname{Spec} B/I$ is naturally a *closed* subset of $\operatorname{Spec} B$. Show that the Zariski topology on $\operatorname{Spec} B/I$ (resp. $\operatorname{Spec} S^{-1}B$) is the subspace topology induced by inclusion in $\operatorname{Spec} B$. (Hint: compare closed subsets.)

**4.4.5.** In particular, if $I \subset \mathfrak{N}$ is an ideal of nilpotents, the bijection $\operatorname{Spec} B/I \to \operatorname{Spec} B$ (Exercise 4.2.N) is a homeomorphism. Thus nilpotents don't affect the topological space. (The difference will be in the structure sheaf.)

**4.4.I.** USEFUL EXERCISE FOR LATER. Suppose $I \subset B$ is an ideal. Show that $f$ vanishes on $V(I)$ if and only if $f \in \sqrt{I}$ (i.e. $f^n \in I$ for some $n$). (If you are stuck, you will get a hint when you see Exercise 4.5.E.)

**4.4.J.** EASY EXERCISE (CF. EXERCISE 4.2.A). Describe the topological space $\operatorname{Spec} k[x]_{(x)}$.

## 4.5 A base of the Zariski topology on $\operatorname{Spec} A$: Distinguished open sets

If $f \in A$, define the **distinguished open set** $D(f) = \{[\mathfrak{p}] \in \operatorname{Spec} A \ : \ f \notin \mathfrak{p}\}$. It is the locus where $f$ doesn't vanish. (I often privately write this as $D(f \neq 0)$ to remind myself of this. I also privately call this a "Doesn't-vanish set" in analogy with $V(f)$ being the Vanishing set.) We have already seen this set when discussing $\operatorname{Spec} A_f$ as a subset of $\operatorname{Spec} A$. For example, we have observed that the Zariski-topology on the distinguished open set $D(f) \subset \operatorname{Spec} A$ coincides with the Zariski topology on $\operatorname{Spec} A_f$ (Exercise 4.4.H).

The reason these sets are important is that they form a particularly nice base for the (Zariski) topology

**4.5.A.** EASY EXERCISE. Show that the distinguished open sets form a base for the (Zariski) topology. (Hint: Given a subset $S \subset A$, show that the complement of $V(S)$ is $\cup_{f \in S} D(f)$.)

Here are some important but not difficult exercises to give you a feel for this concept.

**4.5.B.** EXERCISE. Suppose $f_i \in A$ as $i$ runs over some index set $J$. Show that $\cup_{i \in J} D(f_i) = \operatorname{Spec} A$ if and only if $(f_i) = A$, or equivalently and very usefully, there are $a_i$ ($i \in J$), all but finitely many $0$, such that $\sum_{i \in J} a_i f_i = 1$. (One of the directions will use the fact that any proper ideal of $A$ is contained in some maximal ideal.)

**4.5.C.** EXERCISE. Show that if $\operatorname{Spec} A$ is an infinite union of distinguished open sets $\cup_{j \in J} D(f_j)$, then in fact it is a union of a finite number of these, i.e. there is a finite subset $J'$ so that $\operatorname{Spec} A = \cup_{j \in J'} D(f_j)$. (Hint: exercise 4.5.B.)

**4.5.D.** EASY EXERCISE. Show that $D(f) \cap D(g) = D(fg)$.

**4.5.E.** IMPORTANT EXERCISE (CF. EXERCISE 4.4.I). Show that $D(f) \subset D(g)$ if and only if $f^n \in (g)$ for some $n$, if and only if $g$ is a unit in $A_f$.

We will use Exercise 4.5.E often. You can solve it thinking purely algebraically, but the following geometric interpretation may be helpful. Inside $\operatorname{Spec} A$, we have the closed subset $V(g) = \operatorname{Spec} A/(g)$, where $g$ vanishes, and its complement $D(g)$, where $g$ doesn't vanish. Then $f$ is a function on this closed subset $V(g)$ (or more precisely, on $\operatorname{Spec} A/(g)$), and by assumption it vanishes at all points of the closed subset. Now any function vanishing at every point of the spectrum of a ring must be nilpotent (Theorem 4.2.10). In other words, there is some $n$ such that $f^n = 0$ in $A/(g)$, i.e. $f^n \equiv 0 \pmod{g}$ in $A$, i.e. $f^n \in (g)$.

**4.5.F.** EASY EXERCISE. Show that $D(f) = \varnothing$ if and only if $f \in \mathfrak{N}$.

## 4.6 Topological definitions

A topological space is said to be **irreducible** if it is nonempty, and it is not the union of two proper closed subsets. In other words, $X$ is irreducible if whenever $X = Y \cup Z$ with $Y$ and $Z$ closed, we have $Y = X$ or $Z = X$.

**4.6.A.** EASY EXERCISE. Show that in an irreducible topological space, any nonempty open set is dense. (The moral: unlike in the classical topology, in the Zariski topology, non-empty open sets are all "huge".)

A point of a topological space $x \in X$ is said to be **closed** if $\{x\}$ is a closed subset. In the classical topology on $\mathbb{C}^n$, all points are closed.

**4.6.B.** EXERCISE. Show that the closed points of $\operatorname{Spec} A$ correspond to the maximal ideals.

Thus Hilbert's Nullstellensatz lets us interpret the closed points of $\mathbb{A}^n_{\mathbb{C}}$ as the $n$-tuples of complex numbers. Hence from now on we will say "closed point" instead of "traditional point" and "non-closed point" instead of "bonus" or "new-fangled" point when discussing subsets of $\mathbb{A}^n_{\mathbb{C}}$.

**4.6.1. Quasicompactness.** A topological space $X$ is **quasicompact** if given any cover $X = \cup_{i \in I} U_i$ by open sets, there is a finite subset $S$ of the index set $I$ such that $X = \cup_{i \in S} U_i$. Informally: every cover has a finite subcover. Depending on your definition of "compactness", this is the definition of compactness, minus possibly a Hausdorff condition. We will like this condition, because we are afraid of infinity.

**4.6.C.** EXERCISE. (a) Show that $\operatorname{Spec} A$ is quasicompact. (Hint: Exercise 4.5.C.)
(b) Show that in general $\operatorname{Spec} A$ can have nonquasicompact open sets. (Possible hint: let $A = k[x_1, x_2, x_3, \dots]$ and $\mathfrak{m} = (x_1, x_2, \dots) \subset A$, and consider the complement of $V(\mathfrak{m})$. This example will be useful to construct other enlightening examples later, e.g. Exercises 8.1.B and 8.3.E. In Exercise 4.6.L, we see that such weird behavior doesn't happen for "suitably nice" (Noetherian) rings.)

**4.6.D.** EXERCISE. (a) If $X$ is a topological space that is a finite union of quasicompact spaces, show that $X$ is quasicompact.
(b) Show that every closed subset of a quasicompact topological space is quasicompact.

**4.6.2. Specialization and generization.** Given two points $x, y$ of a topological space $X$, we say that $x$ is a **specialization** of $y$, and $y$ is a **generization** of $x$, if $x \in \overline{\{y\}}$. This now makes precise our hand-waving about "one point containing another". It is of course nonsense for a point to contain another. But it is not nonsense to say that the closure of a point contains another. For example, in $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, $[(y - x^2)]$ is a generization of $(2, 4) = [(x - 2, y - 4)]$, and $(2, 4)$ is a specialization of $[(y - x^2)]$.

**4.6.E.** EXERCISE. If $X = \operatorname{Spec} A$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$.

We say that a point $x \in X$ is a **generic point** for a closed subset $K$ if $\overline{\{x\}} = K$. (Recall that if $S$ is a subset of a topological space, then $\overline{S}$ denotes its closure.)

**4.6.F.** EXERCISE. Verify that $[(y - x^2)] \in \mathbb{A}^2$ is a generic point for $V(y - x^2)$.

We will soon see (Exercise 4.7.E) that there is a natural bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$. You know enough to prove this now, although we will wait until we have developed some convenient terminology.

**4.6.G.** EXERCISE. (a) Suppose $I = (wz - xy, wy - x^2, xz - y^2) \subset k[w, x, y, z]$. Show that $\operatorname{Spec} k[w, x, y, z]/I$ is irreducible, by showing that $I$ is prime. (Possible hint: Show that the quotient ring is a domain, by showing that it is isomorphic to the subring of $k[a, b]$ generated by monomials of degree divisible by 3. There are other approaches as well, some of which we will see later. This is an example of a hard

question: how do you tell if an ideal is prime?) We will later see this as the cone over the *twisted cubic curve* (the twisted cubic curve is defined in Exercise 9.2.A).
(b) Note that the generators of the ideal of part (a) may be rewritten as the equations ensuring that

$$\operatorname{rank}\begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix} \leq 1,$$

i.e., as the determinants of the $2 \times 2$ submatrices. Generalize this to the ideal of rank one $2 \times n$ matrices. This notion will correspond to the cone (§9.2.8) over the *degree* $n$ *rational normal curve* (Exercise 9.2.K).

**4.6.3. Noetherian conditions.**

In the examples we have considered, the spaces have naturally broken up into some obvious pieces. Let's make that a bit more precise.

A topological space $X$ is called **Noetherian** if it satisfies the **descending chain condition** for closed subsets: any sequence $Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots$ of closed subsets eventually stabilizes: there is an $r$ such that $Z_r = Z_{r+1} = \cdots$.

The following exercise may be enlightening.

**4.6.H.** EXERCISE. Show that any decreasing sequence of closed subsets of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ must eventually stabilize. Note that it can take arbitrarily long to stabilize. (The closed subsets of $\mathbb{A}^2_{\mathbb{C}}$ were described in §4.4.3.)

**4.6.4. Noetherian rings.** It turns out that all of the spectra we have considered have this property, but that isn't true of the spectra of all rings. The key characteristic all of our examples have had in common is that the rings were *Noetherian*. A ring is **Noetherian** if every ascending sequence $I_1 \subset I_2 \subset \cdots$ of ideals eventually stabilizes: there is an $r$ such that $I_r = I_{r+1} = \cdots$. (This is called the **ascending chain condition** on ideals.)

Here are some quick facts about Noetherian rings. You should be able to prove them all.

- Fields are Noetherian. $\mathbb{Z}$ is Noetherian.
- If $A$ is Noetherian, and $I$ is any ideal, then $A/I$ is Noetherian.
- If $A$ is Noetherian, and $S$ is any multiplicative set, then $S^{-1}A$ is Noetherian.
- Any submodule of a finitely generated module over a Noetherian ring is finitely generated. (Hint: prove it for $A^{\oplus n}$, and use the next exercise.)

(The notion of a Noetherian *module* will come up in §14.6.)

**4.6.I.** IMPORTANT EXERCISE. Show that a ring $A$ is Noetherian if and only if every ideal of $A$ is finitely generated.

The next fact is non-trivial. (Paul Gordan's comment on the proof: "This is not mathematics, this is theology!")

**4.6.5. The Hilbert basis theorem. —** *If* $A$ *is Noetherian, then so is* $A[x]$.

By the results described above, any polynomial ring over any field, or over the integers, is Noetherian — and also any quotient or localization thereof. Hence for example any finitely-generated algebra over $k$ or $\mathbb{Z}$, or any localization thereof, is Noetherian. Most "nice" rings are Noetherian, but not all rings are Noetherian:

$k[x_1, x_2, \ldots]$ is not, because $\mathfrak{m} = (x_1, x_2, \ldots, )$ is not finitely generated (cf. Exercise 4.6.C(b)).

*Proof of the Hilbert Basis Theorem 4.6.5.* We show that any ideal $I \subset A[x]$ is finitely-generated. We inductively produce a set of generators $f_1, \ldots$ as follows. For $n > 0$, if $I \neq (f_1, \ldots, f_{n-1})$, let $f_n$ be any non-zero element of $I - (f_1, \ldots, f_{n-1})$ of lowest degree. Thus $f_1$ is any element of $I$ of lowest degree, assuming $I \neq (0)$. If this procedure terminates, we are done. Otherwise, let $a_n \in A$ be the initial coefficient of $f_n$ for $n > 0$. Then as $A$ is Noetherian, $(a_1, a_2, \ldots) = (a_1, \ldots, a_N)$ for some $N$. Say $a_{N+1} = \sum_{i=1}^N b_i a_i$. Then

$$f_{N+1} - \sum_{i=1}^{N} b_i f_i x^{\deg f_{N+1} - \deg f_i}$$

is an element of $I$ that is nonzero (as $f_{N+1} \notin (f_1, \ldots, f_N)$) of lower degree than $f_{n+1}$, yielding a contradiction. □

**4.6.J.** UNIMPORTANT EXERCISE. Show that if $A$ is Noetherian, then so is $A[[x]] := \varprojlim A[x]/x^n$, the ring of power series in $x$. (Possible hint: Suppose $I \subset A[[x]]$ is an ideal. Let $I_n \subset A$ be the coefficients of $t^n$ that appear in the elements of $I$. Show that $I_n$ is an ideal. Show that $I_n \subset I_{n+1}$, and that $I$ is determined by $(I_0, I_1, I_2, \ldots)$.)

**4.6.K.** EXERCISE. If $A$ is Noetherian, show that $\operatorname{Spec} A$ is a Noetherian topological space. Describe a ring $A$ such that $\operatorname{Spec} A$ is not a Noetherian topological space. (As an aside, we note that if $\operatorname{Spec} A$ is a Noetherian topological space, $A$ need not be Noetherian.)

**4.6.L.** EXERCISE (PROMISED IN EXERCISE 4.6.C). Show that if $A$ is Noetherian, every open subset of $\operatorname{Spec} A$ is quasicompact.

If $X$ is a topological space, and $Z$ is a maximal irreducible subset (an irreducible closed subset not contained in any larger irreducible closed subset), $Z$ is said to be an **irreducible component** of $X$. We think of these as the "pieces of $X$" (see Figure 4.7).

**4.6.M.** EXERCISE. If $A$ is any ring, show that the irreducible components of $\operatorname{Spec} A$ are in bijection with the minimal primes of $A$. (For example, the only minimal prime of $k[x, y]$ is $(0)$.)

**4.6.N.** EXERCISE. Show that $\operatorname{Spec} A$ is irreducible if and only if $A$ has only one **minimal prime** ideal. (Minimality is with respect to inclusion.) In particular, if $A$ is an integral domain, then $\operatorname{Spec} A$ is irreducible.

**4.6.O.** EXERCISE. What are the minimal primes of $k[x, y]/(xy)$?

**4.6.6. Proposition. —** *Suppose $X$ is a Noetherian topological space. Then every non-empty closed subset $Z$ can be expressed uniquely as a finite union $Z = Z_1 \cup \cdots \cup Z_n$ of irreducible closed subsets, none contained in any other.*

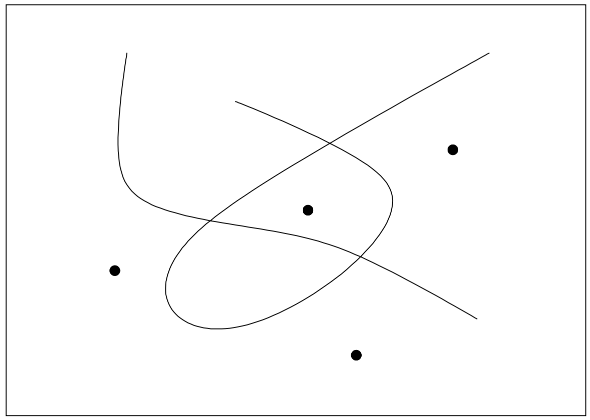


FIGURE 4.7. This closed subset of $\mathbb{A}^2$ has six irreducible components

Translation: any non-empty closed subset Z has a finite number of pieces. As a corollary, this implies that a Noetherian ring A has only finitely many minimal primes.

*Proof.* The following technique is called **Noetherian induction**, for reasons that will become clear.

Consider the collection of nonempty closed subsets of X that *cannot* be expressed as a finite union of irreducible closed subsets. We will show that it is empty. Otherwise, let $Y_1$ be one such. If it properly contains another such, then choose one, and call it $Y_2$. If this one contains another such, then choose one, and call it $Y_3$, and so on. By the descending chain condition, this must eventually stop, and we must have some $Y_r$ that cannot be written as a finite union of irreducible closed subsets, but every closed subset properly contained in it can be so written. But then $Y_r$ is not itself irreducible, so we can write $Y_r = Y' \cup Y''$ where $Y'$ and $Y''$ are both proper closed subsets. Both of these by hypothesis can be written as the union of a finite number of irreducible subsets, and hence so can $Y_r$, yielding a contradiction. Thus each closed subset can be written as a finite union of irreducible closed subsets. We can assume that none of these irreducible closed subsets contain any others, by discarding some of them.

We now show uniqueness. Suppose

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_r = Z_1' \cup Z_2' \cup \cdots \cup Z_s'$$

are two such representations. Then $Z_1' \subset Z_1 \cup Z_2 \cup \cdots \cup Z_r$, so $Z_1' = (Z_1 \cap Z_1') \cup \cdots \cup (Z_r \cap Z_1')$. Now $Z_1'$ is irreducible, so one of these is $Z_1'$ itself, say (without loss of generality) $Z_1 \cap Z_1'$. Thus $Z_1' \subset Z_1$. Similarly, $Z_1 \subset Z_a'$ for some $a$; but because $Z_1' \subset Z_1 \subset Z_a'$, and $Z_1'$ is contained in no other $Z_i'$, we must have $a = 1$, and $Z_1' = Z_1$. Thus each element of the list of $Z$'s is in the list of $Z''$s, and vice versa, so they must be the same list. $\square$

**4.6.7. Definition.** A topological space $X$ is **connected** if it cannot be written as the disjoint union of two non-empty open sets. A subset $Y$ of $X$ is a **connected component** if it is a maximal connected subset.

**4.6.P.** EXERCISE. Show that an irreducible topological space is connected.

**4.6.Q.** EXERCISE. Give (with proof!) an example of a scheme that is connected but reducible. (Possible hint: a picture may help. The symbol "$\times$" has two "pieces" yet is connected.)

**4.6.R.** EXERCISE. If $A$ is a Noetherian ring, show that the connected components of $\operatorname{Spec} A$ are unions of the irreducible components. Show that the connected components of $\operatorname{Spec} A$ are the subsets that are simultaneously open and closed.

**4.6.S.** EXERCISE. If $A = A_1 \times A_2 \times \cdots \times A_n$, describe a homeomorphism $\operatorname{Spec} A = \operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2 \coprod \cdots \coprod \operatorname{Spec} A_n$. Show that each $\operatorname{Spec} A_i$ is a distinguished open subset $D(f_i)$ of $\operatorname{Spec} A$. (Hint: let $f_i = (0, \cdots, 0, 1, 0, \cdots 0)$ where the $1$ is in the $i$th component.) In other words, $\coprod_{i=1}^n \operatorname{Spec} A_i = \operatorname{Spec} \prod_{i=1}^n A_i$.

An extension of the previous exercise (that you can prove if you wish) is that $\operatorname{Spec} A$ is not connected if and only if $A$ is isomorphic to the product of nonzero rings $A_1$ and $A_2$.

**4.6.8.** ⋆ *Fun but irrelevant remark.* The previous exercise shows that $\coprod_{i=1}^n \operatorname{Spec} A_i \cong \operatorname{Spec} \prod_{i=1}^n A_i$, but this can't hold if "$n$ is infinite" as Spec of any ring is quasicompact (Exercise 4.6.C(a)). This leads to an interesting phenomenon. We show that $\operatorname{Spec} \prod_{i=1}^\infty A_i$ is "strictly bigger" than $\coprod_{i=1}^\infty \operatorname{Spec} A_i$ where each $A_i$ is isomorphic to the field $k$. First, we have an inclusion of sets $\coprod_{i=1}^\infty \operatorname{Spec} A_i \hookrightarrow \operatorname{Spec} \prod_{i=1}^\infty A_i$, as there is a maximal ideal of $\prod A_i$ corresponding to each $i$ (precisely those elements $0$ in the $i$th component.) But there are other maximal ideals of $\prod A_i$. Hint: describe a proper ideal not contained in any of these maximal ideals. (One idea: consider elements $\prod a_i$ that are "eventually zero", i.e. $a_i = 0$ for $i \gg 0$.) This leads to the notion of *ultrafilters*, which are very useful, but irrelevant to our current discussion.

**4.6.9.** *Remark.* The notion of constructable and locally closed subsets will be discussed later, see Exercise 8.4.A.

## 4.7 The function $I(\cdot)$, taking subsets of $\operatorname{Spec} A$ to ideals of $A$

We now introduce a notion that is in some sense "inverse" to the vanishing set function $V(\cdot)$. Given a subset $S \subset \operatorname{Spec} A$, $I(S)$ is the set of functions vanishing on $S$.

We make three quick observations:

- $I(S)$ is clearly an ideal.
- $I(\overline{S}) = I(S)$.
- $I(\cdot)$ is inclusion-reversing: if $S_1 \subset S_2$, then $I(S_2) \subset I(S_1)$.

**4.7.A.** EXERCISE. Let $A = k[x, y]$. If $S = \{[(x)], [(x-1, y)]\}$ (see Figure 4.8), then $I(S)$ consists of those polynomials vanishing on the $y$ axis, and at the point $(1, 0)$. Give generators for this ideal.

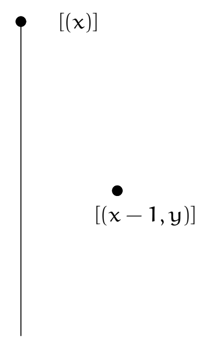



FIGURE 4.8. The set $S$ of Exercise/example 4.7.A, pictured as a subset of $\mathbb{A}^2$

**4.7.B.** TRICKY EXERCISE. Suppose $X \subset \mathbb{A}^3$ is the union of the three axes. (The $x$-axis is defined by $y = z = 0$, and the $y$-axis and $z$-axis are defined analogously.) Give generators for the ideal $I(X)$. Be sure to prove it! We will see in Exercise 13.1.F that this ideal is not generated by less than three elements.

**4.7.C.** EXERCISE. Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set $S$. (Compare this to Exercise 4.7.D.)

Note that $I(S)$ is always a radical ideal — if $f \in \sqrt{I(S)}$, then $f^n$ vanishes on $S$ for some $n > 0$, so then $f$ vanishes on $S$, so $f \in I(S)$.

**4.7.D.** EXERCISE. Prove that if $I \subset A$ is an ideal, then $I(V(I)) = \sqrt{I}$.

This exercise and Exercise 4.7.C suggest that $V$ and $I$ are "almost" inverse. More precisely:

**4.7.1. Theorem. —** $V(\cdot)$ *and* $I(\cdot)$ *give a bijection between closed subsets of* $\operatorname{Spec} A$ *and radical ideals of* $A$ *(where a closed subset gives a radical ideal by* $I(\cdot)$*, and a radical ideal gives a closed subset by* $V(\cdot)$*).*

**4.7.E.** IMPORTANT EXERCISE. Show that $V(\cdot)$ and $I(\cdot)$ give a bijection between *irreducible closed subsets* of $\operatorname{Spec} A$ and *prime* ideals of $A$. From this conclude that in $\operatorname{Spec} A$ there is a bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$ (where a point determines an irreducible closed subset by taking the closure). Hence *each irreducible closed subset of* $\operatorname{Spec} A$ *has precisely one generic point* — any irreducible closed subset $Z$ can be written uniquely as $\overline{\{z\}}$.

CHAPTER 5

# The structure sheaf of an affine scheme, and the definition of schemes in general

## 5.1 The structure sheaf of an affine scheme

The final ingredient in the definition of an affine scheme is the *structure sheaf* $\mathcal{O}_{\operatorname{Spec} A}$, which we think of as the "sheaf of algebraic functions". You should keep in your mind the example of "algebraic functions" on $\mathbb{C}^n$, which you understand well. For example, in $\mathbb{A}^2$, we expect that on the open set $D(xy)$ (away from the two axes), $(3x^4 + y + 4)/x^7y^3$ should be an algebraic function.

These functions will have values at points, but won't be determined by their values at points. But like all sections of sheaves, they will be determined by their germs (see §5.3.3).

It suffices to describe the structure sheaf as a sheaf (of rings) on the base of distinguished open sets (Theorem 3.7.1 and Exercise 4.5.A).

**5.1.1.** *Definition.* Define $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ to be the localization of $A$ at the multiplicative set of all functions that do not vanish outside of $V(f)$ (i.e. those $g \in A$ such that $V(g) \subset V(f)$, or equivalently $D(f) \subset D(g)$). This depends only on $D(f)$, and not on $f$ itself.

**5.1.A.** GREAT EXERCISE. Show that the natural map $A_f \to \mathcal{O}_{\operatorname{Spec} A}(D(f))$ is an isomorphism. (Possible hint: Exercise 4.5.E.)

If $D(f') \subset D(f)$, define the restriction map $\operatorname{res}_{D(f),D(f')} : \mathcal{O}_{\operatorname{Spec} A}(D(f)) \to \mathcal{O}_{\operatorname{Spec} A}(D(f'))$ in the obvious way: the latter ring is a further localization of the former ring. The restriction maps obviously commute: this is a "presheaf on the distinguished base".

**5.1.2. Theorem. —** *The data just described give a sheaf on the distinguished base, and hence determine a sheaf on the topological space* $\operatorname{Spec} A$.

This sheaf is called the **structure sheaf**, and will be denoted $\mathcal{O}_{\operatorname{Spec} A}$, or sometimes $\mathcal{O}$ if the subscript is clear from the context. Such a topological space, with sheaf, will be called an **affine scheme**. The notation $\operatorname{Spec} A$ will hereafter denote the data of a topological space with a structure sheaf.

*Proof.* We must show the base identity and base gluability axioms hold (§3.7). We show that they both hold for the open set that is the entire space $\operatorname{Spec} A$, and leave to you the trick which extends them to arbitrary distinguished open sets (Exercises 5.1.B and 5.1.C). Suppose $\operatorname{Spec} A = \cup_{i \in I} D(f_i)$, or equivalently (Exercise 4.5.B) the ideal generated by the $f_i$ is the entire ring $A$.

We check identity on the base. Suppose that $\operatorname{Spec} A = \cup_{i \in I} D(f_i)$ where $i$ runs over some index set I. Then there is some finite subset of I, which we name $\{1, \dots, n\}$, such that $\operatorname{Spec} A = \cup_{i=1}^n D(f_i)$, i.e. $(f_1, \dots, f_n) = A$ (quasicompactness of $\operatorname{Spec} A$, Exercise 4.5.C). Suppose we are given $s \in A$ such that $\operatorname{res}_{\operatorname{Spec} A, D(f_i)} s = 0$ in $A_{f_i}$ for all $i$. We wish to show that $s = 0$. The fact that $\operatorname{res}_{\operatorname{Spec} A, D(f_i)} s = 0$ in $A_{f_i}$ implies that there is some $m$ such that for each $i \in \{1, \dots, n\}$, $f_i^m s = 0$. Now $(f_1^m, \dots, f_n^m) = A$ (for example, from $\operatorname{Spec} A = \cup D(f_i) = \cup D(f_i^m)$), so there are $r_i \in A$ with $\sum_{i=1}^n r_i f_i^m = 1$ in $A$, from which

$$s = \left( \sum r_i f_i^m \right) s = \sum r_i (f_i^m s) = 0.$$

Thus we have checked the "base identity" axiom for $\operatorname{Spec} A$. (Serre has described this as a "partition of unity" argument, and if you look at it in the right way, his insight is very enlightening.)

**5.1.B.** EXERCISE. Make the tiny changes to the above argument to show base identity for any distinguished open $D(f)$. (Hint: judiciously replace $A$ by $A_f$ in the above argument.)

We next show base gluability. Suppose again $\cup_{i \in I} D(f_i) = \operatorname{Spec} A$, where I is a index set (possibly horribly infinite). Suppose we are given elements in each $A_{f_i}$ that agree on the overlaps $A_{f_i f_j}$. Note that intersections of distinguished open sets are also distinguished open sets.

(Aside: experts might realize that we are trying to show exactness of

$$0 \to A \to \prod_i A_{f_i} \to \prod_{i \neq j} A_{f_i f_j}. \tag{5.1.2.1}$$

Do you understand what the right-hand map is? Base identity corresponds to injectivity at $A$. The composition of the right two morphisms is trivially zero, and gluability is exactness at $\prod_i A_{f_i}$.)

Choose a finite subset $\{1, \dots, n\} \subset I$ with $(f_1, \dots, f_n) = A$ (or equivalently, use quasicompactness of $\operatorname{Spec} A$ to choose a finite subcover by $D(f_i)$). We have elements $a_i / f_i^{l_i} \in A_{f_i}$ agreeing on overlaps $A_{f_i f_j}$. Letting $g_i = f_i^{l_i}$, using $D(f_i) = D(g_i)$, we can simplify notation by considering our elements as of the form $a_i / g_i \in A_{g_i}$.

The fact that $a_i/g_i$ and $a_j/g_j$ "agree on the overlap" (i.e. in $A_{g_i g_j}$) means that for some $m_{ij}$,

$$(g_i g_j)^{m_{ij}} (g_j a_i - g_i a_j) = 0$$

in $A$. By taking $m = \max m_{ij}$ (here we use the finiteness of I), we can simplify notation:

$$(g_i g_j)^m (g_j a_i - g_i a_j) = 0$$

for all $i, j$. Let $b_i = a_i g_i^m$ for all $i$, and $h_i = g_i^{m+1}$ (so $D(h_i) = D(g_i)$). Then we can simplify notation even more: on each $D(h_i)$, we have a function $b_i/h_i$, and the overlap condition is

$$h_j b_i = h_i b_j. \tag{5.1.2.2}$$

Now $\cup_i D(h_i) = \operatorname{Spec} A$, implying that $1 = \sum_{i=1}^n r_i h_i$ for some $r_i \in A$. Define $r = \sum r_i b_i$. This will be the element of $A$ that restricts to each $b_j/h_j$. Indeed, from the overlap condition (5.1.2.2),

$$rh_j = \sum_i r_i b_i h_j = \sum_i r_i h_i b_j = b_j.$$

We are not quite done! We are supposed to have something that restricts to $a_i/f_i^{l_i}$ for *all* $i \in I$, not just $i = 1, \dots, n$. But a short trick takes care of this. We now show that for any $\alpha \in I - \{1, \dots, n\}$, $r$ restricts to the desired element $a_\alpha$ of $A_{f_\alpha}$. Repeat the entire process above with $\{1, \dots, n, \alpha\}$ in place of $\{1, \dots, n\}$, to obtain $r' \in A$ which restricts to $a_\alpha$ for $i \in \{1, \dots, n, \alpha\}$. Then by base identity, $r' = r$. (Note that we use base identity to *prove* base gluability. This is an example of how the identity axiom is "prior" to the gluability axiom.) Hence $r$ restricts to $a_\alpha/f_\alpha^{l_\alpha}$ as desired.

**5.1.C.** EXERCISE. Alter this argument appropriately to show base gluability for any distinguished open $D(f)$.

We have now completed the proof of Theorem 5.1.2. □

The following generalization of Theorem 5.1.2 will be essential for the definition of a quasicoherent sheaf in Chapter 14.

**5.1.D.** IMPORTANT EXERCISE/DEFINITION. Suppose $M$ is an $A$-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. Define $\tilde{M}(D(f))$ to be the localization of $M$ at the multiplicative set of all functions that vanish only in $V(f)$. Define restriction maps $\operatorname{res}_{D(f),D(g)}$ in the analogous way to $\mathcal{O}_{\operatorname{Spec} A}$. Show that this defines a sheaf on the distinguished base, and hence a sheaf on $\operatorname{Spec} A$. Then show that this is an $\mathcal{O}_{\operatorname{Spec} A}$-module. (This sheaf $\tilde{M}$ will be very important soon; it will be an example of a *quasicoherent sheaf*.)

**5.1.3.** *Remark (cf.* (5.1.2.1)*).* In the course of answering the previous exercise, you will show that if $(f_1, \dots, f_r) = A$, $M$ can be identified with a specific submodule of $M_{f_1} \times \cdots \times M_{f_r}$. Even though $M \to M_{f_i}$ may not be an inclusion for any $f_i$, $M \to M_{f_1} \times \cdots \times M_{f_r}$ *is* an inclusion. This will be useful later: we will want to show that if $M$ has some nice property, then $M_f$ does too, which will be easy. We will also want to show that if $(f_1, \dots, f_n) = R$, then if $M_{f_i}$ have this property, then $M$ does too, and we will invoke this.

## 5.2 Visualizing schemes II: nilpotents

In §4.3, we discussed how to visualize the underlying set of schemes,adding in generic points to our previous intuition of "classical" (or closed) points. Our later discussion of the Zariski topology fit well with that picture. In our definition of the "affine scheme" $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, we have the additional information of nilpotents, which are invisible on the level of points (§4.2.9), so now we figure out to picture them. We will then readily be able to glue them together to picture schemes in general, once we've made the appropriate definitions. As we are building intuition, we will not be rigorous or precise.

To begin, we picture $\operatorname{Spec} \mathbb{C}[x]/(x)$ as a closed subset (a point) of $\operatorname{Spec} \mathbb{C}[x]$: to the quotient $\mathbb{C}[x] \to \mathbb{C}[x]/(x)$, we associate the picture of a closed inclusion. The ring map can be interpreted as restriction of functions: to $\mathbb{C}[x]$, we associate its value at $0$ (its residue class modulo $(x)$, by the remainder theorem). The quotient $\mathbb{C}[x]/(x^2)$ should fit in between these rings,

$$\begin{array}{ccccc} \mathbb{C}[x] & \twoheadrightarrow & \mathbb{C}[x]/(x^2) & \twoheadrightarrow & \mathbb{C}[x]/(x) \\ f(x) & & \longmapsto & & f(0), \end{array}$$

and we should picture it in terms of the information the quotient remembers. The image of a polynomial $f(x)$ is the information of its value at $0$, and its derivative (cf. Exercise 4.2.P). We thus picture this as being the point, plus a little bit more — a little bit of "fuzz" on the point (see Figure 5.1). (These will later be examples of *closed subschemes*, the schematic version of closed subsets, §9.1.)

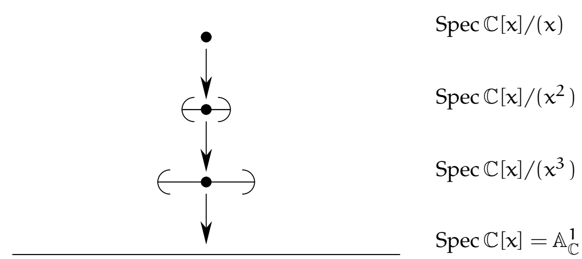



FIGURE 5.1. Picturing quotients of $\mathbb{C}[x]$

Similarly, $\mathbb{C}[x]/(x^3)$ remembers even more information — the second derivative as well. Thus we picture this as the point $0$ plus even more fuzz.

More subtleties arise in two dimensions (see Figure 5.2). Consider $\operatorname{Spec} \mathbb{C}[x,y]/(x,y)^2$, which is sandwiched between two rings we know well:

$$\begin{array}{ccccc} \mathbb{C}[x,y] & \twoheadrightarrow & \mathbb{C}[x,y]/(x,y)^2 & \twoheadrightarrow & \mathbb{C}[x,y]/(x,y) \\ f(x,y) & & \longmapsto & & f(0). \end{array}$$

Again, taking the quotient by $(x,y)^2$ remembers the first derivative, "in both directions". We picture this as fuzz around the point. Similarly, $(x,y)^3$ remembers the second derivative "in all directions".

Consider instead the ideal $(x^2,y)$. What it remembers is the derivative only in the $x$ direction — given a polynomial, we remember its value at $0$, and the coefficient of $x$. We remember this by picturing the fuzz only in the $x$ direction.

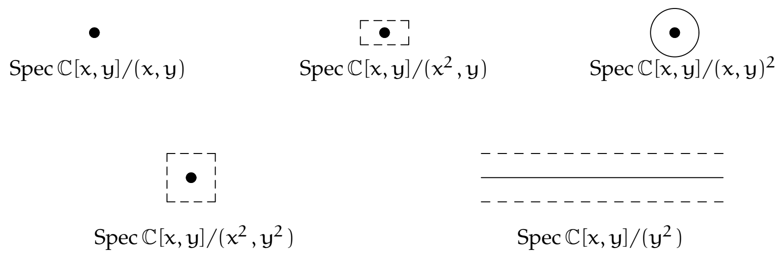



FIGURE 5.2. Picturing quotients of $\mathbb{C}[x,y]$

This gives us some handle on picturing more things of this sort, but now it becomes more an art than a science. For example, $\operatorname{Spec}\mathbb{C}[x,y]/(x^2,y^2)$ we might picture as a fuzzy square around the origin. One feature of this example is that given two ideals $I$ and $J$ of a ring $A$ (such as $\mathbb{C}[x,y]$), your fuzzy picture of $\operatorname{Spec} A/(I,J)$ should be the "intersection" of your picture of $\operatorname{Spec} A/I$ and $\operatorname{Spec} A/J$ in $\operatorname{Spec} A$. (You will make this precise in Exercise 9.1.G(a).) For example, $\operatorname{Spec}\mathbb{C}[x,y]/(x^2,y^2)$ should be the intersection of two thickened lines. (How would you picture $\operatorname{Spec}\mathbb{C}[x,y]/(x^5,y^3)$? $\operatorname{Spec}\mathbb{C}[x,y,z]/(x^3,y^4,z^5,(x+y+z)^2)$? $\operatorname{Spec}\mathbb{C}[x,y]/((x,y)^5,y^3)$?)

This idea captures useful information that you already have some intuition for. For example, consider the intersection of the parabola $y=x^2$ and the $x$-axis (in the $xy$-plane). See Figure 5.3. You already have a sense that the intersection has multiplicity two. In terms of this visualization, we interpret this as intersecting (in $\operatorname{Spec}\mathbb{C}[x,y]$):

$$\operatorname{Spec}\mathbb{C}[x,y]/(y-x^2)\cap\operatorname{Spec}\mathbb{C}[x,y]/(y)=\operatorname{Spec}\mathbb{C}[x,y]/(y-x^2,y)=\operatorname{Spec}\mathbb{C}[x,y]/(y,x^2)$$

which we interpret as the fact that the parabola and line not just meet with multiplicity two, but that the "multiplicity 2" part is in the direction of the $x$-axis. You will make this example precise in Exercise 9.1.G(b).

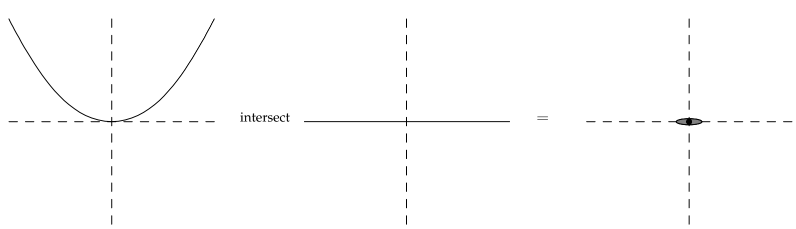



FIGURE 5.3. The scheme-theoretic intersection of the parabola $y=x^2$ and the $x$-axis is a non-reduced scheme (with fuzz in the $x$-direction)

We will later make the location of the fuzz somewhat more precise when we discuss associated points (§6.5). We will see that (in reasonable circumstances, when associated points make sense) the fuzz is concentrated on closed subsets.

## 5.3 Definition of schemes

We can now define *scheme* in general. First, define an **isomorphism of ringed spaces** $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ as (i) a homeomorphism $f : X \to Y$, and (ii) an isomorphism of sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$, considered to be on the same space via $f$. (Part (ii), more precisely, is an isomorphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on $X$, or equivalently $\mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on $Y$.) In other words, we have a "correspondence" of sets, topologies, and structure sheaves. An **affine scheme** is a ringed space that is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some $A$. A **scheme** $(X, \mathcal{O}_X)$ is a ringed space such that any point $x \in X$ has a neighborhood $U$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. The scheme can be denoted $(X, \mathcal{O}_X)$, although it is often denoted $X$, with the structure sheaf implicit.

An **isomorphism of two schemes** $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is an isomorphism as ringed spaces. If $U \subset X$ is an open subset, then $\Gamma(\mathcal{O}_X, U)$ are said to be the **functions on** $U$; this generalizes in an obvious way the definition of functions on an affine scheme, §4.2.1.

**5.3.1.** *Remark.* From this definition of the structure sheaf on an affine scheme, several things are clear. First of all, if we are told that $(X, \mathcal{O}_X)$ is an affine scheme, we may recover its ring (i.e. find the ring $A$ such that $\operatorname{Spec} A = X$) by taking the ring of global sections, as $X = D(1)$, so:

$$\begin{aligned}\Gamma(X, \mathcal{O}_X) &= \Gamma(D(1), \mathcal{O}_{\operatorname{Spec} A}) \quad \text{as } D(1) = \operatorname{Spec} A \\ &= A.\end{aligned}$$

(You can verify that we get more, and can "recognize $X$ as the scheme $\operatorname{Spec} A$": we get an isomorphism $f : (\operatorname{Spec} \Gamma(X, \mathcal{O}_X), \mathcal{O}_{\operatorname{Spec} \Gamma(X, \mathcal{O}_X)}) \to (X, \mathcal{O}_X)$. For example, if $\mathfrak{m}$ is a maximal ideal of $\Gamma(X, \mathcal{O}_X)$, $f([\mathfrak{m}]) = V(\mathfrak{m})$.) The following exercise will give you some practice with these notions.

**5.3.A.** STRANGELY CONFUSING EXERCISE. Describe a bijection between the isomorphisms $\operatorname{Spec} A \to \operatorname{Spec} A'$ and the ring isomorphisms $A' \to A$.

More generally, given $f \in A$, $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) \cong A_f$. Thus under the natural inclusion of sets $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, the Zariski topology on $\operatorname{Spec} A$ restricts to give the Zariski topology on $\operatorname{Spec} A_f$ (Exercise 4.4.H), and the structure sheaf of $\operatorname{Spec} A$ restricts to the structure sheaf of $\operatorname{Spec} A_f$, as the next exercise shows.

**5.3.B.** IMPORTANT BUT EASY EXERCISE. Suppose $f \in A$. Show that under the identification of $D(f)$ in $\operatorname{Spec} A$ with $\operatorname{Spec} A_f$ (§4.5), there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f})$. Hint: notice that distinguished open sets of $\operatorname{Spec} R_f$ are already distinguished open sets in $\operatorname{Spec} R$.

**5.3.C.** EASY EXERCISE. If $X$ is a scheme, and $U$ is *any* open subset, prove that $(U, \mathcal{O}_X|_U)$ is also a scheme.

**5.3.2.** *Definitions.* We say $(U, \mathcal{O}_X|_U)$ is an **open subscheme of** $U$. If $U$ is also an affine scheme, we often say $U$ is an **affine open subset**, or an **affine open subscheme**, or sometimes informally just an **affine open**. For example, $D(f)$ is an affine open subscheme of $\operatorname{Spec} A$.

**5.3.D.** EASY EXERCISE. Show that if $X$ is a scheme, then the affine open sets form a base for the Zariski topology.

**5.3.E.** EASY EXERCISE. The disjoint union of schemes is defined as you would expect: it is the disjoint union of sets, with the expected topology (thus it is the disjoint union of topological spaces), with the expected sheaf. Once we know what morphisms are, it will be immediate (Exercise 10.1.A) that (just as for sets and topological spaces) disjoint union is the coproduct in the category of schemes.
(a) Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: Exercise 4.6.S.)
(b) *(a first example of a non-affine scheme)* Show that an infinite disjoint union of (non-empty) affine schemes is not an affine scheme. (Hint: affine schemes are quasicompact, Exercise 4.6.C(a).)

**5.3.3. Stalks of the structure sheaf: germs, values at a point, and the residue field of a point.** Like every sheaf, the structure sheaf has stalks, and we shouldn't be surprised if they are interesting from an algebraic point of view. In fact, we have seen them before.

**5.3.F.** IMPORTANT EXERCISE. Show that the stalk of $\mathcal{O}_{\operatorname{Spec} A}$ at the point $[\mathfrak{p}]$ is the local ring $A_{\mathfrak{p}}$.

Essentially the same argument will show that the stalk of the sheaf $\tilde{M}$ (defined in Exercise 5.1.D) at $[\mathfrak{p}]$ is $M_{\mathfrak{p}}$. Here is an interesting consequence, or if you prefer, a geometric interpretation of an algebraic fact. A section is determined by its germs (Exercise 3.4.A), meaning that $M \to \prod_{\mathfrak{p}} M_{\mathfrak{p}}$ is an inclusion. So for example an $A$-module is zero if and only if all its localizations at primes are zero.

We say a ringed space is a **local-ringed space** if its stalks are local rings. Thus schemes are local ringed spaces. Manifolds are another example of local-ringed spaces, see §3.1.1. In both cases, taking quotient by the maximal ideal may be interpreted as evaluating at the point. The maximal ideal of the local ring $\mathcal{O}_{X,p}$ is denoted $\mathfrak{m}_{X,p}$ or $\mathfrak{m}_p$, and the **residue field** $\mathcal{O}_{X,p}/\mathfrak{m}_p$ is denoted $\kappa(p)$. Functions on an open subset $U$ of a local-ringed space have **values** at each point of $U$. The value at $p$ of such a function lies in $\kappa(p)$. As usual, we say that a function **vanishes** at a point $p$ if its value at $p$ is $0$.

As an example, consider a point $[\mathfrak{p}]$ of an affine scheme $\operatorname{Spec} A$. (Of course, this example is "universal", as all points may be interpreted in this way, by choosing an affine neighborhood.) The residue field at $[\mathfrak{p}]$ is $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, which is isomorphic to $K(A/\mathfrak{p})$, the fraction field of the quotient domain. It is useful to note that localization at $\mathfrak{p}$ and taking quotient by $\mathfrak{p}$ "commute", i.e. the following diagram commutes.

(5.3.3.1)
$$\begin{tikzcd}
 & A_{\mathfrak{p}} \arrow[dr, "\text{quotient}"] & \\
A \arrow[ur, "\text{localize}"] \arrow[dr, "\text{quotient}"'] & & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = K(A/\mathfrak{p}) \\
 & A/\mathfrak{p} \arrow[ur, "\text{localize, i.e. } K(\cdot)"'] &
\end{tikzcd}$$

For example, consider the scheme $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, where $k$ is a field of characteristic not 2. Then $(x^2 + y^2)/x(y^2 - x^5)$ is a function away from the $y$-axis and the curve $y^2 - x^5$. Its value at $(2, 4)$ (by which we mean $[(x - 2, y - 4)]$) is $(2^2 + 4^2)/(2(4^2 - 2^5))$, as

$$\frac{x^2 + y^2}{x(y^2 - x^5)} \equiv \frac{2^2 + 4^2}{2(4^2 - 2^5)}$$

in the residue field — check this if it seems mysterious. And its value at $[(y)]$, the generic point of the $x$-axis, is $\frac{x^2}{-x^6} = -1/x^4$, which we see by setting $y$ to $0$. This is indeed an element of the fraction field of $k[x, y]/(y)$, i.e. $k(x)$. (If you think you care only about algebraically closed fields, let this example be a first warning: $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ won't be algebraically closed in general, even if $A$ is a finitely generated $\mathbb{C}$-algebra!)

If anything makes you nervous, you should make up an example to make you feel better. Here is one: 27/4 is a function on $\operatorname{Spec} \mathbb{Z} - \{[(2)], [(7)]\}$ or indeed on an even bigger open set. What is its value at $[(5)]$? Answer: $2/(-1) \equiv -2 \pmod 5$. What is its value at the generic point $[(0)]$? Answer: 27/4. Where does it vanish? At $[(3)]$.

## 5.4 Three examples

We now give three extended examples. Our short-term goal is to see that we can really work with the structure sheaf, and can compute the ring of sections of interesting open sets that aren't just distinguished open sets of affine schemes. Our long-term goal is to meet interesting examples that will come up repeatedly in the future.

**5.4.1. Example: The plane minus the origin.** This example will show you that the distinguished base is something that you can work with. Let $A = k[x, y]$, so $\operatorname{Spec} A = \mathbb{A}^2_k$. Let's work out the space of functions on the open set $U = \mathbb{A}^2 - \{(0, 0)\} = \mathbb{A}^2 - \{[(x, y)]\}$.

You can't cut out this set with a single equation (can you see why?), so this isn't a distinguished open set. But in any case, even if we are not sure if this is a distinguished open set, we can describe it as the union of two things which *are* distinguished open sets: $U = D(x) \cup D(y)$. We will find the functions on $U$ by gluing together functions on $D(x)$ and $D(y)$.

The functions on $D(x)$ are, by Definition 5.1.1, $A_x = k[x, y, 1/x]$. The functions on $D(y)$ are $A_y = k[x, y, 1/y]$. Note that $A \hookrightarrow A_x, A_y$. This is because $x$ and $y$ are not zero-divisors. (The ring $A$ is an integral domain — it has no zero-divisors, besides $0$ — so localization is always an inclusion, Exercise 2.3.C.) So we are looking for functions on $D(x)$ and $D(y)$ that agree on $D(x) \cap D(y) = D(xy)$, i.e. they are just the same Laurent polynomial. Which things of this first form are also of the second form? Just traditional polynomials —

$$\Gamma(U, \mathcal{O}_{\mathbb{A}^2}) \equiv k[x, y]. \tag{5.4.1.1}$$

In other words, we get no extra functions by throwing out this point. Notice how easy that was to calculate!

**5.4.2.** *Aside.* Notice that any function on $\mathbb{A}^2 - \{(0,0)\}$ extends over all of $\mathbb{A}^2$. This is an analogue of *Hartogs' Lemma* in complex geometry: you can extend a holomorphic function defined on the complement of a set of codimension at least two on a complex manifold over the missing set. This will work more generally in the algebraic setting: you can extend over points in codimension at least 2 not only if they are smooth, but also if they are mildly singular — what we will call *normal*. We will make this precise in §12.3.7. This fact will be very useful for us.)

**5.4.3.** We now show an interesting fact: $(U, \mathcal{O}_{\mathbb{A}^2}|_U)$ is a scheme, but it is not an affine scheme. (This is confusing, so you will have to pay attention.) Here's why: otherwise, if $(U, \mathcal{O}_{\mathbb{A}^2}|_U) = (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, then we can recover $A$ by taking global sections:

$$A = \Gamma(U, \mathcal{O}_{\mathbb{A}^2}|_U),$$

which we have already identified in (5.4.1.1) as $k[x, y]$. So if $U$ is affine, then $U \cong \mathbb{A}^2_k$. But this bijection between primes in a ring and points of the spectrum is more constructive than that: *given the prime ideal* $I$, *you can recover the point as the generic point of the closed subset cut out by* $I$, *i.e.* $V(I)$, *and given the point* $p$, *you can recover the ideal as those functions vanishing at* $p$, *i.e.* $I(p)$. In particular, the prime ideal $(x, y)$ of $A$ should cut out a point of $\operatorname{Spec} A$. But on $U$, $V(x) \cap V(y) = \varnothing$. Conclusion: $U$ is *not* an affine scheme. (If you are ever looking for a counterexample to something, and you are expecting one involving a non-affine scheme, keep this example in mind!)

**5.4.4. Gluing two copies of $\mathbb{A}^1$ together in two different ways.** We have now seen two examples of non-affine schemes: an infinite disjoint union of non-empty schemes: Exercise 5.3.E and $\mathbb{A}^2 - \{(0,0)\}$. I want to give you two more examples. They are important because they are the first examples of fundamental behavior, the first pathological, and the second central.

First, I need to tell you how to glue two schemes together. Before that, you should review how to glue topological spaces together along isomorphic open sets. Given two topological spaces $X$ and $Y$, and open subsets $U \subset X$ and $V \subset Y$ along with a homeomorphism $U \cong V$, we can create a new topological space $W$, that we think of as gluing $X$ and $Y$ together along $U \cong V$. It is the quotient of the disjoint union $X \coprod Y$ by the equivalence relation $U \cong V$, where the quotient is given the quotient topology. Then $X$ and $Y$ are naturally (identified with) open subsets of $W$, and indeed cover $W$. Can you restate this cleanly with an arbitrary (not necessarily finite) number of topological spaces?

Now that we have discussed gluing topological spaces, let's glue schemes together. Suppose you have two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, and open subsets $U \subset X$ and $V \subset Y$, along with a homeomorphism $f : U \xrightarrow{\sim} V$, and an isomorphism of structure sheaves $\mathcal{O}_X \cong f^* \mathcal{O}_Y$ (i.e. an isomorphism *of schemes* $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_Y|_V)$). Then we can glue these together to get a single scheme. Reason: let $W$ be $X$ and $Y$ glued together using the isomorphism $U \cong V$. Then Exercise 3.7.D shows that the structure sheaves can be glued together to get a sheaf of rings. Note that this is indeed a scheme: any point has a neighborhood that is an affine scheme. (Do you see why?)

**5.4.A.** ESSENTIAL EXERCISE (CF. EXERCISE 3.7.D). For later reference, show that you can glue together an arbitrary collection of schemes together. Suppose we are given:

- schemes $X_i$ (as $i$ runs over some index set $I$, not necessarily finite),
- open subschemes $X_{ij} \subset X_i$,
- isomorphisms $f_{ij} : X_{ij} \to X_{ji}$ with $f_{ii}$ the identity

such that

- (the cocycle condition) the isomorphisms "agree along triple intersections", i.e. $f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{U_{ji} \cap U_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}$.

(The cocycle condition ensures that $f_{ij}$ and $f_{ji}$ are inverses. In fact, the hypothesis that $f_{ii}$ is the identity also follows from the cocycle condition.) Show that there is a unique scheme $X$ (up to unique isomorphism) along with open subset isomorphic to $X_i$ respecting this gluing data in the obvious sense. (Hint: what is $X$ as a set? What is the topology on this set? In terms of your description of the open sets of $X$, what are the sections of this sheaf over each open set?)

I will now give you two non-affine schemes. In both cases, I will glue together two copies of the affine line $\mathbb{A}^1_k$. Let $X = \operatorname{Spec} k[t]$, and $Y = \operatorname{Spec} k[u]$. Let $U = D(t) = \operatorname{Spec} k[t, 1/t] \subset X$ and $V = D(u) = \operatorname{Spec} k[u, 1/u] \subset Y$. We will get both examples by gluing $X$ and $Y$ together along $U$ and $V$. The difference will be in how we glue.

**5.4.5. Extended example: the affine line with the doubled origin.** Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow u$ (cf. Exercise 5.3.A). The resulting scheme is called the **affine line with doubled origin**. Figure 5.4 is a picture of it.

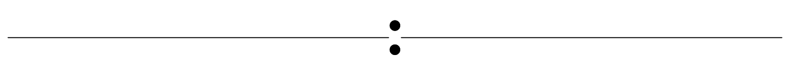


FIGURE 5.4. The affine line with doubled origin

As the picture suggests, intuitively this is an analogue of a failure of Hausdorffness. Now $\mathbb{A}^1$ itself is not Hausdorff, so we can't say that it is a failure of Hausdorffness. We see this as weird and bad, so we will want to make a definition that will prevent this from happening. This will be the notion of *separatedness* (to be discussed in Chapter 11). This will answer other of our prayers as well. For example, on a separated scheme, the "affine base of the Zariski topology" is nice — the intersection of two affine open sets will be affine (Proposition 11.1.8).

**5.4.B.** EXERCISE. Show that the affine line with doubled origin is not affine. Hint: calculate the ring of global sections, and look back at the argument for $\mathbb{A}^2 - \{(0,0)\}$.

**5.4.C.** EASY EXERCISE. Do the same construction with $\mathbb{A}^1$ replaced by $\mathbb{A}^2$. You'll have defined the **affine plane with doubled origin**. Describe two affine open subsets of this scheme whose intersection is not an affine open subset.

**5.4.6. Example 2: the projective line.** Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow 1/u$. Figure 5.5 is a suggestive picture of this gluing. The resulting scheme is called the **projective line over the field** $k$, and is denoted $\mathbb{P}^1_k$.

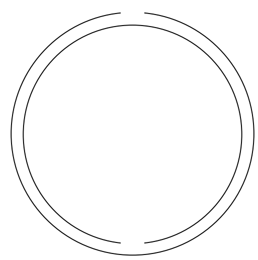


FIGURE 5.5. Gluing two affine lines together to get $\mathbb{P}^1$

Notice how the points glue. Let me assume that $k$ is algebraically closed for convenience. (You can think about how this changes otherwise.) On the first affine line, we have the closed ("traditional") points $[(t-a)]$, which we think of as "$a$ on the $t$-line", and we have the generic point $[(0)]$. On the second affine line, we have closed points that are "$b$ on the $u$-line", and the generic point. Then $a$ on the $t$-line is glued to $1/a$ on the $u$-line (if $a \neq 0$ of course), and the generic point is glued to the generic point (the ideal $(0)$ of $k[t]$ becomes the ideal $(0)$ of $k[t, 1/t]$ upon localization, and the ideal $(0)$ of $k[u]$ becomes the ideal $(0)$ of $k[u, 1/u]$. And $(0)$ in $k[t, 1/t]$ is $(0)$ in $k[u, 1/u]$ under the isomorphism $t \leftrightarrow 1/u$).

We can interpret the closed points of $\mathbb{P}^1$ in the following way, which may make this sound closer to the way you have seen projective space defined earlier. The points are of the form $[a;b]$, where $a$ and $b$ are not both zero, and $[a;b]$ is identified with $[ac;bc]$ where $c \in k^*$. Then if $b \neq 0$, this is identified with $a/b$ on the $t$-line, and if $a \neq 0$, this is identified with $b/a$ on the $u$-line.

**5.4.7. Proposition. —** *$\mathbb{P}^1_k$ is not affine.*

*Proof.* We do this by calculating the ring of global sections. The global sections correspond to sections over $X$ and sections over $Y$ that agree on the overlap. A section on $X$ is a polynomial $f(t)$. A section on $Y$ is a polynomial $g(u)$. If we restrict $f(t)$ to the overlap, we get something we can still call $f(t)$; and similarly for $g(u)$.

Now we want them to be equal: $f(t) = g(1/t)$. But the only polynomials in $t$ that are at the same time polynomials in in $1/t$ are the constants $k$. Thus $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$. If $\mathbb{P}^1$ were affine, then it would be $\operatorname{Spec}\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Spec} k$, i.e. one point. But it isn't — it has lots of points. □

We have proved an analogue of a theorem: the only holomorphic functions on $\mathbb{CP}^1$ are the constants!

**5.4.8. Important example: Projective space.** We now make a preliminary definition of **projective $n$-space over a field** $k$, denoted $\mathbb{P}^n_k$, by gluing together $n+1$ open sets each isomorphic to $\mathbb{A}^n_k$. Judicious choice of notation for these open sets will make our life easier. Our motivation is as follows. In the construction of $\mathbb{P}^1$ above, we thought of points of projective space as $[x_0; x_1]$, where $(x_0, x_1)$ are only determined up to scalars, i.e. $(x_0, x_1)$ is considered the same as $(\lambda x_0, \lambda x_1)$. Then the first patch can be interpreted by taking the locus where $x_0 \neq 0$, and then we consider the points $[1; t]$, and we think of $t$ as $x_1/x_0$; even though $x_0$ and $x_1$ are not well-defined, $x_1/x_0$ *is*. The second corresponds to where $x_1 \neq 0$, and we consider the points $[u; 1]$, and we think of $u$ as $x_0/x_1$. It will be useful to instead use the notation $x_{1/0}$ for $t$ and $x_{0/1}$ for $u$.

For $\mathbb{P}^n$, we glue together $n+1$ open sets, one for each of $i = 0, \dots, n+1$. The $i$th open set $U_i$ will have coordinates $x_{0/i}, \dots, x_{(i-1)/i}, x_{(i+1)/i}, \dots, x_{n/i}$. It will be convenient to write this as

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, \dots, x_{n/i}]/(x_{i/i} - 1)$$

(so we have introduced a "dummy variable" $x_{i/i}$ which we set to 1). We glue the distinguished open set $D(x_{j/i})$ of $U_i$ to the distinguished open set $D(x_{i/j})$ of $U_j$, by identifying these two schemes by describing the identification of rings

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, \dots, x_{n/i}, 1/x_{j/i}]/(x_{i/i} - 1) \cong$$

$$\operatorname{Spec} k[x_{0/j}, x_{1/j}, \dots, x_{n/j}, 1/x_{i/j}]/(x_{j/j} - 1)$$

via $x_{k/i} = x_{k/j}/x_{i/j}$ and $x_{k/j} = x_{k/i}/x_{j/i}$ (which implies $x_{i/j}x_{j/i} = 1$). We need to check that this gluing information agrees over triple overlaps.

**5.4.D.** EXERCISE. Check this, as painlessly as possible. (Possible hint: the triple intersection is affine; describe the corresponding ring.)

Note that our definition does not use the fact that $k$ is a field. Hence we may as well define $\mathbb{P}^n_A$ for any *ring* $A$. This will be useful later.

**5.4.E.** EXERCISE. Show that the only global sections of the structure sheaf are constants, and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine open sets, but you will only need two of your open sets to see this. There is even some geometric intuition behind this: the complement of the union of two open sets has codimension 2. But "Algebraic Hartogs' Lemma" (discussed informally in §5.4.2, to be stated rigorously in Theorem 12.3.7) says that any function defined on this union extends to be a function on all of projective space. Because we are expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

**5.4.F.** EXERCISE. Show that if $k$ is algebraically closed, the closed points of $\mathbb{P}^n_k$ may be interpreted in the traditional way: the points are of the form $[a_0; \dots; a_n]$, where the $a_i$ are not all zero, and $[a_0; \dots; a_n]$ is identified with $[\lambda a_0; \dots; \lambda a_n]$ where $\lambda \in k^*$.

We will later give other definitions of projective space (Definition 5.5.4, §17.4.2). Our first definition here will often be handy for computing things. But there is something unnatural about it — projective space is highly symmetric, and that isn't clear from your point of view.

**5.4.9. Fun aside: The Chinese Remainder Theorem is a *geometric* fact.** The Chinese Remainder theorem is embedded in what we have done, which shouldn't be obvious. I will show this by example, but you should then figure out the general statement. The Chinese Remainder Theorem says that knowing an integer modulo 60 is the same as knowing an integer modulo 3, 4, and 5. Here's how to see this in the language of schemes. What is $\operatorname{Spec} \mathbb{Z}/(60)$? What are the primes of this ring? Answer: those prime ideals containing $(60)$, i.e. those primes dividing 60, i.e. $(2)$, $(3)$, and $(5)$. Figure 5.6 is a sketch of $\operatorname{Spec} \mathbb{Z}/(60)$. They are all closed points, as these are all maximal ideals, so the topology is the discrete topology. What are the stalks? You can check that they are $\mathbb{Z}/4$, $\mathbb{Z}/3$, and $\mathbb{Z}/5$. The nilpotents "at $(2)$" are indicated by the "fuzz" on that point. (We discussed visualizing nilpotents with "infinitesimal fuzz" in §s:visualizingschemesII.) So what are global sections on this scheme? They are sections on this open set $(2)$, this other open set $(3)$, and this third open set $(5)$. In other words, we have a natural isomorphism of rings

$$\mathbb{Z}/60 \to \mathbb{Z}/4 \times \mathbb{Z}/3 \times \mathbb{Z}/5.$$

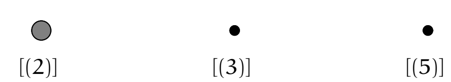



FIGURE 5.6. A picture of the scheme $\operatorname{Spec} \mathbb{Z}/(60)$

**5.4.10.** ⋆ *Example.* Here is an example of a function on an open subset of a scheme that is a bit surprising. On $X = \operatorname{Spec} k[w, x, y, z]/(wx - yz)$, consider the open subset $D(y) \cup D(w)$. Show that the function $x/y$ on $D(y)$ agrees with $z/w$ on $D(w)$ on their overlap $D(y) \cap D(w)$. Hence they glue together to give a section. You may have seen this before when thinking about analytic continuation in complex geometry — we have a "holomorphic" function which has the description $x/y$ on an open set, and this description breaks down elsewhere, but you can still "analytically continue" it by giving the function a different definition on different parts of the space.

Follow-up for curious experts: This function has no "single description" as a well-defined expression in terms of $w, x, y, z$! There is lots of interesting geometry here. This example will be a constant source of interesting examples for us. We will later recognize it as the cone over the quadric surface. Here is a glimpse, in terms of words we have not yet defined. Spec $k[w,x,y,z]$ is $\mathbb{A}^4$, and is, not surprisingly, 4-dimensional. We are looking at the set $X$, which is a hypersurface, and is 3-dimensional. It is a cone over a smooth quadric surface in $\mathbb{P}^3$ (see Figure 5.7). $D(y)$ is $X$ minus some hypersurface, so we are throwing away a codimension 1 locus. $D(z)$ involves throwing away another codimension 1 locus. You might think that their intersection is then codimension 2, and that maybe failure of extending this weird function to a global polynomial comes because of a failure of our Hartogs' Lemma-type theorem, which will be a failure of normality. But that's not true — $V(y) \cap V(z)$ is in fact codimension 1 — so no Hartogs-type theorem holds. Here is what is actually going on. $V(y)$ involves throwing away the (cone over the) union of two lines $\ell$ and $m_1$, one in each "ruling" of the surface, and $V(z)$ also involves throwing away the (cone over the) union of two lines $\ell$ and $m_2$. The intersection is the (cone over the) line $\ell$, which is a codimension 1 set. Neat fact: despite being "pure codimension 1", it is not cut out even set-theoretically by a single equation. (It is hard to get an example of this behavior. This example is the simplest example I know.) This means that any expression $f(w,x,y,z)/g(w,x,y,z)$ for our function cannot correctly describe our function on $D(y) \cup D(z)$ — at some point of $D(y) \cup D(z)$ it must be $0/0$. Here's why. Our function can't be defined on $V(y) \cap V(z)$, so $g$ must vanish here. But $g$ can't vanish just on the cone over $\ell$ — it must vanish elsewhere too. (For the experts among the experts: here is why the cone over $l$ is not cut out set-theoretically by a single equation. If $\ell = V(f)$, then $D(f)$ is affine. Let $\ell'$ be another line in the same ruling as $\ell$, and let $C(\ell)$ (resp. $\ell'$) be the cone over $\ell$ (resp. $\ell'$). Then $C(\ell')$ can be given the structure of a closed subscheme of Spec $k[w,x,y,z]$, and can be given the structure of $\mathbb{A}^2$. Then $C(\ell') \cap V(f)$ is a closed subscheme of $D(f)$. Any closed subscheme of an affine scheme is affine. But $\ell \cap \ell' = \varnothing$, so the cone over $\ell$ intersects the cone over $\ell'$ in a point, so $C(\ell') \cap V(f)$ is $\mathbb{A}^2$ minus a point, which we've seen is not affine, so we have a contradiction.)

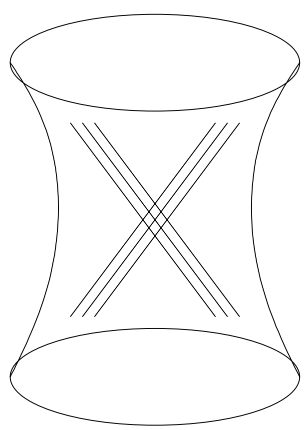


FIGURE 5.7. The two rulings on the quadric surface $V(wz-xy) \subset \mathbb{P}^3$. One ruling contains the line $V(w,x)$ and the other contains the line $V(w,y)$.

## 5.5 Projective schemes

Projective schemes are important for a number of reasons. Here are a few. Schemes that were of "classical interest" in geometry — and those that you would have cared about before knowing about schemes — are all projective or quasiprojective. Moreover, schemes of "current interest" tend to be projective or quasiprojective. In fact, it is very hard to even give an example of a scheme satisfying basic properties — for example, finite type and "Hausdorff" ("separated") over a field — that is provably not quasiprojective. For complex geometers: it is hard to find a compact complex variety that is provably not projective (we will see an example in §24.5.4), and it is quite hard to come up with a complex variety that is provably not an open subset of a projective variety. So projective schemes are really ubiquitous. Also a projective $k$-scheme is a good approximation of the algebro-geometric version of compactness ("properness", see §11.3).

Finally, although projective schemes may be obtained by gluing together affines, and we know that keeping track of gluing can be annoying, there is a simple means of dealing with them without worrying about gluing. Just as there is a rough dictionary between rings and affine schemes, we will have an analogous dictionary between graded rings and projective schemes. Just as one can work with affine schemes by instead working with rings, one can work with projective schemes by instead working with graded rings. To get an initial sense of how this works, consider Example 9.2.1 (which secretly gives the notion of projective $A$-schemes in full generality). Recall that any collection of homogeneous elements of $A[x_0, \dots, x_n]$ describes a closed subscheme of $\mathbb{P}^n_A$. (The $x_0, \dots, x_n$ are called **projective coordinates** on the scheme. Warning: they are not functions on the scheme. Any closed subscheme of $\mathbb{P}^n_A$ cut out by a set of homogeneous polynomials will soon be called a *projective $A$-scheme*.) Thus if $I$ is a **homogeneous ideal** in $A[x_0, \dots, x_n]$ (i.e. generated by homogeneous polynomials), we have defined a closed subscheme of $\mathbb{P}^n_A$ deserving the name $V(I)$. Conversely, given a closed subset $S$ of $\mathbb{P}^n_A$, we can consider those homogeneous polynomials in the projective coordinates, vanishing on $S$. This homogeneous ideal deserves the name $I(S)$.

**5.5.1.** *A motivating picture from classical geometry.* For geometric intuition, we recall how one thinks of projective space "classically" (in the classical topology, over the real numbers). $\mathbb{P}^n$ can be interpreted as the lines through the origin in $\mathbb{R}^{n+1}$. Thus subsets of $\mathbb{P}^n$ correspond to unions of lines through the origin of $\mathbb{R}^{n+1}$, and closed subsets correspond to such unions which are closed. (The same is not true with "closed" replaced by "open"!)

One often pictures $\mathbb{P}^n$ as being the "points at infinite distance" in $\mathbb{R}^{n+1}$, where the points infinitely far in one direction are associated with the points infinitely far in the opposite direction. We can make this more precise using the decomposition

$$\mathbb{P}^{n+1} = \mathbb{R}^{n+1} \coprod \mathbb{P}^n$$

by which we mean that there is an open subset in $\mathbb{P}^{n+1}$ identified with $\mathbb{R}^{n+1}$ (the points with last projective coordinate non-zero), and the complementary closed subset identified with $\mathbb{P}^n$ (the points with last projective coordinate zero).

Then for example any equation cutting out some set $V$ of points in $\mathbb{P}^n$ will also cut out some set of points in $\mathbb{R}^n$ that will be a closed union of lines. We call this the *affine cone* of $V$. These equations will cut out some union of $\mathbb{P}^1$'s in $\mathbb{P}^{n+1}$, and we call this the *projective cone* of $V$. The projective cone is the disjoint union of the affine cone and $V$. For example, the affine cone over $x^2 + y^2 = z^2$ in $\mathbb{P}^2$ is just the "classical" picture of a cone in $\mathbb{R}^2$, see Figure 5.8. We will make this analogy precise in our algebraic setting in §9.2.8. To make a connection with the previous discussion on homogeneous ideals: the homogeneous ideal given by the cone is $(x^2 + y^2 - z^2)$.

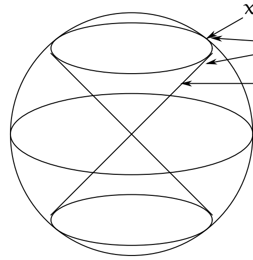



FIGURE 5.8. The affine and projective cone of $x^2 + y^2 = z^2$ in classical geometry

**5.5.2. The** Proj **construction.**

We will now produce a scheme out of a graded ring. A **graded ring** for us is a ring $S_\bullet = \oplus_n S_n$ (the subscript is called the **grading**), where multiplication respects the grading, i.e. sends $S_m \times S_n$ to $S_{m+n}$. (Our graded rings are indexed by $\mathbb{Z}^{\geq 0}$. One can define more general graded rings, but we won't need them.) Note that $S_0$ is a subring, and $S_\bullet$ is a $S_0$-algebra. In our examples so far, we have a graded ring $A[x_0, \dots, x_n]/I$ where $I$ is a homogeneous ideal. We are taking the usual grading on $A[x_0, \dots, x_n]$, where each $x_i$ has weight 1. In most of the examples below, $S_0 = A$, and $S_\bullet$ is generated over $S_0$ by $S_1$.

**5.5.3.** *Graded rings over* $A$*, and finitely generated graded rings.* Fix a ring $A$ (the **base ring**). Our motivating example is $S_\bullet = A[x_0, x_1, x_2]$, with the usual grading. If $S_\bullet$ is graded by $\mathbb{Z}^{\geq 0}$, with $S_0 = A$, we say that $S_\bullet$ is a **graded ring over** $A$. Hence each $S_n$ is an $A$-module. The subset $S_+ := \oplus_{i>0} S_i \subset S_\bullet$ is an ideal, called the **irrelevant ideal**. The reason for the name "irrelevant" will be clearer in a few paragraphs. If the irrelevant ideal $S_+$ is a finitely-generated ideal, we say that $S_\bullet$ is a **finitely generated graded ring over** $A$.

**5.5.A.** EXERCISE. Show that $S_\bullet$ is a finitely-generated graded ring if and only if $S_\bullet$ is a finitely-generated graded $A$-algebra, i.e. generated over $A = S_0$ by a finite number of homogeneous elements of positive degree. (Hint for the forward implication: show that the generators of $S_+$ as an ideal are also generators of $S_\bullet$ as an algebra.)

Motivated by our example of $\mathbb{P}^n_A$ and its closed subschemes, we now define a scheme Proj $S_\bullet$. As we did with Spec of a ring, we will build it first as a set, then as a topological space, and finally as a ringed space. In our preliminary definition of

$\mathbb{P}^n_A$, we glued together $n+1$ well-chosen affine pieces, but we don't want to make any choices, so we do this by simultaneously consider "all possible" affines. Our affine building blocks will be as follows. For each homogeneous $f \in S_+$, consider

$$\operatorname{Spec}((S_\bullet)_f)_0. \tag{5.5.3.1}$$

where $((S_\bullet)_f)_0$ means the 0-graded piece of the graded ring $(S_\bullet)_f$. The notation $((S_\bullet)_f)_0$ is admittedly horrible — the first and third subscripts refer to the grading, and the second refers to localization.

(Before we begin: another possible way of defining $\operatorname{Proj} S_\bullet$ is by gluing together affines, by jumping straight to Exercises 5.5.G, 5.5.H, and 5.5.I. If you prefer that, by all means do so.)

The points of $\operatorname{Proj} S_\bullet$ are set of homogeneous prime ideals of $S_\bullet$ not containing the irrelevant ideal $S_+$ (the "relevant prime ideals").

**5.5.B.** IMPORTANT AND TRICKY EXERCISE. Suppose $f \in S_+$ is homogeneous. Give a bijection between the primes of $((S_\bullet)_f)_0$ and the homogeneous prime ideals of $(S_\bullet)_f$. Describe the latter as a subset of $\operatorname{Proj} S_\bullet$. Hint: From the ring map $((S_\bullet)_f)_0 \to (S_\bullet)_f$, from each homogeneous primes of $(S_\bullet)_f$ we find a homogeneous prime of $((S_\bullet)_f)_0$. The reverse direction is the harder one. Given a prime ideal $P_0 \subset ((S_\bullet)_f)_0$, define $P \subset (S_\bullet)_f$ as generated by the following homogeneous elements: $a \in P$ if and only if $a^{\deg f}/f^{\deg a} \in P_0$. Showing that homogeneous $a$ is in $P$ if and only $a^2 \in P$; show that if $a_1, a_2 \in P$ then $(a_1 + a_2)^2 \in P$ and hence $a_1 + a_2 \in P$; then show that $P$ is an ideal; then show that $P$ is prime.)

The interpretation of the points of $\operatorname{Proj} S_\bullet$ with homogeneous prime ideals helps us picture $\operatorname{Proj} S_\bullet$. For example, if $S_\bullet = k[x,y,z]$ with the usual grading, then we picture the homogeneous prime ideal $(z^2 - x^2 - y^2)$ as a subset of $\operatorname{Spec} S_\bullet$; it is a cone (see Figure 5.8). As in §5.5.1, we picture $\mathbb{P}^2_k$ as the "plane at infinity". Thus we picture this equation as cutting out a conic "at infinity". We will make this intuition somewhat more precise in §9.2.8.

**5.5.C.** EXERCISE (THE ZARISKI TOPOLOGY ON $\operatorname{Proj} S_\bullet$). If $I$ is a homogeneous ideal of $S_+$, define the **vanishing set of** $I$, $V(I) \subset \operatorname{Proj} S_\bullet$, to be those homogeneous prime ideals containing $I$. As in the affine case, let $V(f)$ be $V((f))$, and let $D(f) = \operatorname{Proj} S_\bullet \setminus V(f)$ (the **projective distinguished open set**) be the complement of $V(f)$ (i.e. the open subscheme corresponding to that open set). Show that $D(f)$ is precisely the subset $((S_\bullet)_f)_0$ you described in the previous exercise.

As in the affine case, the $V(I)$'s satisfy the axioms of the closed set of a topology, and we call this the **Zariski topology** on $\operatorname{Proj} S_\bullet$. Many statements about the Zariski topology on Spec of a ring carry over to this situation with little extra work. Clearly $D(f) \cap D(g) = D(fg)$, by the same immediate argument as in the affine case (Exercise 4.5.D). As in the affine case (Exercise 4.5.E), if $D(f) \subset D(g)$, then $f^n \in (g)$ for some $n$, and vice versa.

**5.5.D.** EASY EXERCISE. Verify that the projective distinguished open sets form a base of the Zariski topology.

**5.5.E.** EXERCISE. Fix a graded ring $S_\bullet$.

(a) Suppose $I$ is any homogeneous ideal of $S_\bullet$, and $f$ is a homogeneous element. Show that $f$ vanishes on $V(I)$ if and only if $f^n \in I$ for some $n$. (Hint: Mimic the affine case; see Exercise 4.4.I.)
(b) If $Z \subset \operatorname{Proj} S_\bullet$, define $I(\cdot)$. Show that it is a homogeneous ideal. For any two subsets, show that $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
(c) For any subset $Z \subset \operatorname{Proj} S_\bullet$, show that $V(I(Z)) = \overline{Z}$.

**5.5.F.** EXERCISE (CF. EXERCISE 4.5.B). Fix a graded ring $S_\bullet$. Show that the following are equivalent.
(a) $V(I) = \varnothing$.
(b) for any $f_i$ (as $i$ runs through some index set) generating $I$, $\cup D(f_i) = \operatorname{Proj} S_\bullet$.
(c) $\sqrt{I} \supset S_+$.

This is more motivation for the $S_+$ being "irrelevant": any ideal whose radical contains it is "geometrically irrelevant".

Let's get back to constructing $\operatorname{Proj} S_\bullet$ as a *scheme*.

**5.5.G.** EXERCISE. Suppose some homogeneous $f \in S_\bullet$ is given. Via the inclusion

$$D(f) = \operatorname{Spec}((S_\bullet)_f)_0 \hookrightarrow \operatorname{Proj} S_\bullet,$$

show that the Zariski topology on $\operatorname{Proj} S_\bullet$ restricts to the Zariski topology on $\operatorname{Spec}((S_\bullet)_f)_0$.

Now that we have defined $\operatorname{Proj} S_\bullet$ as a topological space, we are ready to define the structure sheaf. On $D(f)$, we wish it to be the structure sheaf of $\operatorname{Spec}((S_\bullet)_f)_0$. We will glue these sheaves together using Exercise 3.7.D on gluing sheaves.

**5.5.H.** EXERCISE. If $f, g \in S_+$ are homogeneous, describe an isomorphism between $\operatorname{Spec}((S_\bullet)_{fg})_0$ and the distinguished open subset $D(g^{\deg f}/f^{\deg g})$ of $\operatorname{Spec}((S_\bullet)_f)_0$.

Similarly, $\operatorname{Spec}((S_\bullet)_{fg})_0$ is identified with a distinguished open subset of $\operatorname{Spec}((S_\bullet)_g)_0$. We then glue the various $\operatorname{Spec}((S_\bullet)_f)_0$ (as $f$ varies) altogether, using these pairwise gluings.

**5.5.I.** EXERCISE. By checking that these gluings behave well on triple overlaps (see Exercise 3.7.D), finish the definition of the scheme $\operatorname{Proj} S_\bullet$.

**5.5.J.** EXERCISE (SOME WILL FIND THIS ESSENTIAL, OTHERS WILL PREFER TO IGNORE IT). (Re)interpret the structure sheaf of $\operatorname{Proj} S_\bullet$ in terms of compatible stalks.

**5.5.4.** *Definition.* We (re)define **projective space** (over a ring $A$) by $\mathbb{P}^n_A := \operatorname{Proj} A[x_0, \dots, x_n]$. This definition involves no messy gluing, or special choice of patches.

**5.5.K.** EXERCISE. Check that this agrees with our earlier construction of $\mathbb{P}^n_A$ (Definition 5.4.8). (How do you know that the $D(x_i)$ cover $\operatorname{Proj} A[x_0, \dots, x_n]$?)

Notice that with our old definition of projective space, it would have been a nontrivial exercise to show that $D(x^2 + y^2 - z^2) \subset \mathbb{P}^2_k$ (the complement of a plane conic) is affine; with our new perspective, it is immediate — it is $\operatorname{Spec}(k[x, y, z]_{(x^2+y^2-z^2)})_0$.

**5.5.L.** EXERCISE. (a) *(the most important part)* If $S_\bullet$ is generated in degree $1$, and $f \in S_+$ is homogeneous, explain how to define the closed subscheme $V(f)$ of $\operatorname{Proj} S_\bullet$, the **vanishing scheme of** $f$. (Warning: $f$ in general isn't a function on $\operatorname{Proj} S_\bullet$. We will later interpret it as something close: a section of a line bundle.) Hence define $V(I)$ for any homogeneous ideal $I$ of $S_+$.
(b) *(harder)* If $S_\bullet$ is a graded ring over $A$, but not necessarily generated in degree $1$, explain how to define the closed subscheme $V(f)$ of $\operatorname{Proj} S_\bullet$. (Hint: On $D(g)$, let $V(f)$ be cut out by all degree $0$ equations of the form $fh/g^n$, where $n \in \mathbb{Z}^+$, and $h$ is homogeneous. Show that this gives a well defined closed subscheme. Your calculations will mirror those of Exercise 5.5.H.)

**5.5.5. Projective and quasiprojective schemes.**

We call a scheme of the form $\operatorname{Proj} S_\bullet$, where $S_\bullet$ is a *finitely generated* graded ring over $A$, a **projective scheme over** $A$, or a **projective** $A$**-scheme**. A **quasiprojective** $A$**-scheme** is an open subscheme of a projective $A$-scheme. The "$A$" is omitted if it is clear from the context; often $A$ is a field. (Be careful: $\operatorname{Proj} S_\bullet$ makes sense even when $S_\bullet$ is not finitely generated. This can — rarely — be useful. But having this more general construction can make things easier. For example, you will later be able to do Exercise 7.4.D without worrying about Exercise 7.4.H.)

**5.5.6.** *Silly example.* Note that $\mathbb{P}^0_A = \operatorname{Proj} A[T] \cong \operatorname{Spec} A$. Thus "$\operatorname{Spec} A$ is a projective $A$-scheme".

**5.5.7. Example: $\mathbb{P}V$.** We can make this definition of projective space even more choice-free as follows. Let $V$ be an $(n+1)$-dimensional vector space over $k$. (Here $k$ can be replaced by any ring $A$ as usual.) Define

$$\operatorname{Sym}^\bullet V^\vee = k \oplus V^\vee \oplus \operatorname{Sym}^2 V^\vee \oplus \cdots .$$

(The reason for the dual is explained by the next exercise.) If for example $V$ is the dual of the vector space with basis associated to $x_0, \dots, x_n$, we would have $\operatorname{Sym}^\bullet V^\vee = k[x_0, \dots, x_n]$. Then we can define $\mathbb{P}V := \operatorname{Proj} \operatorname{Sym}^\bullet V^\vee$. In this language, we have an interpretation for $x_0, \dots, x_n$: they are the linear functionals on the underlying vector space $V$.

**5.5.M.** UNIMPORTANT EXERCISE. Suppose $k$ is algebraically closed. Describe a natural bijection between one-dimensional subspaces of $V$ and the points of $\mathbb{P}V$. Thus this construction canonically (in a basis-free manner) describes the one-dimensional subspaces of the vector space $\operatorname{Spec} V$.

Unimportant remark: you may be surprised at the appearance of the dual in the definition of $\mathbb{P}V$. This is explained by the previous exercise. Most normal (traditional) people define the projectivization of a vector space $V$ to be the space of one-dimensional subspaces of $V$. Grothendieck considered the projectivization to be the space of one-dimensional *quotients*. One motivation for this is that it gets rid of the annoying dual in the definition above. There are better reasons to, that we won't go into here. In a nutshell, quotients tend to be better-behaved than subobjects for coherent sheaves, which generalize the notion of vector bundle. (We will discuss them in Chapter 14.)

On another note related to Exercise 5.5.M: you can also describe a natural bijection between points of $V$ and the points of $\operatorname{Spec} \operatorname{Sym}^\bullet V^\vee$. This construction respects the affine/projective cone picture of §9.2.8.

**5.5.8.** *The Grassmannian.* At this point, we could describe the fundamental geometric object known as the *Grassmannian*, and give the "wrong" definition of it. We will instead wait until §7.7 to give the wrong definition, when we will know enough to sense that something is amiss. The right definition will be given in §17.6.

CHAPTER 6

# Some properties of schemes

## 6.1 Topological properties

We will now define some useful properties of schemes. The definitions of *irreducible*, *irreducible component*, *closed point*, *specialization*, *generalization*, *generic point*, *connected*, *connected component*, and *quasicompact* were given in §4.5–4.6. You should have pictures in your mind of each of these notions.

Exercise 4.6.N shows that $\mathbb{A}^n$ is irreducible (it was easy). This argument "behaves well under gluing", yielding:

**6.1.A.** EASY EXERCISE. Show that $\mathbb{P}^n_k$ is irreducible.

**6.1.B.** EXERCISE. Exercise 4.7.E showed that there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

**6.1.C.** EASY EXERCISE. Prove that if $X$ is a scheme that has a finite cover $X = \cup_{i=1}^n \operatorname{Spec} A_i$ where $A_i$ is Noetherian, then $X$ is a Noetherian topological space (§4.6.3). (We will soon call such a scheme a *Noetherian scheme*, §6.3.4.)

Thus $\mathbb{P}^n_k$ and $\mathbb{P}^n_{\mathbb{Z}}$ are Noetherian topological spaces: we built them by gluing together a finite number of spectra of Noetherian rings.

**6.1.D.** EASY EXERCISE. Show that a scheme $X$ is quasicompact if and only if it can be written as a finite union of affine schemes. (Hence $\mathbb{P}^n_k$ is quasicompact.)

**6.1.E.** GOOD EXERCISE: QUASICOMPACT SCHEMES HAVE CLOSED POINTS. Show that if $X$ is a quasicompact scheme, then every point has a closed point in its closure. In particular, every nonempty quasicompact scheme has a closed point. (Warning: there exist non-empty schemes with no closed points, so your argument had better use the quasicompactness hypothesis! We will see that in good situations, the closed points are dense, Exercise 6.3.E.)

**6.1.1. Quasiseparatedness.** Quasiseparatedness is a weird notion that comes in handy for certain people. (Warning: we will later realize that this is really a property of *morphisms*, not of schemes §8.3.1.) Most people, however, can ignore this notion, as the schemes they will encounter in real life will all have this property. A topological space is **quasiseparated** if the intersection of any two quasicompact open sets is quasicompact. Thus a scheme is quasiseparated if the intersection of any two affine open subsets is a finite union of affine open subsets.

**6.1.F.** SHORT EXERCISE. Prove this equivalence.

We will see later that this will be a useful hypothesis in theorems (in conjunction with quasicompactness), and that various interesting kinds of schemes (affine, locally Noetherian, separated, see Exercises 6.1.G, 6.3.B, and 11.1.F resp.) are quasiseparated, and this will allow us to state theorems more succinctly (e.g. "if $X$ is quasicompact and quasiseparated" rather than "if $X$ is quasicompact, and either this or that or the other thing hold").

**6.1.G.** EXERCISE. Show that affine schemes are quasiseparated.

"Quasicompact and quasiseparated" means something concrete:

**6.1.H.** EXERCISE. Show that a scheme $X$ is quasicompact and quasiseparated if and only if $X$ can be covered by a finite number of affine open subsets, any two of which have intersection also covered by a finite number of affine open subsets.

So when you see "quasicompact and quasiseparated" as hypotheses in a theorem, you should take this as a clue that you will use this interpretation, and that finiteness will be used in an essential way.

**6.1.I.** EASY EXERCISE. Show that all projective $A$-schemes are quasicompact and quasiseparated. (Hint: use the fact that the graded ring in the definition is finitely generated — those finite number of generators will lead you to a covering set.)

**6.1.2. Dimension.** One very important topological notion is *dimension*. (It is amazing that this is a *topological* idea.) But despite being intuitively fundamental, it is more difficult, so we will put it off until Chapter 12.

## 6.2 Reducedness and integrality

Recall that one of the alarming things about schemes is that functions are not determined by their values at points, and that was because of the presence of nilpotents (§4.2.9).

**6.2.1. Definition.** A ring is said to be *reduced* if it has no nonzero nilpotents (§4.2.11). A scheme $X$ is **reduced** if $\mathcal{O}_X(U)$ is reduced for every open set $U$ of $X$.

An example of a nonreduced affine scheme is $\operatorname{Spec} k[x,y]/(y^2, xy)$. A useful representation of this scheme is given in Figure 6.1, although we will only explain in §6.5 why this is a good picture. The fuzz indicates that there is some nonreducedness going on at the origin. Here are two different functions: $x$ and $x+y$. Their values agree at all points (all closed points $[(x-a, y)] = (a, 0)$ and at the generic point $[(y)]$). They are actually the same function on the open set $D(x)$, which is not surprising, as $D(x)$ is reduced, as the next exercise shows. (This explains why the fuzz is only at the origin, where $y = 0$.)

**6.2.A.** EXERCISE. Show that $\left(k[x,y]/(y^2, xy)\right)_x$ has no nilpotents. (Possible hint: show that it is isomorphic to another ring, by considering the geometric picture. Exercise 4.2.H may give another hint.)

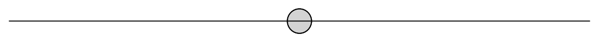


FIGURE 6.1. A picture of the scheme $\operatorname{Spec} k[x,y]/(y^2, xy)$. The fuzz indicates where "the non-reducedness lives".

**6.2.B.** EXERCISE (REDUCEDNESS IS A **stalk-local** PROPERTY, I.E. CAN BE CHECKED AT STALKS). Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if $f$ and $g$ are two functions on a reduced scheme that agree at all points, then $f = g$. (Two hints: $\mathcal{O}_X(U) \hookrightarrow \prod_{x \in U} \mathcal{O}_{X,x}$ from Exercise 3.4.A, and the nilradical is intersection of all prime ideals from Theorem 4.2.10.)

We remark that the fuzz in Figure 6.1 indicates the points where there is nonreducedness.

**6.2.C.** EXERCISE (CF. EXERCISE 6.1.E). If $X$ is a quasicompact scheme, show that it suffices to check reducedness at closed points. (Hint: For show that any point of a quasicompact scheme has a closed point in its closure.)

*Warning for experts:* if a scheme $X$ is reduced, then it is immediate from the definition that its ring of global sections is reduced. However, the converse is not true.

**6.2.D.** EXERCISE. Suppose $X$ is quasicompact, and $f$ is a function (a global section of $\mathcal{O}_X$) that vanishes at all points of $x$. Show that there is some $n$ such that $f^n = 0$. Show that this may fail if $X$ is not quasicompact. (This exercise is less important, but shows why we like quasicompactness, and gives a standard pathology when quasicompactness doesn't hold.) Hint: take an infinite disjoint union of $\operatorname{Spec} A_n$ with $A_n := k[\epsilon]/\epsilon^n$.

**Definition.** A scheme $X$ is **integral** if $\mathcal{O}_X(U)$ is an integral domain for every nonempty open set $U$ of $X$.

**6.2.E.** IMPORTANT EXERCISE. Show that a scheme $X$ is integral if and only if it is irreducible and reduced.

**6.2.F.** EXERCISE. Show that an affine scheme $\operatorname{Spec} A$ is integral if and only if $A$ is an integral domain.

**6.2.G.** EXERCISE. Suppose $X$ is an integral scheme. Then $X$ (being irreducible) has a generic point $\eta$. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of $X$. Show that the stalk at $\eta$, $\mathcal{O}_{X,\eta}$, is naturally $K(A)$, the fraction field of $A$. This is called the **function field** $K(X)$ of $X$. It can be computed on any non-empty open set of $X$, as any such open set contains the generic point.

**6.2.H.** EXERCISE. Suppose $X$ is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \varnothing$. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of $X$ (so $A$ is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = K(A)$ (where $U$ is any non-empty open set) is an inclusion. Thus irreducible varieties (an important example of integral schemes defined later) have the convenient property that sections over different open sets can be considered subsets of the same ring. Thus restriction maps (except to the empty set) are always inclusions, and gluing is easy: functions $f_i$ on a cover $U_i$ of $U$ (as $i$ runs over an index set) glue if and only if they are the same element of $K(X)$. This is one reason why (irreducible) varieties are usually introduced before schemes.

Integrality is not stalk-local (the disjoint union of two integral schemes is not integral, as $\operatorname{Spec} A \coprod \operatorname{Spec} B = \operatorname{Spec} A \times B$, cf. Exercise 4.6.S), but it almost is, as is shown in the following believable exercise.

**6.2.I.** UNIMPORTANT EXERCISE. Show that a locally Noetherian scheme $X$ is integral if and only if $X$ is connected and all stalks $\mathcal{O}_{X,p}$ are integral domains. Thus in "good situations" (when the scheme is Noetherian), integrality is the union of local (stalks are domains) and global (connected) conditions.

## 6.3 Properties of schemes that can be checked "affine-locally"

This section is intended to address something tricky in the definition of schemes. We have defined a scheme as a topological space with a sheaf of rings, that can be covered by affine schemes. Hence we have all of the affine open sets in the cover, but we don't know how to communicate between any two of them. Somewhat more explicitly, if I have an affine cover, and you have an affine cover, and we want to compare them, and I calculate something on my cover, there should be some way of us getting together, and figuring out how to translate my calculation over to your cover. The Affine Communication Lemma 6.3.2 will provide a convenient machine for doing this.

Thanks to this lemma, we can define a host of important properties of schemes. All of these are "affine-local" in that they can be checked on any affine cover, i.e. a covering by open affine sets. We like such properties because we can check them using any affine cover we like. If the scheme in question is quasicompact, then we need only check a finite number of affine open sets.

**6.3.1. Proposition. —** *Suppose* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are affine open subschemes of a scheme* $X$. *Then* $\operatorname{Spec} A \cap \operatorname{Spec} B$ *is the union of open sets that are simultaneously distinguished open subschemes of* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$.

*Proof.* (See Figure 6.2 for a sketch.) Given any point $p \in \operatorname{Spec} A \cap \operatorname{Spec} B$, we produce an open neighborhood of $p$ in $\operatorname{Spec} A \cap \operatorname{Spec} B$ that is simultaneously distinguished in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Let $\operatorname{Spec} A_f$ be a distinguished open subset of $\operatorname{Spec} A$ contained in $\operatorname{Spec} A \cap \operatorname{Spec} B$. Let $\operatorname{Spec} B_g$ be a distinguished open subset of $\operatorname{Spec} B$ contained in $\operatorname{Spec} A_f$. Then $g \in \Gamma(\operatorname{Spec} B, \mathcal{O}_X)$ restricts to an element $g' \in \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X) = A_f$. The points of $\operatorname{Spec} A_f$ where $g$ vanishes are precisely the points of $\operatorname{Spec} A_f$ where $g'$ vanishes, so

$$\begin{aligned} \operatorname{Spec} B_g &= \operatorname{Spec} A_f \setminus \{[\mathfrak{p}] : g' \in \mathfrak{p}\} \\ &= \operatorname{Spec}(A_f)_{g'}. \end{aligned}$$

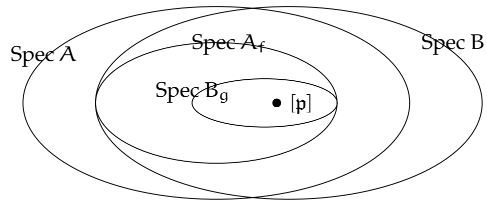



FIGURE 6.2. A trick to show that the intersection of two affine open sets may be covered by open sets that are simultaneously distinguished in both affine open sets

If $g' = g''/f^n$ ($g'' \in A$) then $\operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fg''}$, and we are done. $\square$

The following easy result will be crucial for us.

**6.3.2. Affine Communication Lemma. —** *Let* $P$ *be some property enjoyed by some affine open sets of a scheme* $X$*, such that*

(i) *if an affine open set* $\operatorname{Spec} A \hookrightarrow X$ *has property* $P$ *then for any* $f \in A$, $\operatorname{Spec} A_f \hookrightarrow X$ *does too.*
(ii) *if* $(f_1, \dots, f_n) = A$*, and* $\operatorname{Spec} A_{f_i} \hookrightarrow X$ *has* $P$ *for all* $i$*, then so does* $\operatorname{Spec} A \hookrightarrow X$.

*Suppose that* $X = \cup_{i \in I} \operatorname{Spec} A_i$ *where* $\operatorname{Spec} A_i$ *has property* $P$*. Then every open affine subset of* $X$ *has* $P$ *too.*

We say such a property is **affine-local**. Note that any property that is stalk-local (a scheme has property $P$ if and only if all its stalks have property $Q$) is necessarily affine-local (a scheme has property $P$ if and only if all of its affines have property $R$, where an affine scheme has property $R$ if and only if and only if all its stalks have property $Q$), but it is sometimes not so obvious what the right definition of $Q$ is; see for example the discussion of normality in the next section.

*Proof.* Let $\operatorname{Spec} A$ be an affine subscheme of $X$. Cover $\operatorname{Spec} A$ with a finite number of distinguished open sets $\operatorname{Spec} A_{g_j}$, each of which is distinguished in some $\operatorname{Spec} A_i$. This is possible by Proposition 6.3.1 and the quasicompactness of $\operatorname{Spec} A$ (Exercise 4.6.C(a)). By (i), each $\operatorname{Spec} A_{g_j}$ has $P$. By (ii), $\operatorname{Spec} A$ has $P$. $\square$

By choosing property $P$ appropriately, we define some important properties of schemes.

**6.3.3. Proposition. —** *Suppose* $A$ *is a ring, and* $(f_1, \dots, f_n) = A$.

(a) *If* $A$ *is a Noetherian ring, then so is* $A_{f_i}$*. If each* $A_{f_i}$ *is Noetherian, then so is* $A$.
(b) *If* $A$ *is reduced, then* $A_{f_i}$ *is also reduced. If each* $A_{f_i}$ *is reduced, then so is* $A$.
(c) *Suppose* $B$ *is a ring, and* $A$ *is a* $B$*-algebra. (Hence* $A_g$ *is a* $B$*-algebra for all* $g$*.) If* $A$ *is a finitely generated* $B$*-algebra, then so is* $A_{f_i}$*. If each* $A_{f_i}$ *is a finitely-generated* $B$*-algebra, then so is* $A$.

We will prove these shortly (§6.3.8). But let's first motivate you to read the proof by giving some interesting definitions *assuming* Proposition 6.3.3 is true.

**6.3.4. Important Definition.** Suppose $X$ is a scheme. If $X$ can be covered by affine open sets $\operatorname{Spec} A$ where $A$ is Noetherian, we say that $X$ is a **locally Noetherian scheme**. If in addition $X$ is quasicompact, or equivalently can be covered by finitely many such affine open sets, we say that $X$ is a **Noetherian scheme**. (We will see a number of definitions of the form "if $X$ has this property, we say that it is locally $Q$; if further $X$ is quasicompact, we say that it is $Q$.") By Exercise 6.1.C, the underlying topological space of a Noetherian scheme is Noetherian.

**6.3.A.** EXERCISE. Show that all open subsets of a Noetherian topological space (hence a Noetherian scheme) are quasicompact.

**6.3.B.** EXERCISE. Show that locally Noetherian schemes are quasiseparated.

**6.3.C.** EXERCISE. Show that a Noetherian scheme has a finite number of irreducible components. Show that a Noetherian scheme has a finite number of connected components, each a finite union of irreducible components.

**6.3.D.** EXERCISE. Show that $X$ is reduced if and only if $X$ can be covered by affine open sets $\operatorname{Spec} A$ where $A$ is reduced.

Our earlier definition of reducedness required us to check that the ring of functions over *any* open set is nilpotent-free. Our new definition lets us check a single affine cover. Hence for example $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are reduced.

**6.3.5.** *Schemes over a given field, or more generally over a given ring ($A$-schemes).* You may be particularly interested in working over a particular field, such as $\mathbb{C}$ or $\mathbb{Q}$, or over a ring such as $\mathbb{Z}$. Motivated by this, we define the notion of **$A$-scheme**, or **scheme over $A$**, where $A$ is a ring, as a scheme where all the rings of sections of the structure sheaf (over all open sets) are $A$-algebras, and all restriction maps are maps of $A$-algebras. (Like some earlier notions such as quasiseparatedness, this will later in Exercise 7.3.G be properly understood as a "relative notion"; it is the data of a morphism $X \to \operatorname{Spec} A$.) Suppose now $X$ is an $A$-scheme. If $X$ can be covered by affine open sets $\operatorname{Spec} B_i$ where each $B_i$ is a *finitely generated* $A$-algebra, we say that $X$ is **locally of finite type over $A$**, or that it is a **locally of finite type $A$-scheme**. (This is admittedly cumbersome terminology; it will make more sense later, once we know about morphisms in §8.3.8.) If furthermore $X$ is quasicompact, $X$ is **finite type over $A$**, or a **finite type $A$-scheme**. Note that a scheme locally of finite type over $k$ or $\mathbb{Z}$ (or indeed any Noetherian ring) is locally Noetherian, and similarly a scheme of finite type over any Noetherian ring is Noetherian. As our key "geometric" examples: (i) $\operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]/I$ is a finite-type $\mathbb{C}$-scheme; and (ii) $\mathbb{P}^n_{\mathbb{C}}$ is a finite type $\mathbb{C}$-scheme. (The field $\mathbb{C}$ may be replaced by an arbitrary ring $A$.)

**6.3.6.** *Varieties.* We now make a connection to the classical language of varieties. An affine scheme that is reduced and finite type $k$-scheme is said to be an **affine variety (over $k$)**, or an **affine $k$-variety**. A reduced (quasi-)projective $k$-scheme is a **(quasi-)projective variety (over $k$)**, or an **(quasi-)projective $k$-variety**. (Warning: in the literature, it is sometimes also assumed in the definition of variety that the scheme is irreducible, or that $k$ is algebraically closed.) We will not define varieties in general until §11.1.7; we will need the notion of separatedness first, to exclude abominations like the line with the doubled origin (Example 5.4.5).

**6.3.E.** EXERCISE. Show that a point of a locally finite type $k$-scheme is a closed point if and only if the residue field of the stalk of the structure sheaf at that point is a finite extension of $k$. (Hint: the Nullstellensatz 4.2.3.) Show that the closed points are dense on such a scheme (even though they needn't be quasicompact, cf. Exercise 6.1.E. (For another exercise on closed points, see 6.1.E.)

**6.3.7.** *Definition.* The **degree** of a closed point of a locally finite type $k$-scheme is the degree of this field extension. For example, in $\mathbb{A}^1_k = \operatorname{Spec} k[t]$, the point $[k[t]/p(t)]$ ($p$ irreducible) is $\deg p$. If $k$ is algebraically closed, the degree of every closed point is $1$.

**6.3.8.** *Proof of Proposition 6.3.3.* (a) (i) If $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of $A_f$, then we can verify that $J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of $A$, where

$$J_j = \{r \in A \ : \ r \in I_j\}$$

where $r \in I_j$ means "the image in $A_f$ lies in $I_j$". (We think of this as $I_j \cap A$, except in general $A$ needn't inject into $A_{f_i}$.) Clearly $J_j$ is an ideal of $A$. If $x/f^n \in I_{j+1} \setminus I_j$ where $x \in A$, then $x \in J_{j+1}$, and $x \notin J_j$ (or else $x(1/f)^n \in J_j$ as well). (ii) Suppose $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of $A$. Then for each $1 \leq i \leq n$,

$$I_{i,1} \subset I_{i,2} \subset I_{i,3} \subset \cdots$$

is an increasing chain of ideals in $A_{f_i}$, where $I_{i,j} = I_j \otimes_A A_{f_i}$. It remains to show that for each $j$, $I_{i,j} \subsetneq I_{i,j+1}$ for some $i$; the result will then follow.

**6.3.F.** EXERCISE. Finish this argument.

**6.3.G.** EXERCISE. Prove (b).

(c) (i) is clear: if $A$ is generated over $B$ by $r_1, \dots, r_n$, then $A_f$ is generated over $B$ by $r_1, \dots, r_n, 1/f$.

(ii) Here is the idea. We have generators of $A_i$: $r_{ij}/f_i^j$, where $r_{ij} \in A$. I claim that $\{r_{ij}\}_{ij} \cup \{f_i\}_i$ generate $A$ as a $B$-algebra. Here's why. Suppose you have any $r \in A$. Then in $A_{f_i}$, we can write $r$ as some polynomial in the $r_{ij}$'s and $f_i$, divided by some huge power of $f_i$. So "in each $A_{f_i}$, we have described $r$ in the desired way", except for this annoying denominator. Now use a partition of unity type argument as in the proof of Theorem 5.1.2 to combine all of these into a single expression, killing the denominator. Show that the resulting expression you build still agrees with $r$ in each of the $A_{f_i}$. Thus it is indeed $r$.

**6.3.H.** EXERCISE. Make this argument precise.

This concludes the proof of Proposition 6.3.3 □

**6.3.I.** EASY EXERCISE. Show that $\operatorname{Proj} S_\bullet$ is finite type over $A = S_0$. If $S_0$ is a Noetherian ring, show that $\operatorname{Proj} S_\bullet$ is a Noetherian scheme, and hence that $\operatorname{Proj} S_\bullet$ has a finite number of irreducible components. Show that any quasiprojective scheme is locally of finite type over $A$. If $A$ is Noetherian, show that any quasiprojective $A$-scheme is quasicompact, and hence of finite type over $A$. Show this need not be true if $A$ is not Noetherian. Better: give an example of a quasiprojective $A$-scheme that is not quasicompact, necessarily for some non-Noetherian $A$. (Hint: silly example 5.5.6.)

## 6.4 Normality and factoriality

**6.4.1. Normality.**

We can now define a property of schemes that says that they are "not too far from smooth", called *normality*, which will come in very handy. We will see later that "locally Noetherian normal schemes satisfy Hartogs' Lemma" (Algebraic Hartogs' Lemma 12.3.7 for Noetherian normal schemes): functions defined away form a set of codimension 2 extend over that set. (We saw a first glimpse of this in §5.4.2.) As a consequence, rational functions that have no poles (certain sets of codimension one where the function isn't defined) are defined everywhere. We need definitions of dimension and poles to make this precise.

A scheme $X$ is **normal** if all of its stalks $\mathcal{O}_{X,p}$ are normal, i.e. are integral domains, and integrally closed in their fraction fields. (An integral domain $A$ is **integrally closed** if the only zeros in $K(A)$ to any monic polynomial in $A[x]$ must lie in $A$ itself. The basic example is $\mathbb{Z}$.) As reducedness is a stalk-local property (Exercise 6.2.B), normal schemes are reduced.

**6.4.A.** EXERCISE. Show that integrally closed domains behave well under localization: if $A$ is an integrally closed domain, and $S$ is a multiplicative subset, show that $S^{-1}A$ is an integrally closed domain. (Hint: assume that $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in S^{-1}A$ has a root in the fraction field. Turn this into another equation in $A[x]$ that also has a root in the fraction field.)

It is no fun checking normality at every single point of a scheme. Thanks to this exercise, we know that if $A$ is an integrally closed domain, then $\operatorname{Spec} A$ is normal. Also, for quasicompact schemes, normality can be checked at closed points, thanks to this exercise, and the fact that for such schemes, any point is a generization of a closed point (see Exercise 6.1.E).

It is not true that normal schemes are integral. For example, the disjoint union of two normal schemes is normal. Thus $\operatorname{Spec} k \coprod \operatorname{Spec} k \cong \operatorname{Spec}(k \times k) \cong \operatorname{Spec} k[x]/(x(x-1))$ is normal, but its ring of global sections is not a domain.

**6.4.B.** UNIMPORTANT EXERCISE. Show that a Noetherian scheme is normal if and only if it is the finite disjoint union of integral Noetherian normal schemes. (Hint: Exercise 6.2.I.)

We are close to proving a useful result in commutative algebra, so we may as well go all the way.

**6.4.2. Proposition. —** *If $A$ is an integral domain, then the following are equivalent.*

(1) $A$ *integrally closed.*
(2) $A_{\mathfrak{p}}$ *is integrally closed for all prime ideals* $\mathfrak{p} \subset A$.
(3) $A_{\mathfrak{m}}$ *is integrally closed for all maximal ideals* $\mathfrak{m} \subset A$.

*Proof.* Exercise 6.4.A shows that integral closure is preserved by localization, so (1) implies (2). Clearly (2) implies (3).

It remains to show that (3) implies (1). This argument involves a pretty construction that we will use again. Suppose $A$ is not integrally closed. We show that there is some $\mathfrak{m}$ such that $A_{\mathfrak{m}}$ is also not integrally closed. Suppose

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{6.4.2.1}$$

(with $a_i \in A$) has a solution $s$ in $K(A) \setminus A$. Let I be the **ideal of denominators of** $s$:

$$I := \{r \in A \ : \ rs \in A\}.$$

(Note that I is clearly an ideal of $A$.) Now $I \neq A$, as $1 \notin I$. Thus there is some maximal ideal $\mathfrak{m}$ containing I. Then $s \notin A_{\mathfrak{m}}$, so equation (6.4.2.1) in $A_{\mathfrak{m}}[x]$ shows that $A_{\mathfrak{m}}$ is not integrally closed as well, as desired. □

**6.4.C.** UNIMPORTANT EXERCISE. If $A$ is an integral domain, show that $A = \cap A_{\mathfrak{m}}$, where the intersection runs over all maximal ideals of $A$. (We won't use this exercise, but it gives good practice with the ideal of denominators.)

**6.4.D.** UNIMPORTANT EXERCISE RELATING TO THE IDEAL OF DENOMINATORS. One might naively hope from experience with unique factorization domains that the ideal of denominators is principal. This is not true. As a counterexample, consider our new friend $A = k[a,b,c,d]/(ad - bc)$ (which we last saw in Example 5.4.10, and which we will later recognize as the cone over the quadric surface), and $a/c = b/d \in K(A)$. Show that $I = (c, d)$. We'll soon see that it is not principal (Exercise 13.1.C).

**6.4.3. Factoriality.**

We define a notion which implies normality.

**6.4.4. Definition.** If all the stalks of a scheme $X$ are unique factorization domains, we say that $X$ is **factorial**.

**6.4.E.** EXERCISE. Show that any localization of a unique factorization domain is a unique factorization domain.

Thus if $A$ is a unique factorization domain, then $\operatorname{Spec} A$ is factorial. (The converse need not hold. This property is *not* affine-local, see Exercise 6.4.K. In fact, we will see that elliptic curves are factorial, yet *no* affine open set is the Spec of a unique factorization domain, §21.10.1.) Hence it suffices to check factoriality by finding an appropriate affine cover.

**6.4.5.** ⋆⋆ *How to check if a ring is a unique factorization domain.* We note here that there are very few means of checking that a Noetherian domain is a unique factorization domain. Some useful ones are: (0) elementary means (rings with a euclidean algorithm such as $\mathbb{Z}$, $k[t]$, and $\mathbb{Z}[i]$; polynomial rings over a unique factorization domain, by Gauss' Lemma). (1) Exercise 6.4.E, that the localization of a unique factorization domain is also a unique factorization domain. (2) height 1 primes are principal (Proposition 12.3.9). (3) Nagata's Lemma (Exercise 15.2.S). (4) normal and $\mathrm{Cl} = 0$ (Exercise 15.2.Q).

One of the reasons we like factoriality is that it implies normality.

**6.4.F.** IMPORTANT EXERCISE. Show that unique factorization domains are integrally closed. Hence factorial schemes are normal, and if $A$ is a unique factorization domain, then $\operatorname{Spec} A$ is normal. (However, rings can be integrally closed without being unique factorization domains, as we will see in Exercise 13.1.D. An example without proof: Exercise 6.4.K.)

**6.4.G.** EASY EXERCISE. Show that the following schemes are normal: $\mathbb{A}^n_k$, $\mathbb{P}^n_k$, $\operatorname{Spec} \mathbb{Z}$. (As usual, $k$ is assumed to be a field.)

**6.4.H.** HANDY EXERCISE (YIELDING A NUMBER OF ENLIGHTENING EXAMPLES LATER). Suppose $A$ is a unique factorization domain with 2 invertible, $f \in A$ has no repeated prime factors, and $z^2 - f$ is irreducible in $A[z]$. Show that $\operatorname{Spec} A[z]/(z^2 - f)$ is normal. Show that if $f$ is *not* square-free, then $\operatorname{Spec} A[z]/(z^2 - f)$ is *not* normal. (Hint: $B := A[z]/(z^2 - f)$ is a domain, as $(z^2 - f)$ is prime in $A[z]$. Suppose we have monic $F(T) \in B[T]$ so that $F(T) = 0$ has a root $\alpha$ in $K(B)$. Then by replacing $F(T)$ by $\overline{F}(T)F(T)$, we can assume $F(T) \in A[T]$. Also, $\alpha = g + hz$ where $g, h \in K(A)$. Now $\alpha$ is the root of $Q(T) = 0$ for monic $Q(T) = T^2 - 2gT + (g^2 - h^2 f)T \in K(A)[T]$, so we can factor $F(T) = P(T)Q(T)$ in $K(A)[T]$. By Gauss' lemma, $2g$, $g^2 - h^2 f \in A$. Say $g = r/2$, $h = s/t$ ($s$ and $t$ have no common factors, $r, s, t \in A$). Then $g^2 - h^2 f = (r^2 t^2 - 4s^2 f)/4t^2$. Then $t$ is a unit, and $r$ is even.)

**6.4.I.** EXERCISE. Show that the following schemes are normal:

(a) $\operatorname{Spec} \mathbb{Z}[x]/(x^2 - n)$ where $n$ is a square-free integer congruent to $3 \pmod 4$;
(b) $\operatorname{Spec} k[x_1, \dots, x_n]/(x_1^2 + x_2^2 + \cdots + x_m^2)$ where $\operatorname{char} k \neq 2$, $m \geq 3$;
(c) $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ where $\operatorname{char} k \neq 2$ and $k$ is algebraically closed. This is our cone over a quadric surface example from Exercises 5.4.10 and 6.4.D. (Hint: the side remark below may help.)

**6.4.6.** *Side remark: diagonalizing quadrics.* Suppose $k$ is an algebraically closed field of characteristic not 2. Then any quadratic form in $n$ variables can be "diagonalized" by changing coordinates to be a sum of squares (e.g. $uw - v^2 = ((u+v)/2)^2 + (i(u-v)/2)^2 + (iv)^2)$, and the number of such squares (the **rank** of the quadratic form) is invariant of the change of coordinates. (Reason: write the quadratic form on $x_1, \dots, x_n$ as

$$\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where $M$ is a symmetric matrix — here you are using characteristic $\neq 2$. Then diagonalize $M$ — here you are using algebraic closure.)

**6.4.J.** EXERCISE. Suppose $A$ is a $k$-algebra where $\operatorname{char} k = 0$, and $l/k$ is a finite field extension. Show that if $A \otimes_k l$ is normal (and in particular a domain) then $A$ is normal. (This is a case of a more general fact, and stated correctly, the converse is true.) Show that $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ is normal if $k$ has characteristic 0. Possible hint: reduce to the case where $l/k$ is Galois.

**6.4.K.** EXERCISE (FACTORIALITY IS NOT AFFINE-LOCAL). Let $A = (\mathbb{Q}[x, y]_{x^2+y^2})_0$ denote the homogeneous degree 0 part of the ring $\mathbb{Q}[x, y]_{x^2+y^2}$. In other words, it consists of quotients $f(x,y)/(x^2+y^2)^n$, where $f$ has pure degree $2n$. Show that the distinguished open sets $D(\frac{x^2}{x^2+y^2})$ and $D(\frac{y^2}{x^2+y^2})$ cover $\operatorname{Spec} A$. (Hint: the sum of those two fractions is 1.) Show that $A_{\frac{x^2}{x^2+y^2}}$ and $A_{\frac{y^2}{x^2+y^2}}$ are unique factorization domains. (Hint for the first: show that each ring is isomorphic to $\mathbb{Q}[t]_{t^2+1}$, where $t = y/x$; this is a localization of the unique factorization domain $\mathbb{Q}[t]$.) Finally, show that $A$ is not a unique factorization domain. Possible hint:

$$\left(\frac{xy}{x^2+y^2}\right)^2 = -\left(\frac{y^2}{x^2+y^2}\right)^2.$$

(This example didn't come out of thin air; we will see $\operatorname{Spec} A$ later as an example of a scheme with Picard group — or class group — $\mathbb{Z}/2$.)

## 6.5 Associated points of (locally Noetherian) schemes, and drawing fuzzy pictures

(This important topic won't be used in an essential way for some time, certainly until we talk about dimension in Chapter 12, so it may be best skipped on a first reading. Better: read this section considering only the case where $A$ is an integral domain, or possibly a reduced Noetherian ring, thereby bypassing some of the annoyances. Then you will at least be comfortable with the notion of a rational function in these situations.)

Recall from just after Definition 6.2.1 (of *reduced*) our "fuzzy" pictures of the non-reduced scheme $\operatorname{Spec} k[x,y]/(y^2, xy)$ (see Figure 6.1). When this picture was introduced, we mentioned that the "fuzz" at the origin indicated that the non-reduced behavior was concentrated there. This was verified in Exercise 6.2.A, and indeed the origin is the only point where the stalk of the structure sheaf is non-reduced.

You might imagine that in a bigger scheme, we might have different closed subsets with different amount of "non-reducedness". This intuition will be made precise in this section. We will define *associated points* of a scheme, which will be the most important points of a scheme, encapsulating much of the interesting behavior of the structure sheaf. For example, in Figure 6.1, the associated points are the generic point of the $x$-axis, and the origin (where "the nonreducedness lives").

The primes corresponding to the associated points of an affine scheme $\operatorname{Spec} A$ will be called *associated primes of* $A$. In fact this is backwards; we will define associated primes first, and then define associated points.

The properties about associated points that it will be most important to remember are as follows. Frankly, it is much more important to remember these facts than it is to remember their proofs. But we will, of course, prove these statements.

**(0)** They will exist for any locally Noetherian scheme, and for integral schemes. There are a finite number in any affine open set (and hence in any quasicompact open set). This will come for free.

**(1)** *The generic points of the irreducible components are associated points.* The other associated points are called **embedded points**. Thus in Figure 6.1, the origin is the only embedded point.

**(2)** *If $X$ is reduced, then $X$ has no embedded points.* (This jibes with the intuition of the picture of associated points described earlier.) It follows from (1) and (2) that if $X$ is integral (i.e. irreducible and reduced, Exercise 6.2.E), then the generic point is the only associated point.

**(3)** Recall that one nice property of integral schemes $X$ (such as irreducible affine varieties) not shared by all schemes is that for any non-empty open $U \subset X$, the natural map $\Gamma(U, \mathcal{O}_X) \to K(X)$ is an inclusion (Exercise 6.2.H). Thus all sections over any non-empty open set, and stalks, can be thought of as lying in a single field $K(X)$, which is the stalk at the generic point.

More generally, *if $X$ is a locally Noetherian scheme, then for any $U \subset X$, the natural map*

$$\Gamma(U, \mathcal{O}_X) \to \prod_{\textit{associated } p \textit{ in } U} \mathcal{O}_{X,p} \tag{6.5.0.1}$$

*is an injection*.

We define a **rational function** on a scheme with associated points to be an element of the image of $\Gamma(U, \mathcal{O}_U)$ in (6.5.0.1) for some $U$ containing all the associated points. Equivalently, it is the colimit of $\mathcal{O}_X(U)$ over all open sets containing the associated points. Thus if $X$ is integral, the rational functions are the elements of the stalk at the generic point, and even if there are more than one associated points, it is helpful to think of them in this stalk-like manner. For example, in Figure 6.1, we think of $\frac{x-2}{(x-1)(x-3)}$ as a rational function, but not $\frac{x-2}{x(x-1)}$. The rational functions form a ring, called the **total fraction ring** of $X$, denoted $Q(X)$. If $X = \operatorname{Spec} A$ is affine, then this ring is called the **total fraction ring** of $A$, $Q(A)$. If $X$ is integral, this is the function field $K(X)$, so this extends our earlier Definition 6.2.G of $K(\cdot)$. It can be more conveniently interpreted as follows, using the injectivity of (6.5.0.1). A rational function is a function defined on an open set containing all associated points, i.e. an ordered pair $(U, f)$, where $U$ is an open set containing all associated points, and $f \in \Gamma(U, \mathcal{O}_X)$. Two such data $(U, f)$ and $(U', f')$ define the same open rational function if and only if the restrictions of $f$ and $f'$ to $U \cap U'$ are the same. If $X$ is reduced, this is the same as requiring that they are defined on an open set of each of the irreducible components. A rational function has a maximal **domain of definition**, because any two actual functions on an open set (i.e. sections of the structure sheaf over that open set) that agree as "rational functions" (i.e. on small enough open sets containing associated points) must be the same function, by the injectivity of (6.5.0.1). We say that a rational function $f$ is **regular** at a point $p$ if $p$ is contained in this maximal domain of definition (or equivalently, if there is some open set containing $p$ where $f$ is defined). For example, in Figure 6.1, the rational function $\frac{x-2}{(x-1)(x-3)}$ has domain of definition everything but 1 and 3 (i.e. $[(x-1)]$ and $[(x-3)]$), and is regular away from those two points.

The previous facts are intimately related to the following one.

**(4)** *A function on $X$ is a zero divisor if and only if it vanishes at an associated point of $X$.*

Motivated by the above four properties, when sketching (locally Noetherian) schemes, we will draw the irreducible components (the closed subsets corresponding to maximal associated points), and then draw "additional fuzz" precisely at the closed subsets corresponding to embedded points. All of our earlier sketches were of this form. (See Figure 6.3.)

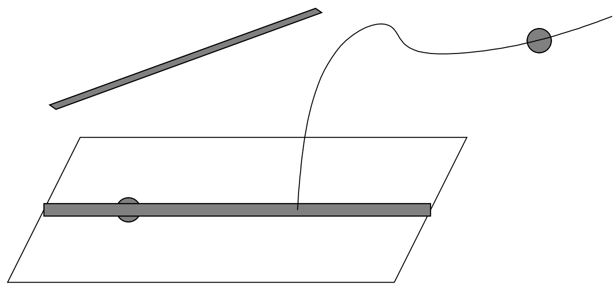


FIGURE 6.3. This scheme has 6 associated points, of which 3 are embedded points. A function is a zero-divisor if it vanishes at one of these six points. It is nilpotent if it vanishes at all six of these points. (In fact, it suffices to vanish at the non-embedded associated points.)

We now finally define associated points, and show that they have the desired properties **(1)** through **(4)**.

We work more generally with modules $M$ over a ring $A$. A prime $\mathfrak{p} \subset A$ is **associated** to $M$ if $\mathfrak{p}$ is the annihilator of an element $m$ of $M$ ($\mathfrak{p} = \{a \in A : am = 0\}$). The set of primes associated to $M$ is denoted $\operatorname{Ass} M$ (or $\operatorname{Ass}_A M$). Awkwardly, if $I$ is an ideal of $A$, the associated primes of the module $A/I$ are said to be the associated prime of $I$. This is not my fault.

**6.5.A.** EASY EXERCISE. Show that $\mathfrak{p}$ is associated to $M$ if and only if $M$ has a submodule isomorphic to $A/\mathfrak{p}$.

**6.5.1. Theorem (properties of associated primes). —** *Suppose $A$ is a Noetherian ring, and $M \neq 0$ is finitely generated.*

(a) *The set $\operatorname{Ass} M$ is finite and nonempty.*
(b) *The natural map $M \to \prod_{\mathfrak{p} \in \operatorname{Ass} M} \prod M_{\mathfrak{p}}$ is an injection.*
(c) *The set is zero-divisors of $M$, union $0$, is $\cup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$.*
(d) *(association commutes with localization) If $S$ is a multiplicative set, then*

$$\operatorname{Ass}_{S^{-1}A} S^{-1}M = \operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A$$

*($= \{\mathfrak{p} \in \operatorname{Ass}_A M : \mathfrak{p} \cap S = \varnothing\}$).*
(e) *The set $\operatorname{Ass} M$ contains the primes minimal among those containing $\operatorname{ann} M := \{a \in A : aM = 0\}$.*

We define the **associated points** of a locally Noetherian scheme $X$ to be those points $p \in X$ such that, on any affine open set $\operatorname{Spec} A$ containing $p$, $p$ corresponds to an associated prime of $A$. This notion is independent of choice of affine neighborhood $\operatorname{Spec} A$: if $p$ has two affine open neighborhoods $\operatorname{Spec} A$ and $\operatorname{Spec} B$ (say corresponding to primes $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$ respectively), then $p$ corresponds to an associated prime of $A$ if and only if it corresponds to an associated prime of $A_{\mathfrak{p}} = \mathcal{O}_{X,p} = B_{\mathfrak{q}}$ if and only if it corresponds to an associated prime of $B$, by Theorem 6.5.1(d).

**6.5.B.** IMPORTANT EXERCISE. Show how Theorem 6.5.1 implies properties **(0)–(4)**. (By **(3)**, I mean the injectivity of (6.5.0.1). The trickiest is probably **(2)**.)

We now prove Theorem 6.5.1.

**6.5.C.** EXERCISE. Suppose $M \neq 0$ is an $A$-module. Show that if $I \subset A$ is maximal among all ideals that are annihilators of elements of $M$, then $I$ is prime, and hence $I \in \operatorname{Ass} M$. Thus if $A$ is Noetherian, then $\operatorname{Ass} M$ is nonempty (part of Theorem 6.5.1(a)).

**6.5.D.** EXERCISE. Suppose that $M$ is a module over a Noetherian ring $A$. Show that $m = 0$ if and only if $m$ is $0$ in $M_{\mathfrak{p}}$ for each of the maximal associated primes of $M$. (Hint: use the previous exercise.)

This immediately implies Theorem 6.5.1(b). It also implies Theorem 6.5.1(c): Any nonzero element of $\cup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$ is clearly a zero-divisor. Conversely, if $a$ annihilates a nonzero element of $M$, then $r$ is contained in a maximal annihilator ideal.

**6.5.E.** EXERCISE. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $A$-modules, show that

$$\operatorname{Ass} M' \subset \operatorname{Ass} M \subset \operatorname{Ass} M' \cup \operatorname{Ass} M''.$$

(Possible hint for the second containment: if $m \in M$ has annihilator $\mathfrak{p}$, then $Am = A/\mathfrak{p}$, cf. Exercise 6.5.A.)

**6.5.F.** EXERCISE. If $M$ is a finitely generated module over Noetherian $A$, show that $M$ has a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

where $M_{i+1}/M_i \cong R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. Show that the associated primes are among the $\mathfrak{p}_i$, and thus prove Theorem 6.5.1(a).

**6.5.G.** EXERCISE. Prove Theorem 6.5.1(d) as follows.
(a) Show that $\operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A \subset \operatorname{Ass}_{S^{-1}A} S^{-1}M$. (Hint: suppose $\mathfrak{p} \in \operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A$, with $\mathfrak{p} = \operatorname{ann} m$ for $m \in M$.)
(b) Suppose $\mathfrak{q} \in \operatorname{Ass}_{S^{-1}A} S^{-1}M$, which corresponds to $\mathfrak{p} \in A$ (i.e. $\mathfrak{q} = \mathfrak{p}(S^{-1}A)$). Then $\mathfrak{q} = \operatorname{ann}_{S^{-1}A} m$ ($m \in S^{-1}M$), which yields a nonzero element of

$$\operatorname{Hom}_{S^{-1}A}(S^{-1}A/\mathfrak{q}, S^{-1}M).$$

Argue that this group is isomorphic to $S^{-1}\operatorname{Hom}_A(A/\mathfrak{p}, M)$ (see Exercise 2.6.G), and hence $\operatorname{Hom}_A(A/\mathfrak{p}, M) \neq 0$.

**6.5.H.** EXERCISE. Prove Theorem 6.5.1(e) as follows. If $\mathfrak{p}$ is minimal over $\operatorname{ann} M$, localize at $\mathfrak{p}$, so that $\mathfrak{p}$ is the *only* prime containing $\operatorname{ann} M$. Use Theorem 6.5.1(d).

**6.5.2.** *Aside: Primary ideals.* The notion of primary ideals is important, although we won't use it. (An ideal $I \subset A$ in a ring is **primary** if $I \neq A$ and if $xy \in I$ implies either $x \in I$ or $y^n \in I$ for some $n > 0$.) The associated primes of an ideal turn out to be precisely those primes appearing in its primary decomposition. See [**E**, §3.3], for example, for more on this topic.

# Part III

# Morphisms of schemes

CHAPTER 7

# Morphisms of schemes

## 7.1 Introduction

We now describe the morphisms between schemes. We will define some easy-to-state properties of morphisms, but leave more subtle properties for later.

Recall that a scheme is (i) a set, (ii) with a topology, (iii) and a (structure) sheaf of rings, and that it is sometimes helpful to think of the definition as having three steps. In the same way, the notion of morphism of schemes $X \to Y$ may be defined (i) as a map of sets, (ii) that is continuous, and (iii) with some further information involving the sheaves of functions. In the case of affine schemes, we have already seen the map as sets (§4.2.7) and later saw that this map is continuous (Exercise 4.4.G).

Here are two motivations for how morphisms should behave. The first is algebraic, and the second is geometric.

*(a) Algebraic motivation.* We'll want morphisms of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ to be precisely the ring maps $A \to B$. We have already seen that ring maps $A \to B$ induce maps of topological spaces in the opposite direction (Exercise 4.4.G); the main new ingredient will be to see how to add the structure sheaf of functions into the mix. Then a morphism of schemes should be something that "on the level of affines, looks like this".

*(b) Geometric motivation.* Motivated by the theory of differentiable manifolds (§4.1.1), which like schemes are ringed spaces, we want morphisms of schemes at the very least to be morphisms of ringed spaces; we now describe what these are. Notice that if $\pi : X \to Y$ is a map of differentiable manifolds, then a differentiable function on $Y$ pulls back to a differentiable function on $X$. More precisely, given an open subset $U \subset Y$, there is a natural map $\Gamma(U, \mathcal{O}_Y) \to \Gamma(\pi^{-1}(U), \mathcal{O}_X)$. This behaves well with respect to restriction (restricting a function to a smaller open set and pulling back yields the same result as pulling back and then restricting), so in fact we have a map of sheaves on $Y$: $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. Similarly a morphism of schemes $X \to Y$ should induce a map $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. But in fact in the category of differentiable manifolds a continuous map $X \to Y$ is a map of differentiable manifolds precisely when differentiable functions on $Y$ pull back to differentiable functions on $X$ (i.e. the pullback map from differentiable functions on $Y$ to *functions* on $X$ in fact lies in the subset of *differentiable functions*, i.e. the continuous map $X \to Y$ induces a pullback of differential functions $\mathcal{O}_Y \to \mathcal{O}_X$), so this map of sheaves *characterizes* morphisms in the differentiable category. So we could use this as the *definition* of morphism in the differentiable category.

But how do we apply this to the category of schemes? In the category of differentiable manifolds, a continuous map $X \to Y$ *induces* a pullback of (the sheaf of) functions, and we can ask when this induces a pullback of *differentiable* functions. However, functions are odder on schemes, and we can't recover the pullback map just from the map of topological spaces. A reasonable patch is to hardwire this into the definition of morphism, i.e. to have a continuous map $f : X \to Y$, along with a pullback map $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$. This leads to the definition of the *category* of ringed spaces.

One might hope to define morphisms of schemes as morphisms of ringed spaces. This isn't quite right, as then motivation (a) isn't satisfied: as desired, to each morphism $A \to B$ there is a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, but there can be additional morphisms of ringed spaces $\operatorname{Spec} B \to \operatorname{Spec} A$ not arising in this way (see Exercise 7.2.E). A revised definition as morphisms of ringed spaces that locally looks of this form will work, but this is awkward to work with, and we take a different approach. However, we will check that our eventual definition actually is equivalent to this (Exercise 7.3.C).

We begin by formally defining morphisms of ringed spaces.

## 7.2 Morphisms of ringed spaces

**7.2.1. Definition.** A **morphism** $\pi : X \to Y$ **of ringed spaces** is a continuous map of topological spaces (which we unfortunately also call $\pi$) along with a "pullback map" $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. By adjointness (§3.6.1), this is the same as a map $\pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. There is an obvious notion of composition of morphisms, so ringed spaces form a category. Hence we have notion of automorphisms and isomorphisms. You can easily verify that an isomorphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a homeomorphism $f : X \to Y$ along with an isomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ (or equivalently $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$).

If $U \subset Y$ is an open subset, then there is a natural morphism of ringed spaces $(U, \mathcal{O}_Y|_U) \to (Y, \mathcal{O}_Y)$. (Check this! The $f^{-1}$ interpretation is cleaner to use here.) More precisely, if $U \to Y$ is an isomorphism of $U$ with an open subset $V$ of $Y$, and we are given an isomorphism $(U, \mathcal{O}_U) \cong (V, \mathcal{O}_Y|_V)$ (via the isomorphism $U \cong V$), then the resulting map of ringed spaces is called an **open immersion** of ringed spaces.

**7.2.A.** EXERCISE (MORPHISMS OF RINGED SPACES GLUE). Suppose $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces, $X = \cup_i U_i$ is an open cover of $X$, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (Exercise 3.2.F essentially showed this for topological spaces.)

**7.2.B.** EASY IMPORTANT EXERCISE: $\mathcal{O}$-MODULES PUSH FORWARD. Given a morphism of ringed spaces $f : X \to Y$, show that sheaf pushforward induces a functor $\mathit{Mod}_{\mathcal{O}_X} \to \mathit{Mod}_{\mathcal{O}_Y}$.

**7.2.C.** EASY IMPORTANT EXERCISE. Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

# Bibliography


[ACGH] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of Algebraic Curves I*, Springer-Verlag, New York, 1985.

[AM] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publ. Co., Reading, Mass.-London-Don Mils, On. 1969.

[DF] D. Dummit and R. Foote *Abstract Algebra*, third ed., Wiley and Sons, Inc., 2004.

[E] D. Eisenbud, *Commutative Algebra with a View to Algebraic Geometry*, GTM 150, Springer-Verlag, New York, 1995.

[EH] D. Eisenbud and J. Harris, *The Geometry of Schemes*, GTM 197, Springer-Verlag, New York, 2000.

[EGA] A. Grothendieck and J. Dieudonné, *Éléments de Géometrie Algébrique*, Publ. Math. IHES, Part I: 4 (1960); Part II: 8 (1961); Part III: 11 (1961), 17 (1963); Part IV: 20 (1964), 24 (1965), 28 (1966), 32 (1967).

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[Gr] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** 1957, 119–221.

[H] R. Hartshorne, *Algebraic Geometry*, GTM 52, Springer-Verlag, New York-Heidelberg, 1977.

[Kl] S. Kleiman, *The transversality of a general translate*, Compositio Math. **28**, no. 3 (1974), p. 287–297.

[KS] M. Kashiwara and P. Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften 332, Springer-Verlag, Berlin, 2006.

[L] S. Lang, *Algebra* (rev. 3rd ed.), GTM 211, Springer-Verlag, New York, 2002.

[M-CA] H. Matsumura, *Commutative Algebra*.

[M-CRT] H. Matsumura, *Commutative Ring Theory*, M. Reid trans., 2nd ed, Cambridge Studies in Advanced Mathematics 8, Cambridge U.P., Cambridge, 1989.

[M-Red] D. Mumford, *The Red Book of Varieties and Schemes* (first ed.), LNM 1358, Springer-Verlag, Berlin, 1988.

[M-CAS] D. Mumford, *Lectures on Curves on an Algebraic Surface*, Princeton U.P., Princeton NJ, 1966.

[MAV] D. Mumford, *Abelian Varieties*, TIFR Studies in Math. 5, Oxford U.P., London, 1970.

[N] These Notes.

[S-FAC] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278.

[S-GAGA] J.-P. Serre, GAGA.

[SGA2] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux (SGA2)*, North-Holland, Amsterdam, 1968.

[Stacks] The Stacks Project Authors, *Stacks Project*, `http://math.columbia.edu/algebraic_geometry/stacks-git`.

[W] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics 38, Cambridge U.P., Cambridge, 1994.

# Index