8.4 Images of morphisms: Chevalley's theorem and elimination theory

In this section, we will answer a question that you may have wondered about long before hearing the phrase "algebraic geometry". If you have a number of polynomial equations in a number of variables with indeterminate coefficients, and you wonder if they have a solution, you would reasonably ask what conditions there are on the coefficients for a common solution to exist. Given the algebraic nature of the problem, you might hope that the answer should be purely algebraic in nature — it shouldn't be "random", or involve bizarre functions like exponentials or cosines. This is indeed the case, and was first understood via the nineteenth century theory of "elimination of variables". It can be profitably interpreted as a question about images of maps of varieties or schemes, in which guise it is answered by Chevalley's theorem.

8.4.1. Images of morphisms, and Chevalley's theorem.

If $f : X \to Y$ is a morphism of schemes, the notion of the image of f as *sets* is clear: we just take the points in Y that are the image of points in X. We know that the image can be open (open immersions), and we will soon see that it can be closed (closed immersions), and hence locally closed (locally closed immersions). But it can be weirder still: consider the morphism $\mathbb{A}^2_k \to \mathbb{A}^2_k$ given by $(x, y) \mapsto (x, xy)$. The image is the plane, with the x-axis removed, but the origin put back in. This isn't so horrible. We make a definition to capture this phenomenon. A **constructible subset** of a Noetherian topological space is a subset which belongs to the smallest family of subsets such that (i) every open set is in the family, (ii) a finite intersection of family members is in the family, and (iii) the complement of a family member is also in the family. For example the image of $(x, y) \mapsto (x, xy)$ is constructible.

8.4.A. EXERCISE: CONSTRUCTIBLE SUBSETS IN TERMS OF LOCALLY CLOSED SUBSETS. Suppose X is a topological space. Recall that a subset of X is *locally closed* if it is the intersection of an open subset and a closed subset. (Equivalently, it is an open subset of a closed subset, or a closed subset of an open subset. We will later have trouble extending this to open and closed and locally closed subschemes, see Exercise 9.1.K.) Show that a subset of X is constructible if and only if it is the finite disjoint union of locally closed subsets. As a consequence, if $X \to Y$ is a continuous map of topological spaces, then the preimage of a constructible set is a constructible set.

One nice property of constructible subsets of schemes is that there is a short criterion for openness: a constructible subset is open if it is "closed under generization" (see Exercise 24.2.N).

The image of a morphism of schemes can be stranger than constructible. Indeed if S is *any* subset of a scheme Y, it can be the image of a morphism: let X be the disjoint union of spectra of the residue fields of all the points of S, and let $f : X \to Y$ be the natural map. This is quite pathological, but in any reasonable situation, the image is essentially no worse than arose in the previous example of $(x, y) \mapsto (x, xy)$. This is made precise by Chevalley's theorem.

8.4.2. Chevalley's theorem. — *If $\pi : X \to Y$ is a finite type morphism of Noetherian schemes, the image of any constructible set is constructible. In particular, the image of π is constructible.*

To prove Chevalley's theorem, we prove the following classical result.

8.4.3. Theorem (Fundamental Theorem of Elimination Theory). — *The morphism $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ is closed (sends closed sets to closed sets).*

Theorem 8.4.3 is simpler than Chevalley's theorem; and there are only equalities, no inequalities. A great deal of classical algebra and geometry is contained in it as special cases. Here are some examples.

First, let $A = k[a, b, c, \dots, i]$, and consider the closed subscheme of $\mathbb{P}^2_A$ (taken with coordinates x, y, z) corresponding to $ax + by + cz = 0$, $dx + ey + fz = 0$, $gx + hy + iz = 0$. Then we are looking for the locus in $\operatorname{Spec} A$ where these equations have a non-trivial solution. This indeed corresponds to a Zariski-closed set — where

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 0.$$

Thus the idea of the determinant is embedded in elimination theory.

As a second example, let $A = k[a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n]$. Now consider the closed subscheme of $\mathbb{P}^1_A$ (taken with coordinates x and y) corresponding to $a_0x^m + a_1x^{m-1}y + \cdots + a_my^m = 0$ and $b_0x^n + b_1x^{m-1}y + \cdots + b_ny^n = 0$. Then there is a polynomial on the coefficients $a_0, \dots, b_n$ (an element of A) which vanishes if and only if these two polynomials have a common non-zero root — this polynomial is called the *resultant*.

More generally, this question boils down to the following question. Given a number of homogeneous equations in $n + 1$ variables with indeterminate coefficients, Proposition 8.4.3 implies that one can write down equations in the coefficients that will precisely determine when the equations have a nontrivial solution.

Proof of the Fundamental Theorem of Elimination Theory 8.4.3. Suppose $Z \hookrightarrow \mathbb{P}^n_A$ is a closed *subset*. We wish to show that $\pi(Z)$ is closed. (See Figure 8.6.)

Suppose $y \notin \pi(Z)$ is a *closed* point of $\operatorname{Spec} A$. We will check that there is a distinguished open neighborhood $D(f)$ of y in $\operatorname{Spec} A$ such that $D(f)$ doesn't meet $\pi(Z)$. (If we could show this for *all* points of $\pi(Z)$, we would be done. But I prefer to concentrate on closed points first for simplicity.) Suppose y corresponds to the maximal ideal $\mathfrak{m}$ of A. We seek $f \in A - \mathfrak{m}$ such that $\pi^* f$ vanishes on Z.

Let $U_0, \dots, U_n$ be the usual affine open cover of $\mathbb{P}^n_A$. The closed subsets $\pi^{-1}y$ and Z do not intersect. On the affine open set U_i, we have two closed subsets $Z \cap U_i$ and $\pi^{-1}y \cap U_i$ that do not intersect, which means that the ideals corresponding to the two closed sets generate the unit ideal, so in the ring of functions $A[x_{0/i}, x_{1/i}, \dots, x_{n/i}]/(x_{i/i} - 1)$ on U_i, we can write

$$1 = a_i + \sum m_{ij} g_{ij}$$

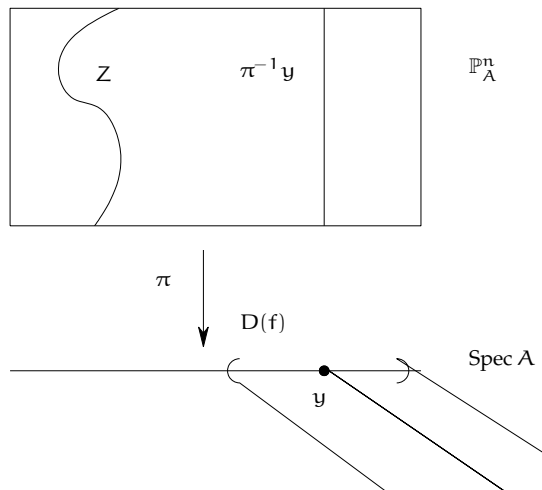

FIGURE 8.6.

where $m_{ij} \in \mathfrak{m}$, and a_i vanishes on Z. Note that $a_i, g_{ij} \in A[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$, so by multiplying by a sufficiently high power x_i^n of x_i, we have an equality

$$x_i^N = a_i' + \sum m_{ij} g_{ij}'$$

in $S_\bullet = A[x_0, \ldots, x_n]$. We may take N large enough so that it works for all i. Thus for N' sufficiently large, we can write any monomial in $x_1, \ldots, x_n$ of degree N' as something vanishing on Z plus a linear combination of elements of $\mathfrak{m}$ times other polynomials. Hence

$$S_{N'} = I(Z)_{N'} + \mathfrak{m} S_{N'}$$

where $I(Z)_\bullet$ is the graded ideal of functions vanishing on Z. By Nakayama's lemma (version 1, Lemma 8.2.8), taking $M = S_{N'}/I(Z)_{N'}$, we see that there exists $f \in A - \mathfrak{m}$ such that

$$f S_{N'} \subset I(Z)_{N'}.$$

Thus we have found our desired f.

We now tackle Theorem 8.4.3 in general, by simply extending the above argument so that y need not be a *closed* point. Suppose $y = [\mathfrak{p}]$ not in the image of Z. Applying the above argument in $\operatorname{Spec} A_\mathfrak{p}$, we find $S_{N'} \otimes A_\mathfrak{p} = I(Z)_{N'} \otimes A_\mathfrak{p} + \mathfrak{m} S_{N'} \otimes A_\mathfrak{p}$, from which $g(S_{N'}/I(Z)_{N'}) \otimes A_\mathfrak{p} = 0$ for some $g \in A_\mathfrak{p} - \mathfrak{p}A_\mathfrak{p}$, from which $(S_{N'}/I(Z)_{N'}) \otimes A_\mathfrak{p} = 0$. As $S_{N'}$ is a finitely generated A-module, there is some $f \in A - \mathfrak{p}$ with $f S_N \subset I(Z)$ (if the module-generators of $S_{N'}$, and $f_1, \ldots, f_a$ are annihilate the generators respectively, then take $f = \prod f_i$), so once again we have found $D(f)$ containing $\mathfrak{p}$, with (the pullback of) f vanishing on Z. □

Notice that projectivity was crucial to the proof: we used graded rings in an essential way.

Proof of Chevalley's Theorem 8.4.2. We begin with a series of reductions.

8.4.B. EXERCISE. (a) Reduce to the case where Y is affine, say $Y = \operatorname{Spec} B$.
(b) Reduce further to the case where X is affine.

(c) Reduce further to the case where $X = \mathbb{A}^n_B = \operatorname{Spec} B[t_1, \dots, t_n]$.
(d) By induction on n, reduce further to the case where $X = \mathbb{A}^1_B = \operatorname{Spec} B[t]$.
(e) Reduce to showing that for any Noetherian ring B, and any locally closed subset $Z \subset \mathbb{A}^1_B$, the image of Z under the projection $\pi : \mathbb{A}^1_B \to \operatorname{Spec} B$ is constructible.
(f) Reduce to showing that for any Noetherian ring B, and any *irreducible closed* subset $Z \subset \mathbb{A}^1_B$, the image of Z under the projection $\pi : \mathbb{A}^1_B \to \operatorname{Spec} B$ is constructible.

Now Z is cut out (in $\mathbb{A}^1_B$) by (finitely many) polynomials $f_1, \dots, f_m \in B[t]$. Let F be the subset of $\operatorname{Spec} B$ consisting of points of $\operatorname{Spec} B$ where Z contains the entire fiber above such points.

8.4.C. EXERCISE. If $a_1, \dots, a_M \in B$ are the coefficients of the polynomials $f_1, \dots, f_m$, show that $F = V(a_1, \dots, a_M)$, and hence that F is closed.

8.4.D. EXERCISE. By covering $\operatorname{Spec} B \setminus F$ with finitely many affines ($\operatorname{Spec} B \setminus F$ is quasicompact by Exercise 4.6.M), show that it suffices to prove the statement given in Exercise 8.4.B(f) in the case where Z contains *no* fibers of π.

We now consider $\mathbb{A}^1_B$ as a standard open subset of $\mathbb{P}^1_B$, with complement H ("the hyperplane at infinity"). We still denote the structure morphism $\mathbb{P}^1_B \to \operatorname{Spec} B$ by π. Let $\overline{Z}$ be the closure of Z in $\mathbb{P}^1_B$, and let $Z_\infty = \overline{Z} \setminus Z$. By the Fundamental Theorem of Elimination Theory 8.4.3, $\pi(Z_\infty)$ and $\pi(\overline{Z})$ are both closed in $\operatorname{Spec} B$.

8.4.E. EXERCISE. Show that $\pi(\overline{Z})$ is irreducible.

8.4.F. SUBTLE EXERCISE. Show that $\pi(Z_\infty)$ is *strictly* contained in $\pi(\overline{Z})$. Hint: show that the fiber above the generic point of $\pi(\overline{Z})$ cannot meet Z_∞.

Using Noetherian induction on B, we may assume the desired result over any proper closed subset of $\operatorname{Spec} B$, so applying this to $\pi(Z_\infty)$, we see that $\pi(Z) \cap \pi(Z_\infty)$ is constructible. But also $\pi(Z) \setminus \pi(Z_\infty) = \pi(\overline{Z}) \setminus \pi(Z_\infty)$ is also constructible (it is the difference of two closed sets), so we are done. □

8.4.G. ⋆⋆ EXERCISE (CHEVALLEY'S THEOREM FOR LOCALLY FINITELY PRESENTED MORPHISMS). If you are macho and are embarrassed by Noetherian rings, the following extension of Chevalley's theorem will give you a sense of one of the standard ways of removing Noetherian hypotheses.
(a) Suppose that A is a finitely presented B-algebra, so $A = B[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Show that the image of $\operatorname{Spec} A \to \operatorname{Spec} B$ is a constructible subset of $\operatorname{Spec} B$. Hint: describe $\operatorname{Spec} A \to \operatorname{Spec} B$ as the base change of

$$\operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n, a_1, \dots, a_N]/(g_1, \dots, g_n) \to \operatorname{Spec} \mathbb{Z}[a_1, \dots, a_N],$$

where the images of a_i in $\operatorname{Spec} B$ are the coefficients of the f_j (there is one a_i for each coefficient of each f_j), and $g_i \mapsto f_i$.
(b) Show that if $\pi : X \to Y$ is a quasicompact locally finitely presented morphism, and Y is quasicompact, then $\pi(X)$ is constructible. (For hardened experts only: [**EGA**, 0_{III}.9.1] gives a definition of constructability, and local constructability, in more generality. The general form of Chevalley's constructibility theorem [**EGA**,

IV$_1$.1.8.4] is that the image of a locally constructible set, under a finitely presented map, is also locally constructible.)

8.4.4. $\star$ **Elimination of quantifiers.** A basic sort of question that arises in any number of contexts is when a system of equations has a solution. Suppose for example you have some polynomials in variables $x_1, \dots, x_n$ over an algebraically closed field $\overline{k}$, some of which you to set to be zero, and some of which you set to be nonzero. (This question is of fundamental interest even before you know any scheme theory!) Then there is an algebraic condition on the coefficients which will tell you if this is the case. Define the Zariski topology on $\overline{k}^n$ in the obvious way: closed subsets are cut out by equations.

8.4.H. EXERCISE (ELIMINATION OF QUANTIFIERS, OVER AN ALGEBRAICALLY CLOSED FIELD). Fix an algebraically closed field $\overline{k}$. Suppose

$$f_1, \dots, f_p, g_1, \dots, g_q \in \overline{k}[A_1, \dots, A_m, X_1, \dots X_n]$$

are given. Show that there is a Zariski-constructible subset Y of $\overline{k}^m$ such that

$$f_1(a_1, \dots, a_m, X_1, \dots, X_n) = \cdots = f_p(a_1, \dots, a_m, X_1, \dots, X_n) = 0 \tag{8.4.4.1}$$

and

$$g_1(a_1, \dots, a_m, X_1, \dots, X_n) \neq 0 \quad \cdots \quad g_p(a_1, \dots, a_m, X_1, \dots, X_n) \neq 0 \tag{8.4.4.2}$$

has a solution $(X_1, \dots, X_n) = (x_1, \dots, x_n) \in \overline{k}^n$ if and only if $(a_1, \dots, a_m) \in Y$. Hints: if Z is a finite type scheme over $\overline{k}$, and the closed points are denoted Z^{cl} ("cl" is for either "closed" or "classical"), then the inclusion of topological spaces $Z^{\mathrm{cl}} \hookrightarrow Z$, the Zariski topology on Z induces the Zariski topology on Z^{cl}. Note that we can identify $(\mathbb{A}^p_{\overline{k}})^{\mathrm{cl}}$ with $\overline{k}^p$ by the Nullstellensatz (Exercise 6.3.E). If X is the locally closed subset of $\mathbb{A}^{m+n}$ cut out by the equalities and inequalities (8.4.4.1) and (8.4.4.2), we have the diagram

$$\begin{tikzcd}
X^{\mathrm{cl}} \arrow[r, hook] \arrow[d, "\pi^{\mathrm{cl}}"'] & X \arrow[r, hook, "\text{l. cl. imm.}"] \arrow[d, "\pi"] & \mathbb{A}^{m+n} \arrow[dl] \\
\overline{k}^m \arrow[r, hook] & \mathbb{A}^m &
\end{tikzcd}$$

where $Y = \operatorname{im} \pi^{\mathrm{cl}}$. By Chevalley's theorem 8.4.2, $\operatorname{im} \pi$ is constructible, and hence so is $(\operatorname{im} \pi) \cap \overline{k}^m$. It remains to show that $(\operatorname{im} \pi) \cap \overline{k}^m = Y$ $(= \operatorname{im} \pi^{\mathrm{cl}})$. You might use the Nullstellensatz.

This is called "elimination of quantifiers" because it gets rid of the quantifier "there exists a solution". The analogous statement for real numbers, where inequalities are also allowed, is a special case of Tarski's celebrated theorem of elimination of quantifiers for real closed fields.