MATH 216: FOUNDATIONS OF ALGEBRAIC GEOMETRY

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June 27, 2011 draft

Contents

CHAPTER 1

Introduction

I can illustrate the .... approach with the ... image of a nut to be opened. The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you let time pass. The shell becomes more flexible through weeks and months — when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado!

A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration ... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it ... yet finally it surrounds the resistant substance.

— Alexander Grothendieck, Récoltes et Semailles *p. 552-3, translation by Colin McLarty*

1.1 Goals

These are an updated version of notes accompanying a hard year-long class taught at Stanford in 2009-2010. I am currently editing them and adding a few more sections, and I hope to post a reasonably complete (if somewhat rough) version over the 2010-11 academic year at the site `http://math216.wordpress.com/`.

In any class, choices must be made as to what the course is about, and who it is for — there is a finite amount of time, and any addition of material or explanation or philosophy requires a corresponding subtraction. So these notes are highly inappropriate for most people and most classes. Here are my goals. (I do not claim that these goals are achieved; but they motivate the choices made.)

These notes currently have a very particular audience in mind: Stanford Ph.D. students, postdocs and faculty in a variety of fields, who may want to use algebraic geometry in a sophisticated way. This includes algebraic and arithmetic geometers, but also topologists, number theorists, symplectic geometers, and others.

The notes deal purely with the algebraic side of the subject, and completely neglect analytic aspects.

They assume little prior background (see §1.2), and indeed most students have little prior background. Readers with less background will necessarily have to work harder. It would be great if the reader had seen varieties before, but many students haven't, and the course does not assume it — and similarly for category theory, homological algebra, more advanced commutative algebra, differential geometry, ... Surprisingly often, what we need can be developed quickly from scratch. The cost is that the course is much denser; the benefit is that more people can follow it; they don't reach a point where they get thrown. (On the other hand, people who already have some familiarity with algebraic geometry, but want to understand the foundations more completely should not be bored, and will focus on more subtle issues.)

The notes seek to cover everything that one should see in a first course in the subject, including theorems, proofs, and examples.

They seek to be complete, and not leave important results as black boxes pulled from other references.

There are lots of exercises. I have found that unless I have some problems I can think through, ideas don't get fixed in my mind. Some are trivial — that's okay, and even desirable. A very few necessary ones may be hard, but the reader should have the background to deal with them — they are not just an excuse to push material out of the text.

There are optional starred ($\star$) sections of topics worth knowing on a second or third (but not first) reading. You should not read double-starred sections ($\star\star$) unless you really really want to, but you should be aware of their existence.

The notes are intended to be readable, although certainly not easy reading.

In short, after a year of hard work, students should have a broad familiarity with the foundations of the subject, and be ready to attend seminars, and learn more advanced material. They should not just have a vague intuitive understanding of the ideas of the subject; they should know interesting examples, know why they are interesting, and be able to prove interesting facts about them.

I have greatly enjoyed thinking through these notes, and teaching the corresponding classes, in a way I did not expect. I have had the chance to think through the structure of algebraic geometry from scratch, not blindly accepting the choices made by others. (Why do we need this notion? Aha, this forces us to consider this other notion earlier, and now I see why this third notion is so relevant...) I have repeatedly realized that ideas developed in Paris in the 1960's are simpler than I initially believed, once they are suitably digested.

1.1.1. Implications. We will work with as much generality as we need for most readers, and no more. In particular, we try to have hypotheses that are as general as possible without making proofs harder. The right hypotheses can make a proof easier, not harder, because one can remember how they get used. As an inflammatory example, the notion of quasiseparated comes up early and often. The cost is that one extra word has to be remembered, on top of an overwhelming number of other words. But once that is done, it is not hard to remember that essentially every scheme anyone cares about is quasiseparated. Furthermore, whenever the hypotheses "quasicompact and quasiseparated" turn up, the reader will likely immediately see a key idea of the proof.

Similarly, there is no need to work over an algebraically closed field, or even a field. Geometers needn't be afraid of arithmetic examples or of algebraic examples; a central insight of algebraic geometry is that the same formalism applies without change.

1.1.2. Costs. Choosing these priorities requires that others be shortchanged, and it is best to be up front about these. Because of our goal is to be comprehensive, and to understand everything one should know after a first course, it will necessarily take longer to get to interesting sample applications. You may be misled into thinking that one has to work this hard to get to these applications — it is not true!

1.2 Background and conventions

All rings are assumed to be commutative unless explicitly stated otherwise. All rings are assumed to contain a unit, denoted 1. Maps of rings must send 1 to 1. We don't require that $0 \neq 1$; in other words, the "0-ring" (with one element) is a ring. (There is a ring map from any ring to the 0-ring; the 0-ring only maps to itself. The 0-ring is the final object in the category of rings.) The definition of "integral domain" includes $1 \neq 0$, so the 0-ring is not an integral domain. We accept the axiom of choice. In particular, any proper ideal in a ring is contained in a maximal ideal. (The axiom of choice also arises in the argument that the category of A-modules has enough injectives, see Exercise 24.2.F.)

The reader should be familiar with some basic notions in commutative ring theory, in particular the notion of ideals (including prime and maximal ideals) and localization. For example, the reader should be able to show that if S is a multiplicative set of a ring A (which we assume to contain 1), then the primes of $S^{-1}A$ are in natural bijection with those primes of A not meeting S (§4.2.6). Tensor products and exact sequences of A-modules will be important. We will use the notation $(A, \mathfrak{m})$ or $(A, \mathfrak{m}, k)$ for local rings (rings with a unique maximal ideal) — A is the ring, $\mathfrak{m}$ its maximal ideal, and $k = A/\mathfrak{m}$ its residue field. We will use (in Proposition 14.7.3) the structure theorem for finitely generated modules over a principal ideal domain A: any such module can be written as the direct sum of principal modules $A/(a)$.

Algebra is the offer made by the devil to the mathematician ... All you need to do is give me your soul: give up geometry.
— *Michael Atiyah*

1.2.1. *Caution about on foundational issues.* We will not concern ourselves with subtle foundational issues (set-theoretic issues, universes, etc.). It is true that some people should be careful about these issues. But is that really how you want to spend your life? (If you are one of these rare people, a good start is [**KS**, §1.1].)

1.2.2. *Further background.* It may be helpful to have books on other subjects handy that you can dip into for specific facts, rather than reading them in advance. In commutative algebra, Eisenbud [**E**] is good for this. Other popular choices are Atiyah-Macdonald [**AM**] and Matsumura [**M-CRT**]. For homological algebra, Weibel [**W**] is simultaneously detailed and readable.

Background from other parts of mathematics (topology, geometry, complex analysis) will of course be helpful for developing intuition.

Finally, it may help to keep the following quote in mind.

[Algebraic geometry] seems to have acquired the reputation of being esoteric, exclusive, and very abstract, with adherents who are secretly plotting to take over all the rest of mathematics! In one respect this last point is accurate ...
— *David Mumford, 1975* [**M-Red2**, p. 227]

Part I

Preliminaries

CHAPTER 2

Some category theory

That which does not kill me, makes me stronger. — Nietzsche

2.1 Motivation

Before we get to any interesting geometry, we need to develop a language to discuss things cleanly and effectively. This is best done in the language of categories. There is not much to know about categories to get started; it is just a very useful language. Like all mathematical languages, category theory comes with an embedded logic, which allows us to abstract intuitions in settings we know well to far more general situations.

Our motivation is as follows. We will be creating some new mathematical objects (such as schemes, and certain kinds of sheaves), and we expect them to act like objects we have seen before. We could try to nail down precisely what we mean by "act like", and what minimal set of things we have to check in order to verify that they act the way we expect. Fortunately, we don't have to — other people have done this before us, by defining key notions, such as *abelian categories*, which behave like modules over a ring.

Our general approach will be as follows. I will try to tell what you need to know, and no more. (This I promise: if I use the word "topoi", you can shoot me.) I will begin by telling you things you already know, and describing what is essential about the examples, in a way that we can abstract a more general definition. We will then see this definition in less familiar settings, and get comfortable with using it to solve problems and prove theorems.

For example, we will define the notion of *product* of schemes. We could just give a definition of product, but then you should want to know why this precise definition deserves the name of "product". As a motivation, we revisit the notion of product in a situation we know well: (the category of) sets. One way to define the product of sets U and V is as the set of ordered pairs $\{(u,v) : u \in U, v \in V\}$. But someone from a different mathematical culture might reasonably define it as the set of symbols $\{{}^{u}_{v} : u \in U, v \in V\}$. These notions are "obviously the same". Better: there is "an obvious bijection between the two".

This can be made precise by giving a better definition of product, in terms of a *universal property*. Given two sets M and N, a product is a set P, along with maps $\mu : P \to M$ and $\nu : P \to N$, such that for *any set* P' *with maps* $\mu' : P' \to M$ *and*

$\nu' : P' \to N$, these maps must factor *uniquely* through P:

(2.1.0.1)
$$\begin{tikzcd} P' \arrow[dr, dotted, "\exists!"] \arrow[drr, "\nu'"] \arrow[ddr, "\mu'"'] & & \\ & P \arrow[r, "\nu"'] \arrow[d, "\mu"] & N \\ & M & \end{tikzcd}$$

(The symbol $\exists$ means "there exists", and the symbol ! here means "unique".) Thus a **product** is a *diagram*

$$\begin{tikzcd} P \arrow[r, "\nu"] \arrow[d, "\mu"'] & N \\ M & \end{tikzcd}$$

and not just a set P, although the maps μ and ν are often left implicit.

This definition agrees with the traditional definition, with one twist: there isn't just a single product; but any two products come with a *unique* isomorphism between them. In other words, the product is unique up to unique isomorphism. Here is why: if you have a product

$$\begin{tikzcd} P_1 \arrow[r, "\nu_1"] \arrow[d, "\mu_1"'] & N \\ M & \end{tikzcd}$$

and I have a product

$$\begin{tikzcd} P_2 \arrow[r, "\nu_2"] \arrow[d, "\mu_2"'] & N \\ M & \end{tikzcd}$$

then by the universal property of my product (letting (P_2, μ_2, ν_2) play the role of (P, μ, ν), and (P_1, μ_1, ν_1) play the role of (P', μ', ν') in (2.1.0.1)), there is a unique map $f : P_1 \to P_2$ making the appropriate diagram commute (i.e. $\mu_1 = \mu_2 \circ f$ and $\nu_1 = \nu_2 \circ f$). Similarly by the universal property of your product, there is a unique map $g : P_2 \to P_1$ making the appropriate diagram commute. Now consider the universal property of my product, this time letting (P_2, μ_2, ν_2) play the role of both (P, μ, ν) and (P', μ', ν') in (2.1.0.1). There is a unique map $h : P_2 \to P_2$ such that

$$\begin{tikzcd} P_2 \arrow[dr, "h"] \arrow[drr, "\nu_2"] \arrow[ddr, "\mu_2"'] & & \\ & P_2 \arrow[r, "\nu_2"'] \arrow[d, "\mu_2"] & N \\ & M & \end{tikzcd}$$

commutes. However, I can name two such maps: the identity map id_{P_2}, and $g \circ f$. Thus $g \circ f = \mathrm{id}_{P_2}$. Similarly, $f \circ g = \mathrm{id}_{P_1}$. Thus the maps f and g arising from the universal property are bijections. In short, there is a unique bijection between P_1 and P_2 preserving the "product structure" (the maps to M and N). This gives us the right to name any such product $M \times N$, since any two such products are uniquely identified.

This definition has the advantage that it works in many circumstances, and once we define categories, we will soon see that the above argument applies verbatim in any category to show that products, if they exist, are unique up to unique isomorphism. Even if you haven't seen the definition of category before, you can verify that this agrees with your notion of product in some category that you have seen before (such as the category of vector spaces, where the maps are taken to be linear maps; or the category of smooth manifolds, where the maps are taken to be *submersions*, i.e. differentiable maps whose differential is everywhere surjective).

This is handy even in cases that you understand. For example, one way of defining the product of two manifolds M and N is to cut them both up into charts, then take products of charts, then glue them together. But if I cut up the manifolds in one way, and you cut them up in another, how do we know our resulting manifolds are the "same"? We could wave our hands, or make an annoying argument about refining covers, but instead, we should just show that they are "categorical products" and hence canonically the "same" (i.e. isomorphic). We will formalize this argument in §2.3.

Another set of notions we will abstract are categories that "behave like modules". We will want to define kernels and cokernels for new notions, and we should make sure that these notions behave the way we expect them to. This leads us to the definition of *abelian categories*, first defined by Grothendieck in his Tôhoku paper **[Gr]**.

In this chapter, we will give an informal introduction to these and related notions, in the hope of giving just enough familiarity to comfortably use them in practice.

2.2 Categories and functors

We begin with an informal definition of categories and functors.

2.2.1. Categories.

A **category** consists of a collection of **objects**, and for each pair of objects, a set of maps, or **morphisms** (or **arrows**), between them. (For experts: technically, this is the definition of a *locally small category*. In the correct definition, the morphisms need only form a class, not necessarily a set, but see Caution 1.2.1.) The collection of objects of a category $\mathcal{C}$ are often denoted $\mathrm{obj}(\mathcal{C})$, but we will usually denote the collection also by $\mathcal{C}$. If $A, B \in \mathcal{C}$, then the set of morphisms from A to B is denoted $\mathrm{Mor}(A, B)$. A morphism is often written $f : A \to B$, and A is said to be the **source** of f, and B the **target** of f. (Of course, $\mathrm{Mor}(A, B)$ is taken to be disjoint from $\mathrm{Mor}(A', B')$ unless $A = A'$ and $B = B'$.)

Morphisms compose as expected: there is a composition $\mathrm{Mor}(B, C) \times \mathrm{Mor}(A, B) \to \mathrm{Mor}(A, C)$, and if $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, then their composition is denoted $g \circ f$. Composition is associative: if $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, and $h \in \mathrm{Mor}(C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$. For each object $A \in \mathcal{C}$, there is always an **identity morphism** $\mathrm{id}_A : A \to A$, such that when you (left- or right-)compose a morphism with the identity, you get the same morphism. More precisely, for any morphisms $f : A \to B$ and $g : B \to C$, $f \circ \mathrm{id}_B = f$ and $\mathrm{id}_B \circ g = g$. (If you wish, you may check that "identity morphisms are unique": there is only one morphism deserving the name id_A.)

If we have a category, then we have a notion of **isomorphism** between two objects (a morphism $f : A \to B$ such that there exists some — necessarily unique — morphism $g : B \to A$, where $f \circ g$ and $g \circ f$ are the identity on B and A respectively), and a notion of **automorphism** of an object (an isomorphism of the object with itself).

2.2.2. *Example.* The prototypical example to keep in mind is the category of sets, denoted *Sets*. The objects are sets, and the morphisms are maps of sets. (Because Russell's paradox shows that there is no set of all sets, we did not say earlier that there is a set of all objects. But as stated in §1.2, we are deliberately omitting all set-theoretic issues.)

2.2.3. *Example.* Another good example is the category Vec_k of vector spaces over a given field k. The objects are k-vector spaces, and the morphisms are linear transformations. (What are the isomorphisms?)

2.2.A. UNIMPORTANT EXERCISE. A category in which each morphism is an isomorphism is called a **groupoid**. (This notion is not important in these notes. The point of this exercise is to give you some practice with categories, by relating them to an object you know well.)
(a) A perverse definition of a group is: a groupoid with one object. Make sense of this.
(b) Describe a groupoid that is not a group.

2.2.B. EXERCISE. If A is an object in a category $\mathcal{C}$, show that the invertible elements of $\mathrm{Mor}(A, A)$ form a group (called the **automorphism group of** A, denoted $\mathrm{Aut}(A)$). What are the automorphism groups of the objects in Examples 2.2.2 and 2.2.3? Show that two isomorphic objects have isomorphic automorphism groups. (For readers with a topological background: if X is a topological space, then the fundamental groupoid is the category where the objects are points of X, and the morphisms $x \to y$ are paths from x to y, up to homotopy. Then the automorphism group of x_0 is the (pointed) fundamental group $\pi_1(X, x_0)$. In the case where X is connected, and $\pi_1(X)$ is not abelian, this illustrates the fact that for a connected groupoid — whose definition you can guess — the automorphism groups of the objects are all isomorphic, but not canonically isomorphic.)

2.2.4. *Example: abelian groups.* The abelian groups, along with group homomorphisms, form a category *Ab*.

2.2.5. *Important example: modules over a ring.* If A is a ring, then the A-modules form a category Mod_A. (This category has additional structure; it will be the prototypical example of an *abelian category*, see §2.6.) Taking $A = k$, we obtain Example 2.2.3; taking $A = \mathbb{Z}$, we obtain Example 2.2.4.

2.2.6. *Example: rings.* There is a category *Rings*, where the objects are rings, and the morphisms are morphisms of rings (which send 1 to 1 by our conventions, §1.2).

2.2.7. *Example: topological spaces.* The topological spaces, along with continuous maps, form a category *Top*. The isomorphisms are homeomorphisms.

In all of the above examples, the objects of the categories were in obvious ways sets with additional structure. This needn't be the case, as the next example shows.

2.2.8. *Example: partially ordered sets.* A **partially ordered set**, or **poset**, is a set S along with a binary relation $\geq$ on S satisfying:

(i) $x \geq x$ (reflexivity),
(ii) $x \geq y$ and $y \geq z$ imply $x \geq z$ (transitivity), and
(iii) if $x \geq y$ and $y \geq x$ then $x = y$.

A partially ordered set $(S, \geq)$ can be interpreted as a category whose objects are the elements of S, and with a single morphism from x to y if and only if $x \geq y$ (and no morphism otherwise).

A trivial example is $(S, \geq)$ where $x \geq y$ if and only if $x = y$. Another example is

(2.2.8.1)
$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Here there are three objects. The identity morphisms are omitted for convenience, and the two non-identity morphisms are depicted. A third example is

(2.2.8.2)
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Here the "obvious" morphisms are again omitted: the identity morphisms, and the morphism from the upper left to the lower right. Similarly,

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

depicts a partially ordered set, where again, only the "generating morphisms" are depicted.

2.2.9. *Example: the category of subsets of a set, and the category of open sets in a topological space.* If X is a set, then the subsets form a partially ordered set, where the order is given by inclusion. Informally, if $U \subset V$, then we have exactonly one more morphism $U \to V$ in the category (and otherwise none). Similarly, if X is a topological space, then the *open* sets form a partially ordered set, where the order is given by inclusion.

2.2.10. *Example.* A **subcategory** $\mathcal{A}$ of a category $\mathcal{B}$ has as its objects some of the objects of $\mathcal{B}$, and some of the morphisms, such that the morphisms of $\mathcal{A}$ include the identity morphisms of the objects of $\mathcal{A}$, and are closed under composition. (For example, (2.2.8.1) is in an obvious way a subcategory of (2.2.8.2). Also, we have an obvious "inclusion functor" $i : \mathcal{A} \to \mathcal{B}$.)

2.2.11. Functors.

A **covariant functor** F from a category $\mathcal{A}$ to a category $\mathcal{B}$, denoted $F : \mathcal{A} \to \mathcal{B}$, is the following data. It is a map of objects $F : \text{obj}(\mathcal{A}) \to \text{obj}(\mathcal{B})$, and for each A_1, $A_2 \in \mathcal{A}$, and morphism $m : A_1 \to A_2$, a morphism $F(m) : F(A_1) \to F(A_2)$ in $\mathcal{B}$. We require that F preserves identity morphisms (for $A \in \mathcal{A}$, $F(\text{id}_A) = \text{id}_{F(A)}$), and that F preserves composition ($F(m_2 \circ m_1) = F(m_2) \circ F(m_1)$). (You may wish to verify that covariant functors send isomorphisms to isomorphisms.) A trivial example is the **identity functor** $\text{id} : \mathcal{A} \to \mathcal{A}$, whose definition you can guess.

If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are covariant functors, then we define a functor $G \circ F : \mathcal{A} \to \mathcal{C}$ (the **composition** of $\mathcal{G}$ and $\mathcal{F}$) in the obvious way. Composition of functors is associative in an evident sense.

2.2.12. *Example: a forgetful functor.* Consider the functor from the category of vector spaces (over a field k) Vec_k to *Sets*, that associates to each vector space its underlying set. The functor sends a linear transformation to its underlying map of sets. This is an example of a **forgetful functor**, where some additional structure is forgotten. Another example of a forgetful functor is $\mathit{Mod}_A \to \mathit{Ab}$ from A-modules to abelian groups, remembering only the abelian group structure of the A-module.

2.2.13. *Topological examples.* Examples of covariant functors include the fundamental group functor π_1, which sends a topological space X with choice of a point $x_0 \in X$ to a group $\pi_1(X, x_0)$ (what are the objects and morphisms of the source category?), and the ith homology functor $\mathit{Top} \to \mathit{Ab}$, which sends a topological space X to its ith homology group $H_i(X, \mathbb{Z})$. The covariance corresponds to the fact that a (continuous) morphism of pointed topological spaces $f : X \to Y$ with $f(x_0) = y_0$ induces a map of fundamental groups $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, and similarly for homology groups.

2.2.14. *Example.* Suppose A is an object in a category $\mathcal{C}$. Then there is a functor $h^A : \mathcal{C} \to \mathit{Sets}$ sending $B \in \mathcal{C}$ to $\text{Mor}(A, B)$, and sending $f : B_1 \to B_2$ to $\text{Mor}(A, B_1) \to \text{Mor}(A, B_2)$ described by

$$[g : A \to B_1] \mapsto [f \circ g : A \to B_1 \to B_2].$$

This seemingly silly functor ends up surprisingly being an important concept, and will come up repeatedly for us.

2.2.15. Full and faithful functors. A covariant functor $F : \mathcal{A} \to \mathcal{B}$ is **faithful** if for all $A, A' \in \mathcal{A}$, the map $\text{Mor}_{\mathcal{A}}(A, A') \to \text{Mor}_{\mathcal{B}}(F(A), F(A'))$ is injective, and **full** if it is surjective. A functor that is full and faithful is **fully faithful**. A subcategory $i : \mathcal{A} \to \mathcal{B}$ is a **full subcategory** if i is full. Thus a subcategory $\mathcal{A}'$ of $\mathcal{A}$ is full if and only if for all $A, B \in \text{obj}(\mathcal{A}')$, $\text{Mor}_{\mathcal{A}'}(A, B) = \text{Mor}_{\mathcal{A}}(A, B)$.

2.2.16. Definition. A **contravariant functor** is defined in the same way as a covariant functor, except the arrows switch directions: in the above language, $F(A_1 \to A_2)$ is now an arrow from $F(A_2)$ to $F(A_1)$. (Thus $\mathcal{F}(m_2 \circ m_1) = \mathcal{F}(m_1) \circ \mathcal{F}(m_2)$, not $\mathcal{F}(m_2) \circ \mathcal{F}(m_1)$.)

It is wise to always state whether a functor is covariant or contravariant. If it is not stated, the functor is often assumed to be covariant.

(Sometimes people describe a contravariant functor $\mathcal{C} \to \mathcal{D}$ as a covariant functor $\mathcal{C}^{\text{opp}} \to \mathcal{D}$, where $\mathcal{C}^{\text{opp}}$ is the same category as $\mathcal{C}$ except that the arrows go in the opposite direction. Here $\mathcal{C}^{\text{opp}}$ is said to be the **opposite category** to $\mathcal{C}$.) One can define fullness, etc. for contravariant functors, and you should do so.

2.2.17. *Linear algebra example.* If Vec_k is the category of k-vector spaces (introduced in Example 2.2.3), then taking duals gives a contravariant functor $\cdot^\vee : Vec_k \to Vec_k$. Indeed, to each linear transformation $f : V \to W$, we have a dual transformation $f^\vee : W^\vee \to V^\vee$, and $(f \circ g)^\vee = g^\vee \circ f^\vee$.

2.2.18. *Topological example (cf. Example 2.2.13) for those who have seen cohomology.* The ith cohomology functor $H^i(\cdot, \mathbb{Z}) : Top \to Ab$ is a contravariant functor.

2.2.19. *Example.* There is a contravariant functor $Top \to Rings$ taking a topological space X to the real-valued continuous functions on X. A morphism of topological spaces $X \to Y$ (a continuous map) induces the pullback map from functions on Y to maps on X.

2.2.20. *Example (the functor of points, cf. Example 2.2.14).* Suppose A is an object of a category $\mathcal{C}$. Then there is a contravariant functor $h_A : \mathcal{C} \to Sets$ sending $B \in \mathcal{C}$ to $\text{Mor}(B, A)$, and sending the morphism $f : B_1 \to B_2$ to the morphism $\text{Mor}(B_2, A) \to \text{Mor}(B_1, A)$ via

$$[g : B_2 \to A] \mapsto [g \circ f : B_1 \to B_2 \to A].$$

This example initially looks weird and different, but Examples 2.2.17 and 2.2.19 may be interpreted as special cases; do you see how? What is A in each case? This functor might reasonably be called the *functor of maps* (to A), but is actually known as the **functor of points**. We will meet this functor again (in the category of schemes) in Definition 7.3.6.

2.2.21. $\star$ **Natural transformations (and natural isomorphisms) of covariant functors, and equivalences of categories.**

(This notion won't come up in an essential way until at least Chapter 7, so you shouldn't read this section until then.) Suppose F and G are two covariant functors from $\mathcal{A}$ to $\mathcal{B}$. A **natural transformation of covariant functors** $F \to G$ is the data of a morphism $m_A : F(A) \to G(A)$ for each $A \in \mathcal{A}$ such that for each $f : A \to A'$ in $\mathcal{A}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "m_A"'] & F(A') \arrow[d, "m_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

commutes. A **natural isomorphism** of functors is a natural transformation such that each m_A is an isomorphism. The data of functors $F : \mathcal{A} \to \mathcal{B}$ and $F' : \mathcal{B} \to \mathcal{A}$ such that $F \circ F'$ is naturally isomorphic to the identity functor $I_{\mathcal{B}}$ on $\mathcal{B}$ and $F' \circ F$ is naturally isomorphic to $I_{\mathcal{A}}$ is said to be an **equivalence of categories**. "Equivalence of categories" is an equivalence relation on categories. The right notion of when two categories are "essentially the same" is not *isomorphism* (a functor giving bijections of objects and morphisms) but *equivalence*. Exercises 2.2.C and 2.2.D might give you some vague sense of this. Later exercises (for example, that "rings" and "affine schemes" are essentially the same, once arrows are reversed, Exercise 7.3.D) may help too.

Two examples might make this strange concept more comprehensible. The double dual of a finite-dimensional vector space V is *not* V, but we learn early to say that it is canonically isomorphic to V. We can make that precise as follows. Let $\mathit{f.d.Vec}_k$ be the category of finite-dimensional vector spaces over k. Note that this category contains oodles of vector spaces of each dimension.

2.2.C. EXERCISE. Let $\cdot^{\vee\vee} : \mathit{f.d.Vec}_k \to \mathit{f.d.Vec}_k$ be the double dual functor from the category of finite-dimensional vector spaces over k to itself. Show that $\cdot^{\vee\vee}$ is naturally isomorphic to the identity functor on $\mathit{f.d.Vec}_k$. (Without the finite-dimensional hypothesis, we only get a natural transformation of functors from id to $\cdot^{\vee\vee}$.)

Let $\mathcal{V}$ be the category whose objects are the k-vector spaces k^n for each $n \geq 0$ (there is one vector space for each n), and whose morphisms are linear transformations. This latter space can be thought of as vector spaces with bases, and the morphisms are honest matrices. There is an obvious functor $\mathcal{V} \to \mathit{f.d.Vec}_k$, as each k^n *is* a finite-dimensional vector space.

2.2.D. EXERCISE. Show that $\mathcal{V} \to \mathit{f.d.Vec}_k$ gives an equivalence of categories, by describing an "inverse" functor. (Recall that we are being cavalier about set-theoretic assumptions, see Caution 1.2.1, so feel free to simultaneously choose bases for each vector space in $\mathit{f.d.Vec}_k$. To make this precise, you will need to use Godel-Bernays set theory or else replace $\mathit{f.d.Vec}_k$ with a very similar small category, but we won't worry about this.)

2.2.22. $\star\star$ *Aside for experts.* Your argument for Exercise 2.2.D will show that (modulo set-theoretic issues) this definition of equivalence of categories is the same as another one commonly given: a covariant functor $F : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories if it is fully faithful and every object of $\mathcal{B}$ is isomorphic to an object of the form $F(a)$ for some $a \in \mathcal{A}$ (F is *essentially surjective*). Indeed, one can show that such a functor has a *quasiinverse*, i.e., a functor $G : \mathcal{B} \to \mathcal{A}$ (necessarily also an equivalence and unique up to unique isomorphism) for which $G \circ F \cong \mathrm{id}_A$ and $F \circ G \cong \mathrm{id}_B$, and conversely, any functor that has a quasiinverse is an equivalence.

2.3 Universal properties determine an object up to unique isomorphism

Given some category that we come up with, we often will have ways of producing new objects from old. In good circumstances, such a definition can be made using the notion of a *universal property*. Informally, we wish that there were an object with some property. We first show that if it exists, then it is essentially unique, or more precisely, is unique up to unique isomorphism. Then we go about constructing an example of such an object to show existence.

Explicit constructions are sometimes easier to work with than universal properties, but with a little practice, universal properties are useful in proving things quickly and slickly. Indeed, when learning the subject, people often find explicit constructions more appealing, and use them more often in proofs, but as they become more experienced, they find universal property arguments more elegant and insightful.

2.3.1. Products were defined by universal property. We have seen one important example of a universal property argument already in §2.1: products. You should go back and verify that our discussion there gives a notion of product in any category, and shows that products, *if they exist*, are unique up to unique isomorphism.

2.3.2. Initial, final, and zero objects. Here are some simple but useful concepts that will give you practice with universal property arguments. An object of a category $\mathcal{C}$ is an **initial object** if it has precisely one map to every object. It is a **final object** if it has precisely one map from every object. It is a **zero object** if it is both an initial object and a final object.

2.3.A. EXERCISE. Show that any two initial objects are uniquely isomorphic. Show that any two final objects are uniquely isomorphic.

In other words, *if* an initial object exists, it is unique up to unique isomorphism, and similarly for final objects. This (partially) justifies the phrase "*the* initial object" rather than "*an* initial object", and similarly for "*the* final object" and "*the* zero object".

2.3.B. EXERCISE. What are the initial and final objects in *Sets*, *Rings*, and *Top* (if they exist)? How about in the two examples of §2.2.9?

2.3.3. Localization of rings and modules. Another important example of a definition by universal property is the notion of *localization* of a ring. We first review a constructive definition, and then reinterpret the notion in terms of universal property. A multiplicative subset S of a ring A is a subset closed under multiplication containing 1. We define a ring $S^{-1}A$. The elements of $S^{-1}A$ are of the form a/s where $a \in A$ and $s \in S$, and where $a_1/s_1 = a_2/s_2$ if (and only if) *for some* $s \in S$, $s(s_2a_1 - s_1a_2) = 0$. We define $(a_1/s_1) + (a_2/s_2) = (s_2a_1 + s_1a_2)/(s_1s_2)$, and $(a_1/s_1) \times (a_2/s_2) = (a_1a_2)/(s_1s_2)$. (If you wish, you may check that this equality of fractions really is an equivalence relation and the two binary operations on fractions are well-defined on equivalence classes and make $S^{-1}A$ into a ring.) We have a canonical ring map $A \to S^{-1}A$ given by $a \mapsto a/1$. Note that if $0 \in S$, $S^{-1}A$ is the 0-ring.

There are two particularly important flavors of multiplicative subsets. The first is $\{1, f, f^2, \dots\}$, where $f \in A$. This localization is denoted A_f. The second is $A - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. This localization $S^{-1}A$ is denoted $A_{\mathfrak{p}}$. (Notational warning: If $\mathfrak{p}$ is a prime ideal, then $A_{\mathfrak{p}}$ means you're allowed to divide by elements not in $\mathfrak{p}$. However, if $f \in A$, A_f means you're allowed to divide by f. This can be confusing. For example, if (f) is a prime ideal, then $A_f \neq A_{(f)}$.)

Warning: sometimes localization is first introduced in the special case where A is an integral domain and $0 \notin S$. In that case, $A \hookrightarrow S^{-1}A$, but this isn't always true, as shown by the following exercise. (But we will see that noninjective localizations needn't be pathological, and we can sometimes understand them geometrically, see Exercise 4.2.I.)

2.3.C. EXERCISE. Show that $A \to S^{-1}A$ is injective if and only if S contains no zerodivisors. (A **zerodivisor** of a ring A is an element a such that there is a non-zero element b with $ab = 0$. The other elements of A are called **non-zerodivisors**. For example, a unit is never a zerodivisor. Counter-intuitively, 0 is a zerodivisor in every ring but the 0-ring.)

If A is an integral domain and $S = A - \{0\}$, then $S^{-1}A$ is called the **fraction field** of A, which we denote $K(A)$. The previous exercise shows that A is a subring of its fraction field $K(A)$. We now return to the case where A is a general (commutative) ring.

2.3.D. EXERCISE. Verify that $A \to S^{-1}A$ satisfies the following universal property: $S^{-1}A$ is initial among A-algebras B where every element of S is sent to a unit in B. (Recall: the data of "an A-algebra B" and "a ring map $A \to B$" are the same.) Translation: any map $A \to B$ where every element of S is sent to a unit must factor uniquely through $A \to S^{-1}A$. Another translation: a ring map out of $S^{-1}A$ is the same thing as a ring map from A that sends every element of S to a unit. Furthermore, an $S^{-1}A$-module is the same thing as an A-module for which $s \times \cdot : M \to M$ is an A-module isomorphism for all $s \in S$.

In fact, it is cleaner to *define* $A \to S^{-1}A$ by the universal property, and to show that it exists, and to use the universal property to check various properties $S^{-1}A$ has. Let's get some practice with this by *defining* localizations of modules by universal property. Suppose M is an A-module. We define the A-module map $\phi : M \to S^{-1}M$ as being initial among A-module maps $M \to N$ such that elements of S are invertible in N ($s \times \cdot : N \to N$ is an isomorphism for all $s \in S$). More precisely, any such map $\alpha : M \to N$ factors uniquely through ϕ:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "\alpha"'] & S^{-1}M \arrow[d, dotted, "\exists!"] \\ & N \end{tikzcd}$$

(Translation: $M \to S^{-1}M$ is universal (initial) among A-module maps from M to modules that are actually $S^{-1}A$-modules. Can you make this precise by defining clearly the objects and morphisms in this category?)

Notice: (i) this determines $\phi : M \to S^{-1}M$ up to unique isomorphism (you should think through what this means); (ii) we are defining not only $S^{-1}M$, but also the map ϕ at the same time; and (iii) essentially by definition the A-module structure on $S^{-1}M$ extends to an $S^{-1}A$-module structure.

2.3.E. EXERCISE. Show that $\phi : M \to S^{-1}M$ exists, by constructing something satisfying the universal property. Hint: define elements of $S^{-1}M$ to be of the form m/s where $m \in M$ and $s \in S$, and $m_1/s_1 = m_2/s_2$ if and only if for some $s \in S$, $s(s_2m_1 - s_1m_2) = 0$. Define the additive structure by $(m_1/s_1) + (m_2/s_2) = (s_2m_1 + s_1m_2)/(s_1s_2)$, and the $S^{-1}A$-module structure (and hence the A-module structure) is given by $(a_1/s_1) \circ (m_2/s_2) = (a_1m_2)/(s_1s_2)$.

2.3.F. EXERCISE. Show that localization commutes with finite products. In other words, if $M_1, \dots, M_n$ are A-modules, describe an isomorphism (of A-modules, and of S^{-1}-modules) $S^{-1}(M_1 \times \cdots \times M_n) \to S^{-1}M_1 \times \cdots \times S^{-1}M_n$. Show that "localization does not necessarily commute with infinite products": the obvious map $S^{-1}(\prod_i M_i) \to \prod_i S^{-1}M_i$ induced by the universal property of localization is not always an isomorphism. (Hint: $(1, 1/2, 1/3, 1/4, \dots) \in \mathbb{Q} \times \mathbb{Q} \times \cdots$.)

2.3.4. *Remark.* Localization does not necessarily commute with Hom, see Example 2.6.7. But Exercise 2.6.G will show that in good situations (if the first argument of Hom is *finitely presented*, localization *does* commute with Hom.

2.3.5. Tensor products. Another important example of a universal property construction is the notion of a **tensor product** of A-modules

$$\begin{array}{rccc} \otimes_A : & \mathrm{obj}(Mod_A) \times \mathrm{obj}(Mod_A) & \longrightarrow & \mathrm{obj}(Mod_A) \\ & (M, N) & \longmapsto & M \otimes_A N \end{array}$$

The subscript A is often suppressed when it is clear from context. The tensor product is often defined as follows. Suppose you have two A-modules M and N. Then elements of the tensor product $M \otimes_A N$ are finite A-linear combinations of symbols $m \otimes n$ ($m \in M$, $n \in N$), subject to relations $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$, $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$, $a(m \otimes n) = (am) \otimes n = m \otimes (an)$ (where $a \in A$, $m_1, m_2 \in M$, $n_1, n_2 \in N$). More formally, $M \otimes_A N$ is the free A-module generated by $M \times N$, quotiented by the submodule generated by $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$, $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$, $a(m, n) - (am, n)$, and $a(m, n) - (m, an)$ for $a \in A$, $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$. The image of (m, n) in this quotient is $m \otimes n$.

If A is a field k, we recover the tensor product of vector spaces.

2.3.G. EXERCISE (IF YOU HAVEN'T SEEN TENSOR PRODUCTS BEFORE). Show that $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \cong \mathbb{Z}/(2)$. (This exercise is intended to give some hands-on practice with tensor products.)

2.3.H. IMPORTANT EXERCISE: RIGHT-EXACTNESS OF $\cdot \otimes_A N$. Show that $\cdot \otimes_A N$ gives a covariant functor $Mod_A \to Mod_A$. Show that $\cdot \otimes_A N$ is a **right-exact functor**, i.e. if

$$M' \to M \to M'' \to 0$$

is an exact sequence of A-modules (which means $f : M \to M''$ is surjective, and M' surjects onto the kernel of f; see §2.6), then the induced sequence

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

is also exact. This exercise is repeated in Exercise 2.6.F, but you may get a lot out of doing it now. (You will be reminded of the definition of right-exactness in §2.6.4.)

The constructive definition $\otimes$ is a weird definition, and really the "wrong" definition. To motivate a better one: notice that there is a natural A-bilinear map $M \times N \to M \otimes_A N$. (If $M, N, P \in Mod_A$, a map $f : M \times N \to P$ is A-**bilinear** if $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$, $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$, and $f(am, n) = f(m, an) = af(m, n)$.) *Any* A-bilinear map $M \times N \to P$ factors through the tensor product uniquely: $M \times N \to M \otimes_A N \to P$. (Think this through!)

We can take this as the *definition* of the tensor product as follows. It is an A-module T along with an A-bilinear map $t : M \times N \to T$, such that given any A-bilinear map $t' : M \times N \to T'$, there is a unique A-linear map $f : T \to T'$ such that $t' = f \circ t$.

$$\begin{tikzcd} M \times N \arrow[rr, "t"] \arrow[dr, "t'"'] & & T \arrow[dl, dotted, "\exists ! f"] \\ & T' & \end{tikzcd}$$

2.3.I. EXERCISE. Show that $(T, t : M \times N \to T)$ is unique up to unique isomorphism. Hint: first figure out what "unique up to unique isomorphism" means for such pairs. Then follow the analogous argument for the product.

In short: given M and N, there is an A-bilinear map $t : M \times N \to M \otimes_A N$, unique up to unique isomorphism, defined by the following universal property: for any A-bilinear map $t' : M \times N \to T'$ there is a unique A-linear map $f : M \otimes_A N \to T'$ such that $t' = f \circ t$.

As with all universal property arguments, this argument shows uniqueness *assuming existence*. To show existence, we need an explicit construction.

2.3.J. EXERCISE. Show that the construction of §2.3.5 satisfies the universal property of tensor product.

The two exercises below are some useful facts about tensor products with which you should be familiar.

2.3.K. IMPORTANT EXERCISE. (a) If M is an A-module and $A \to B$ is a morphism of rings, give $B \otimes_A M$ the structure of a B-module (this is part of the exercise). Show that this describes a functor $Mod_A \to Mod_B$.
(b) If further $A \to C$ is a morphism of rings, show that $B \otimes_A C$ has the structure of a ring. Hint: multiplication will be given by $(b_1 \otimes c_1)(b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2)$. (Exercise 2.3.T will interpret this construction as a fibered coproduct.)

2.3.L. IMPORTANT EXERCISE. If S is a multiplicative subset of A and M is an A-module, describe a natural isomorphism $(S^{-1}A) \otimes_A M \cong S^{-1}M$ (as $S^{-1}A$-modules *and* as A-modules).

2.3.6. Important Example: Fibered products. (This notion will be essential later.) Suppose we have morphisms $f : X \to Z$ and $g : Y \to Z$ (in *any* category). Then the **fibered product** is an object $X \times_Z Y$ along with morphisms $\pi_X : X \times_Z Y \to X$ and $\pi_Y : X \times_Z Y \to Y$, where the two compositions $f \circ \pi_X, g \circ \pi_Y : X \times_Z Y \to Z$ agree, such that given any object W with maps to X and Y (whose compositions to

Z agree), these maps factor through some unique $W \to X \times_Z Y$:

$$\begin{tikzcd}
W \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

(Warning: the definition of the fibered product depends on f and g, even though they are omitted from the notation $X \times_Z Y$.)

By the usual universal property argument, if it exists, it is unique up to unique isomorphism. (You should think this through until it is clear to you.) Thus the use of the phrase "the fibered product" (rather than "a fibered product") is reasonable, and we should reasonably be allowed to give it the name $X \times_Z Y$. We know what maps to it are: they are precisely maps to X and maps to Y that agree as maps to Z.

Depending on your religion, the diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

is called a **fibered/pullback/Cartesian diagram/square** (six possibilities).

The right way to interpret the notion of fibered product is first to think about what it means in the category of sets.

2.3.M. EXERCISE. Show that in *Sets*,

$$X \times_Z Y = \{(x, y) \in X \times Y \ : \ f(x) = g(y)\}.$$

More precisely, show that the right side, equipped with its evident maps to X and Y, satisfies the universal property of the fibered product. (This will help you build intuition for fibered products.)

2.3.N. EXERCISE. If X is a topological space, show that fibered products always exist in the category of open sets of X, by describing what a fibered product is. (Hint: it has a one-word description.)

2.3.O. EXERCISE. If Z is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, show that "$X \times_Z Y = X \times Y$": "the" fibered product over Z is uniquely isomorphic to "the" product. Assume all relevant (fibered) products exist. (This is an exercise about unwinding the definition.)

2.3.P. USEFUL EXERCISE: TOWERS OF FIBER DIAGRAMS ARE FIBER DIAGRAMS. If the two squares in the following commutative diagram are fiber diagrams, show that the "outside rectangle" (involving U, V, Y, and Z) is also a fiber diagram.

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

2.3.Q. EXERCISE. Given morphisms $X_1 \to Y$, $X_2 \to Y$, and $Y \to Z$, show that there is a natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$, assuming that both fibered products exist. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

2.3.R. USEFUL EXERCISE: THE MAGIC DIAGRAM. Suppose we are given morphisms $X_1, X_2 \to Y$ and $Y \to Z$. Describe the natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$. Show that the following diagram is a fibered square.

$$\boxed{\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}}$$

Assume all relevant (fibered) products exist. This diagram is surprisingly useful — so useful that we will call it the **magic diagram**.

2.3.7. *Coproducts.* Define **coproduct** in a category by reversing all the arrows in the definition of product. Define **fibered coproduct** in a category by reversing all the arrows in the definition of fibered product.

2.3.S. EXERCISE. Show that coproduct for *Sets* is disjoint union. This is why we use the notation $\coprod$ for disjoint union.

2.3.T. EXERCISE. Suppose $A \to B, C$ are two ring morphisms, so in particular B and C are A-modules. Recall (Exercise 2.3.K) that $B \otimes_A C$ has a ring structure. Show that there is a natural morphism $B \to B \otimes_A C$ given by $b \mapsto b \otimes 1$. (This is not necessarily an inclusion, see Exercise 2.3.G.) Similarly, there is a natural morphism $C \to B \otimes_A C$. Show that this gives a fibered coproduct on rings, i.e. that

$$\begin{tikzcd} B \otimes_A C & C \arrow[l] \\ B \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

satisfies the universal property of fibered coproduct.

2.3.8. Monomorphisms and epimorphisms.

2.3.9. Definition. A morphism $f : X \to Y$ is a **monomorphism** if any two morphisms $g_1, g_2 : Z \to X$ such that $f \circ g_1 = f \circ g_2$ must satisfy $g_1 = g_2$. In other words, for any other object Z, the natural map $\mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ is an injection. This a generalization of an injection of sets. In other words, there is at most one way of filling in the dotted arrow so that the following diagram commutes.

$$\begin{tikzcd} Z \arrow[d, dotted, "\leq 1"'] \arrow[dr] & \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Intuitively, it is the categorical version of an injective map, and indeed this notion generalizes the familiar notion of injective maps of sets. (The reason we don't use the word "injective" is that in some contexts, "injective" will have an intuitive meaning which may not agree with "monomorphism". This is also the case with "epimorphism" vs. "surjective".)

2.3.U. EXERCISE. Show that the composition of two monomorphisms is a monomorphism.

2.3.V. EXERCISE. Prove that a morphism $X \to Y$ is a monomorphism if and only if the fibered product $X \times_Y X$ exists, and the induced morphism $X \to X \times_Y X$ is an isomorphism. We may then take this as the definition of monomorphism. (Monomorphisms aren't central to future discussions, although they will come up again. This exercise is just good practice.)

2.3.W. EASY EXERCISE. We use the notation of Exercise 2.3.Q. Show that if $Y \to Z$ is a monomorphism, then the morphisms $X_1 \times_Y X_2 \to X_1 \times_Z X_2$ you described in Exercise 2.3.Q is an isomorphism. We will use this later when talking about fibered products. (Hint: for any object V, give a natural bijection between maps from V to the first and maps from V to the second. It is also possible to use the magic diagram, Exercise 2.3.R.)

The notion of an **epimorphism** is "dual" to the definition of monomorphism, where all the arrows are reversed. This concept will not be central for us, although it turns up in the definition of an abelian category. Intuitively, it is the categorical version of a surjective map.

2.3.10. Representable functors and Yoneda's lemma. Much of our discussion about universal properties can be cleanly expressed in terms of representable functors, under the rubric of "Yoneda's Lemma". Yoneda's lemma is an easy fact stated in a complicated way. Informally speaking, you can essentially recover an object in a category by knowing the maps into it. For example, we have seen that the data of maps to $X \times Y$ are naturally (canonically) the data of maps to X and to Y. Indeed, we have now taken this as the *definition* of $X \times Y$.

Recall Example 2.2.20. Suppose A is an object of category $\mathcal{C}$. For any object $C \in \mathcal{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

$$\mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A), \tag{2.3.10.1}$$

by composition: given a map from C to A, we get a map from B to A by precomposing with $f : B \to C$. Hence this gives a contravariant functor $h_A : \mathcal{C} \to$ *Sets*.

Yoneda's Lemma states that the functor h_A determines A up to unique isomorphism. More precisely:

2.3.X. IMPORTANT EXERCISE THAT YOU SHOULD DO ONCE IN YOUR LIFE (YONEDA'S LEMMA). Given two objects A and A' in a category $\mathcal{C}$, and bijections

$$i_C : \mathrm{Mor}(C, A) \to \mathrm{Mor}(C, A') \tag{2.3.10.2}$$

that commute with the maps (2.3.10.1). Prove i_C is induced from a unique isomorphism $g : A \to A'$. More precisely, show that there is a unique isomorphism $g : A \to A'$ such that for all $C \in \mathcal{C}$, i_C is $u \mapsto g \circ u$. (Hint: This sounds hard, but it really is not. This statement is so general that there are really only a couple of things that you could possibly try. For example, if you're hoping to find an isomorphism $A \to A'$, where will you find it? Well, you are looking for an element $\mathrm{Mor}(A, A')$. So just plug in $C = A$ to (2.3.10.2), and see where the identity goes. You will quickly find the desired morphism; show that it is an isomorphism, then show that it is unique.)

There is an analogous statement with the arrows reversed, where instead of maps into A, you think of maps *from* A. The role of the contravariant functor h_A of Example 2.2.20 is played by the covariant functor h^A of Example 2.2.14. Because the proof is the same (with the arrows reversed), you needn't think it through.

Yoneda's lemma properly refers to a more general statement. Although it looks more complicated, it is no harder to prove.

2.3.Y. ⋆ EXERCISE.
(a) Suppose A and B are objects in a category $\mathcal{C}$. Give a bijection between the natural transformations $h^A \to h^B$ of covariant functors $\mathcal{C} \to$ *Sets* (see Example 2.2.14 for the definition) and the morphisms $B \to A$.
(b) State and prove the corresponding fact for contravariant functors h_A (see Example 2.2.20). Remark: A contravariant functor F from $\mathcal{C}$ to *Sets* is said to be **representable** if there is a natural isomorphism

$$\xi : F \xrightarrow{\sim} h_A .$$

Thus the representing object A is determined up to unique isomorphism by the pair (F, ξ). There is a similar definition for covariant functors. (We will revisit this in §7.6, and this problem will appear again as Exercise 7.6.B.)
(c) **Yoneda's lemma.** Suppose F is a covariant functor $\mathcal{C} \to$ *Sets*, and $A \in \mathcal{C}$. Give a bijection between the natural transformations $h^A \to F$ and $F(A)$. (The corresponding fact for contravariant functors is essentially Exercise 10.1.D.)

In fancy terms, Yoneda's lemma states the following. Given a category $\mathcal{C}$, we can produce a new category, called the *functor category* of $\mathcal{C}$, where the objects are contravariant functors $\mathcal{C} \to$ *Sets*, and the morphisms are natural transformations of such functors. We have a functor (which we can usefully call h) from $\mathcal{C}$ to its functor category, which sends A to h_A. Yoneda's Lemma states that this is a fully faithful functor, called the *Yoneda embedding*. (Fully faithful functors were defined in §2.2.15.)

2.4 Limits and colimits

Limits and colimits are two important definitions determined by universal properties. They generalize a number of familiar constructions. I will give the definition first, and then show you why it is familiar. For example, fractions will be motivating examples of colimits (Exercise 2.4.B(a)), and the p-adic integers (Example 2.4.3) will be motivating examples of limits.

2.4.1. Limits. We say that a category is a **small category** if the objects and the morphisms are sets. (This is a technical condition intended only for experts.) Suppose $\mathcal{I}$ is any small category, and $\mathcal{C}$ is any category. Then a functor $F : \mathcal{I} \to \mathcal{C}$ (i.e. with an object $A_i \in \mathcal{C}$ for each element $i \in \mathcal{I}$, and appropriate commuting morphisms dictated by $\mathcal{I}$) is said to be a **diagram indexed by** $\mathcal{I}$. We call $\mathcal{I}$ an **index category**. Our index categories will be partially ordered sets (Example 2.2.8), in which in particular there is at most one morphism between any two objects. (But other examples are sometimes useful.) For example, if $\square$ is the category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

and $\mathcal{A}$ is a category, then a functor $\square \to \mathcal{A}$ is precisely the data of a commuting square in $\mathcal{A}$.

Then the **limit** is an object $\varprojlim_{\mathcal{I}} A_i$ of $\mathcal{C}$ along with morphisms $f_j : \varprojlim_{\mathcal{I}} A_i \to A_j$ for each $j \in \mathcal{I}$, such that if $m : j \to k$ is a morphism in $\mathcal{I}$, then

(2.4.1.1)
$$\begin{tikzcd} \varprojlim_{\mathcal{I}} A_i \arrow[d, "f_j"'] \arrow[dr, "f_k"] & \\ A_j \arrow[r, "F(m)"] & A_k \end{tikzcd}$$

commutes, and this object and maps to each A_i are universal (final) with respect to this property. More precisely, given any other object W along with maps $g_i : W \to A_i$ commuting with the $F(m)$ (if $m : j \to k$ is a morphism in $\mathcal{I}$, then $g_k = F(m) \circ g_j$), then there is a unique map $g : W \to \varprojlim_{\mathcal{I}} A_i$ so that $g_i = f_i \circ g$ for all i. (In some cases, the limit is sometimes called the **inverse limit** or **projective limit**. We won't use this language.) By the usual universal property argument, if the limit exists, it is unique up to unique isomorphism.

2.4.2. *Examples: products.* For example, if $\mathcal{I}$ is the partially ordered set

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

we obtain the fibered product.

If $\mathcal{I}$ is

$$\bullet \qquad \bullet$$

we obtain the product.

If $\mathcal{I}$ is a set (i.e. the only morphisms are the identity maps), then the limit is called the **product** of the A_i, and is denoted $\prod_i A_i$. The special case where $\mathcal{I}$ has two elements is the example of the previous paragraph.

If $\mathcal{I}$ has an initial object e, then A_e is the limit, and in particular the limit always exists.

2.4.3. *Unimportant Example: the* p*-adic integers.* For a prime number p, the **p-adic integers** (or more informally, **p-adics**), $\mathbb{Z}_p$, are often described informally (and somewhat unnaturally) as being of the form $\mathbb{Z}_p = ? + ?p + ?p^2 + ?p^3 + \cdots$. They are an example of a limit in the category of rings:

$$\begin{tikzcd} \mathbb{Z}_p \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\ \cdots \arrow[r] & \mathbb{Z}/p^3 \arrow[r] & \mathbb{Z}/p^2 \arrow[r] & \mathbb{Z}/p. \end{tikzcd}$$

(Warning: $\mathbb{Z}_p$ is sometimes is used to denote the integers modulo p, but $\mathbb{Z}/(p)$ or $\mathbb{Z}/p\mathbb{Z}$ is better to use for this, to avoid confusion. Worse: by §2.3.3, $\mathbb{Z}_p$ also denotes those rationals whose denominators are a power of p. Hopefully the meaning of $\mathbb{Z}_p$ will be clear from the context.)

Limits do not always exist for any index category $\mathcal{I}$. However, you can often easily check that limits exist if the objects of your category can be interpreted as sets with additional structure, and arbitrary products exist (respecting the set-like structure).

2.4.A. IMPORTANT EXERCISE. Show that in the category *Sets*,

$$\left\{ (a_i)_{i\in I} \in \prod_i A_i : F(m)(a_j) = a_k \text{ for all } m \in \mathrm{Mor}_{\mathcal{I}}(j,k) \in \mathrm{Mor}(\mathcal{I}) \right\},$$

along with the obvious projection maps to each A_i, is the limit $\varprojlim_{\mathcal{I}} A_i$.

This clearly also works in the category Mod_A of A-modules (in particular Vec_k and *Ab*), as well as *Rings*.

From this point of view, $2 + 3p + 2p^2 + \cdots \in \mathbb{Z}_p$ can be understood as the sequence $(2, 2+3p, 2+3p+2p^2, \dots)$.

2.4.4. Colimits. More immediately relevant for us will be the dual (arrow-reversed version) of the notion of limit (or inverse limit). We just flip the arrows f_i in (2.4.1.1), and get the notion of a **colimit**, which is denoted $\varinjlim_{\mathcal{I}} A_i$. (You should draw the corresponding diagram.) Again, if it exists, it is unique up to unique isomorphism. (In some cases, the colimit is sometimes called the **direct limit**, **inductive limit**, or **injective limit**. We won't use this language. I prefer using limit/colimit in analogy with kernel/cokernel and product/coproduct. This is more than analogy, as kernels and products may be interpreted as limits, and similarly with cokernels and coproducts. Also, I remember that kernels "map to", and cokernels are "mapped to", which reminds me that a limit maps *to* all the objects in the big commutative diagram indexed by $\mathcal{I}$; and a colimit has a map *from* all the objects.)

Even though we have just flipped the arrows, colimits behave quite differently from limits.

2.4.5. *Example.* The group $5^{-\infty}\mathbb{Z}$ of rational numbers whose denominators are powers of 5 is a colimit $\varinjlim 5^{-i}\mathbb{Z}$. More precisely, $5^{-\infty}\mathbb{Z}$ is the colimit of the groups

$$\mathbb{Z} \longrightarrow 5^{-1}\mathbb{Z} \longrightarrow 5^{-2}\mathbb{Z} \longrightarrow \cdots$$

The colimit over an index *set* I is called the **coproduct**, denoted $\coprod_i A_i$, and is the dual (arrow-reversed) notion to the product.

2.4.B. EXERCISE. (a) Interpret the statement "$\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$". (b) Interpret the union of the some subsets of a given set as a colimit. (Dually, the intersection can be interpreted as a limit.) The objects of the category in question are the subsets of the given set.

Colimits don't always exist, but there are two useful large classes of examples for which they do.

2.4.6. Definition. A nonempty partially ordered set $(S, \geq)$ is **filtered** (or is said to be a **filtered set**) if for each $x, y \in S$, there is a z such that $x \geq z$ and $y \geq z$. More generally, a nonempty category $\mathcal{I}$ is **filtered** if:

(i) for each $x, y \in \mathcal{I}$, there is a $z \in \mathcal{I}$ and arrows $x \to z$ and $y \to z$, and
(ii) for every two arrows $u, v : x \to y$, there is an arrow $w : y \to z$ such that $w \circ u = w \circ v$.

(Other terminologies are also commonly used, such as "directed partially ordered set" and "filtered index category", respectively.)

2.4.C. EXERCISE. Suppose $\mathcal{I}$ is filtered. (We will almost exclusively use the case where $\mathcal{I}$ is a filtered set.) Show that any diagram in *Sets* indexed by $\mathcal{I}$ has the following, with the obvious maps to it, as a colimit:

$$\left\{ a \in \coprod_{i \in \mathcal{I}} A_i \right\} \;/\; (\,(a_i, i) \sim (a_j, a_j)\text{ if and only if there are } f : A_i \to A_k \text{ and } g : A_j \to A_k \text{ in the diagram for whic}$$

This idea applies to many categories whose objects can be interpreted as sets with additional structure (such as abelian groups, A-modules, groups, etc.). For example, in Example 2.4.5, each element of the colimit is an element of something upstairs, but you can't say in advance what it is an element of. For example, 17/125 is an element of the $5^{-3}\mathbb{Z}$ (or $5^{-4}\mathbb{Z}$, or later ones), but not $5^{-2}\mathbb{Z}$.

More generally, the colimit $\varinjlim M_i$ in the category of A-modules Mod_A can be described as follows. The set underlying $\varinjlim M_i$ is defined as in Exercise 2.4.C. To add the elements $m_i \in M_i$ and $m_j \in M_j$, choose an $\ell \in \mathcal{I}$ with arrows $u : i \to \ell$ and $v : j \to \ell$, and then define the sum of m_i and m_j to be $F(u)(m_i) + F(v)(m_j) \in M_\ell$. The element $m_i \in M_i$ is 0 if and only if there is some arrow $u : i \to k$ for which $F(u)(m_i) = 0$, i.e. if it becomes 0 "later in the diagram". Last, multiplication by an element of A is defined in the obvious way.

2.4.D. EXERCISE. Verify that the A-module described above is indeed the colimit. (Make sure you verify that addition is well-defined, i.e. is independent of the choice of representatives m_i and m_j, the choice of ℓ, and the choice of arrows u and v. Similarly, make sure that scalar multiplication is well-defined.)

2.4.E. USEFUL EXERCISE (LOCALIZATION AS A COLIMIT). Generalize Exercise 2.4.B(a) to interpret localization of an integral domain as a colimit over a filtered set: suppose S is a multiplicative set of A, and interpret $S^{-1}A = \varinjlim \frac{1}{s}A$ where the limit is over $s \in S$, and in the category of A-modules. (Aside: Can you make some version of this work even if A isn't an integral domain, e.g. $S^{-1}A = \varinjlim A_s$? This will work in the category of A-algebras.)

A variant of this construction works without the filtered condition, if you have another means of "connecting elements in different objects of your diagram". For example:

2.4.F. EXERCISE: COLIMITS OF A-MODULES WITHOUT THE FILTERED CONDITION. Suppose you are given a diagram of A-modules indexed by $\mathcal{I}$: $F : \mathcal{I} \to Mod_A$, where we let $M_i := F(i)$. Show that the colimit is $\oplus_{i \in \mathcal{I}} M_i$ modulo the relations $m_i - F(n)(m_i)$ for every $n : i \to j$ in $\mathcal{I}$ (i.e. for every arrow in the diagram). (Somewhat more precisely: "modulo" means "quotiented by the submodule generated by".)

2.4.7. Summary. One useful thing to informally keep in mind is the following. In a category where the objects are "set-like", an element of a limit can be thought of as an element in each object in the diagram, that are "compatible" (Exercise 2.4.A). And an element of a colimit can be thought of ("has a representative that is") an element of a single object in the diagram (Exercise 2.4.C). Even though the definitions of limit and colimit are the same, just with arrows reversed, these interpretations are quite different.

2.4.8. *Joke.* A comathematician is a system for turning ffee into cotheorems.

2.5 Adjoints

We next come to a very useful construction closely related to universal properties. Just as a universal property "essentially" (up to unique isomorphism) determines an object in a category (assuming such an object exists), "adjoints" essentially determine a functor (again, assuming it exists). Two *covariant* functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are **adjoint** if there is a natural bijection for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

(2.5.0.1) $$\tau_{AB} : \mathrm{Mor}_{\mathcal{B}}(F(A), B) \to \mathrm{Mor}_{\mathcal{A}}(A, G(B)).$$

We say that (F, G) form an **adjoint pair**, and that F is **left-adjoint** to G (and G is **right-adjoint** to F). By "natural" we mean the following. For all $f : A \to A'$ in $\mathcal{A}$, we require

(2.5.0.2)
$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{B}}(F(A'), B) \arrow[r, "Ff^*"] \arrow[d, "\tau_{A'B}"] & \mathrm{Mor}_{\mathcal{B}}(F(A), B) \arrow[d, "\tau_{AB}"] \\
\mathrm{Mor}_{\mathcal{A}}(A', G(B)) \arrow[r, "f^*"] & \mathrm{Mor}_{\mathcal{A}}(A, G(B))
\end{tikzcd}$$

to commute, and for all $g : B \to B'$ in $\mathcal{B}$ we want a similar commutative diagram to commute. (Here f^* is the map induced by $f : A \to A'$, and Ff^* is the map induced by $Ff : F(A) \to F(A')$.)

2.5.A. EXERCISE. Write down what this diagram should be. (Hint: do it by extending diagram (2.5.0.2) above.)

2.5.B. EXERCISE. Show that the map τ_{AB} (2.5.0.1) has the following properties. For each A there is a map $\eta_A : A \to GF(A)$ so that for any $g : F(A) \to B$, the corresponding $\tau_{AB}(g) : A \to G(B)$ is given by the composition

$$A \xrightarrow{\eta_A} GF(A) \xrightarrow{Gg} G(B).$$

Similarly, there is a map $\epsilon_B : FG(B) \to B$ for each B so that for any $f : A \to G(B)$, the corresponding map $\tau_{AB}^{-1}(f) : F(A) \to B$ is given by the composition

$$F(A) \xrightarrow{Ff} FG(B) \xrightarrow{\epsilon_B} B.$$

Here is an example of an adjoint pair.

2.5.C. EXERCISE. Suppose M, N, and P are A-modules. Describe a bijection $\mathrm{Hom}_A(M \otimes_A N, P) \leftrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$. (Hint: try to use the universal property.)

2.5.D. EXERCISE. Show that $\cdot \otimes_A N$ and $\mathrm{Hom}_A(N, \cdot)$ are adjoint functors.

2.5.1. ⋆ *Fancier remarks we won't use.* You can check that the left adjoint determines the right adjoint up to natural isomorphism, and vice versa. The maps η_A and ϵ_B of Exercise 2.5.B are called the **unit** and **counit** of the adjunction. This leads to a different characterization of adjunction. Suppose functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are given, along with natural transformations $\eta : \mathrm{id}_{\mathcal{A}} \to GF$ and $\epsilon : FG \to \mathrm{id}_{\mathcal{B}}$ with the property that $G\epsilon \circ \eta G = \mathrm{id}_G$ (for each $B \in \mathcal{B}$, the composition of $\eta_{G(B)} : G(B) \to GFG(B)$ and $G(\epsilon_B) : GFG(B) \to G(B)$ is the identity) and $\epsilon F \circ F\eta = \mathrm{id}_F$. Then you can check that F is left adjoint to G. These facts aren't hard to check, so if you want to use them, you should verify everything for yourself.

2.5.2. *Examples from other fields.* For those familiar with representation theory: Frobenius reciprocity may be understood in terms of adjoints. Suppose V is a finite-dimensional representation of a finite group G, and W is a representation of a subgroup $H < G$. Then induction and restriction are an adjoint pair $(\mathrm{Ind}_H^G, \mathrm{Res}_H^G)$ between the category of G-modules and the category of H-modules.

Topologists' favorite adjoint pair may be the suspension functor and the loop space functor.

2.5.3. Example: groupification. Here is another motivating example: getting an abelian group from an abelian semigroup. (An **abelian semigroup** is just like an abelian group, except you don't require an identity or an inverse. Morphisms of abelian semigroups are maps of sets preserving the binary operation.) One example is the non-negative integers $0, 1, 2, \ldots$ under addition. Another is the positive integers under multiplication $1, 2, \ldots$. From an abelian semigroup, you can create an abelian group. Here is a formalization of that notion. A **groupification** of a semigroup S is a map of semigroups $\pi : S \to G$ such that G is a group, and any map of semigroups from S to a *group* G' factors *uniquely* through G.

$$\begin{tikzcd} S \arrow[r, "\pi"] \arrow[dr] & G \arrow[d, dotted, "\exists!"] \\ & G' \end{tikzcd}$$

2.5.E. EXERCISE. Construct groupification H from the category of non-empty abelian semigroups to the category of abelian groups. (One possibility of a construction: given an abelian semigroup S, the elements of its groupification $H(S)$ are ordered pairs $(a, b) \in S \times S$, which you may think of as $a - b$, with the equivalence that $(a, b) \sim (c, d)$ if $a + d + e = b + c + e$ for some $e \in S$. Describe addition in this group, and show that it satisfies the properties of an abelian group. Describe the semigroup map $S \to H(S)$.) Let F be the forgetful functor from the category of abelian groups *Ab* to the category of abelian semigroups. Show that H is left-adjoint to F.

(Here is the general idea for experts: We have a full subcategory of a category. We want to "project" from the category to the subcategory. We have

$$\mathrm{Mor}_{\text{category}}(S, H) = \mathrm{Mor}_{\text{subcategory}}(G, H)$$

automatically; thus we are describing the left adjoint to the forgetful functor. How the argument worked: we constructed something which was in the smaller category, which automatically satisfies the universal property.)

2.5.F. EXERCISE (A GROUP IF GROUPIFIED BY ITSELF). Show that if a semigroup is *already* a group then the identity morphism is the groupification. (More correct: the identity morphism is *a* groupification.)

2.5.G. EXERCISE. The purpose of this exercise is to give you some practice with "adjoints of forgetful functors", the means by which we get groups from semigroups, and sheaves from presheaves. Suppose A is a ring, and S is a multiplicative subset. Then $S^{-1}A$-modules are a fully faithful subcategory (§2.2.15) of the category of A-modules (via the obvious inclusion $\mathit{Mod}_{S^{-1}A} \hookrightarrow \mathit{Mod}_A$). Then $\mathit{Mod}_A \to \mathit{Mod}_{S^{-1}A}$ can be interpreted as an adjoint to the forgetful functor $\mathit{Mod}_{S^{-1}A} \to \mathit{Mod}_A$. Figure out the correct statement, and prove that it holds.

(Here is the larger story. Every $S^{-1}A$-module is an A-module, and this is an injective map, so we have a covariant forgetful functor $F : \mathit{Mod}_{S^{-1}A} \to \mathit{Mod}_A$. In fact this is a fully faithful functor: it is injective on objects, and the morphisms between any two $S^{-1}A$-modules *as A-modules* are just the same when they are considered as $S^{-1}A$-modules. Then there is a functor $G : \mathit{Mod}_A \to \mathit{Mod}_{S^{-1}A}$, which might reasonably be called "localization with respect to S", which is left-adjoint to the forgetful functor. Translation: If M is an A-module, and N is an $S^{-1}A$-module, then $\mathrm{Mor}(GM, N)$ (morphisms as $S^{-1}A$-modules, which are the same as morphisms as A-modules) are in natural bijection with $\mathrm{Mor}(M, FN)$ (morphisms as A-modules).)

Here is a table of adjoints that will come up for us.

situation	category $\mathcal{A}$	category $\mathcal{B}$	left-adjoint $F : \mathcal{A} \to \mathcal{B}$	right-adjoint $G : \mathcal{B} \to \mathcal{A}$
A-modules (Ex. 2.5.D)			$\cdot \otimes_A N$	$\mathrm{Hom}_A(N, \cdot)$
ring maps $A \to B$ (e.g. Ex. 2.5.G)	Mod_A	Mod_B	$\cdot \otimes_A B$ (extension of scalars)	forgetful (restriction of scalars)
(pre)sheaves on a topological space X (Ex. 3.4.L)	presheaves on X	sheaves on X	sheafification	forgetful
(semi)groups (§2.5.3)	semigroups	groups	groupification	forgetful
sheaves, $f : X \to Y$ (Ex. 3.6.B)	sheaves on Y	sheaves on X	f^{-1}	f_*
sheaves of abelian groups or $\mathcal{O}$-modules, open immersions $f : U \hookrightarrow Y$ (Ex. 3.6.G)	sheaves on U	sheaves on Y	$f_!$	f^{-1}
quasicoherent sheaves, $f : X \to Y$ (Prop. 17.3.5)	quasicoherent sheaves on Y	quasicoherent sheaves on X	f^*	f_*

Other examples will also come up, such as the adjoint pair $(\sim, \Gamma_\bullet)$ between graded modules over a graded ring, and quasicoherent sheaves on the corresponding projective scheme (§16.4).

2.5.4. *Useful comment for experts.* One last comment only for people who have seen adjoints before: If (F, G) is an adjoint pair of functors, then F commutes with colimits, and G commutes with limits. Also, limits commute with limits and colimits commute with colimits. We will prove these facts (and a little more) in §2.6.11.

2.6 (Co-)kernels, and exact sequences (an introduction to abelian categories)

The introduction of the digit 0 or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps...

— Alexander Grothendieck

Since learning linear algebra, you have been familiar with the notions and behaviors of kernels, cokernels, etc. Later in your life you saw them in the category of abelian groups, and later still in the category of A-modules. Each of these notions generalizes the previous one.

We will soon define some new categories (certain sheaves) that will have familiar-looking behavior, reminiscent of that of modules over a ring. The notions of kernels, cokernels, images, and more will make sense, and they will behave "the way we expect" from our experience with modules. This can be made precise through the notion of an *abelian category*. Abelian categories are the right general setting in which one can do "homological algebra", in which notions of kernel, cokernel, and so on are used, and one can work with complexes and exact sequences.

We will see enough to motivate the definitions that we will see in general: monomorphism (and subobject), epimorphism, kernel, cokernel, and image. But in these notes we will avoid having to show that they behave "the way we expect" in a general abelian category because the examples we will see are directly interpretable in terms of modules over rings. In particular, it is not worth memorizing the definition of abelian category.

Two central examples of an abelian category are the category *Ab* of abelian groups, and the category Mod_A of A-modules. The first is a special case of the second (just take $A = \mathbb{Z}$). As we give the definitions, you should verify that Mod_A is an abelian category.

We first define the notion of *additive category*. We will use it only as a stepping stone to the notion of an abelian category.

2.6.1. Definition. A category $\mathcal{C}$ is said to be **additive** if it satisfies the following properties.

- Ad1. For each $A, B \in \mathcal{C}$, $\operatorname{Mor}(A, B)$ is an abelian group, such that composition of morphisms distributes over addition. (You should think about what this means — it translates to two distinct statements).
- Ad2. $\mathcal{C}$ has a zero object, denoted 0. (This is an object that is simultaneously an initial object and a final object, Definition 2.3.2.)
- Ad3. It has products of two objects (a product $A \times B$ for any pair of objects), and hence by induction, products of any finite number of objects.

In an additive category, the morphisms are often called homomorphisms, and Mor is denoted by Hom. In fact, this notation Hom is a good indication that you're working in an additive category. A functor between additive categories preserving the additive structure of Hom, is called an **additive functor**.

2.6.2. *Remarks.* It is a consequence of the definition of additive category that finite direct products are also finite direct sums (coproducts) — the details don't matter to us. The symbol $\oplus$ is used for this notion. Also, it is quick to show that additive functors send zero objects to zero objects (show that a is a 0-object if and only if $\mathrm{id}_a = 0_a$; additive functors preserve both id and 0), and preserves products.

One motivation for the name 0-object is that the 0-morphism in the abelian group $\operatorname{Hom}(A, B)$ is the composition $A \to 0 \to B$.

Real (or complex) Banach spaces are an example of an additive category. The category of free A-modules is another. The category of A-modules Mod_A is also an example, but it has even more structure, which we now formalize as an example of an abelian category.

2.6.3. Definition. Let $\mathcal{C}$ be an additive category. A **kernel** of a morphism $f : B \to C$ is a map $i : A \to B$ such that $f \circ i = 0$, and that is universal with respect to this property. Diagramatically:

$$\begin{tikzcd}
Z \arrow[d, dotted, "\exists!"'] \arrow[dr] \arrow[drr, "0"] & & \\
A \arrow[r, "i"] \arrow[rr, bend right, "0"'] & B \arrow[r, "f"] & C
\end{tikzcd}$$

(Note that the kernel is not just an object; it is a morphism of an object to B.) Hence it is unique up to unique isomorphism by universal property nonsense. A **cokernel** is defined dually by reversing the arrows — do this yourself. The kernel of $f : B \to C$ is the limit (§2.4) of the diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

and similarly the cokernel is a colimit.

If $i : A \to B$ is a monomorphism, then we say that A is a **subobject** of B, where the map i is implicit. Dually, there is the notion of **quotient object**, defined dually to subobject.

An **abelian category** is an additive category satisfying three additional properties.

(1) Every map has a kernel and cokernel.
(2) Every monomorphism is the kernel of its cokernel.
(3) Every epimorphism is the cokernel of its kernel.

It is a nonobvious (and imprecisely stated) fact that every property you want to be true about kernels, cokernels, etc. follows from these three. (Warning: in part of the literature, additional hypotheses are imposed as part of the definition.)

The **image** of a morphism $f : A \to B$ is defined as $\operatorname{im}(f) = \ker(\operatorname{coker} f)$. The morphism $f : A \to B$ factors uniquely through $\operatorname{im} f \to B$, and $A \to \operatorname{im} f$ is an epimorphism, and is a cokernel of $\ker f \to A$. The reader may want to verify this as an exercise. It is unique up to unique isomorphism. The cokernel of a monomorphism is called the **quotient**. The quotient of a monomorphism $A \to B$ is often denoted B/A (with the map from B implicit).

We will leave the foundations of abelian categories untouched. The key thing to remember is that if you understand kernels, cokernels, images and so on in the category of modules over a ring Mod_A, you can manipulate objects in any abelian category. This is made precise by Freyd-Mitchell Embedding Theorem. (The Freyd-Mitchell Embedding Theorem: If $\mathcal{A}$ is an abelian category such that $\operatorname{Hom}(a, a')$ is a set for all $a, a' \in \mathcal{A}$, then there is a ring A and an exact, fully faithful functor from $\mathcal{A}$ into Mod_A, which embeds $\mathcal{A}$ as a full subcategory. A proof is sketched in [**W**, §1.6], and references to a complete proof are given there. A proof is also given in [**KS**, §9.7]. The upshot is that to prove something about a diagram in some abelian category, we may assume that it is a diagram of modules over some ring, and we may then "diagram-chase" elements. Moreover, any fact about kernels, cokernels, and so on that holds in Mod_A holds in any abelian category.) However, the abelian categories we will come across will obviously be related to modules, and our intuition will clearly carry over, so we needn't invoke a theorem whose proof we haven't read. For example, we will show that sheaves of abelian groups on a topological space X form an abelian category (§3.5), and the interpretation in terms of "compatible germs" will connect notions of kernels, cokernels etc. of sheaves of abelian groups to the corresponding notions of abelian groups.

2.6.4. Complexes, exactness, and homology.

We say a sequence

$$\cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \cdots \tag{2.6.4.1}$$

is a **complex** at B if $g \circ f = 0$, and is **exact** at B if $\ker g = \operatorname{im} f$. A sequence is a complex if it is a complex at each (internal) term. (For example: $0 \longrightarrow A \longrightarrow 0$ is a complex if and only if $A = 0$; $0 \longrightarrow A \xrightarrow{f} B$ is a complex if and only if f is a monomorphism; and $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$ is a complex if and only if f is an isomorphism.) An exact sequence with five terms, the first and last of which are 0, is a **short exact sequence**. Note that $A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$ being exact is equivalent to describing C as a cokernel of f (with a similar statement for $0 \longrightarrow A \longrightarrow B \xrightarrow{g} C$).

If you would like practice in playing with these notions before thinking about homology, you can prove the Snake Lemma (stated in Example 2.7.5, with a stronger version in Exercise 2.7.B), or the Five Lemma (stated in Example 2.7.6, with a stronger version in Exercise 2.7.C). (I would do this in the category of A-modules, but see [**KS**, Lem. 12.1.1, Lem. 8.3.13] for proofs in general.)

If (2.6.4.1) is a complex, then its **homology** (often denoted H) is $\ker g / \operatorname{im} f$. We say that the $\ker g$ are the **cycles**, and $\operatorname{im} f$ are the **boundaries** (so homology is "cycles mod boundaries"). If the complex is indexed in decreasing order, the indices are often written as subscripts, and H_i is the homology at $A_{i+1} \to A_i \to A_{i-1}$. If the complex is indexed in increasing order, the indices are often written as superscripts, and the homology H^i at $A^{i-1} \to A^i \to A^{i+1}$ is often called **cohomology**.

An exact sequence

$$A^\bullet : \qquad \cdots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots \tag{2.6.4.2}$$

can be "factored" into short exact sequences

$$0 \longrightarrow \ker f^i \longrightarrow A^i \longrightarrow \ker f^{i+1} \longrightarrow 0$$

which is helpful in proving facts about long exact sequences by reducing them to facts about short exact sequences.

More generally, if (2.6.4.2) is assumed only to be a complex, then it can be "factored" into short exact sequences.

$$0 \longrightarrow \ker f^i \longrightarrow A^i \longrightarrow \operatorname{im} f^i \longrightarrow 0 \tag{2.6.4.3}$$

$$0 \longrightarrow \operatorname{im} f^{i-1} \longrightarrow \ker f^i \longrightarrow H^i(A^\bullet) \longrightarrow 0$$

2.6.A. EXERCISE. Describe exact sequences

$$0 \longrightarrow \operatorname{im} f^i \longrightarrow A^{i+1} \longrightarrow \operatorname{coker} f^i \longrightarrow 0 \tag{2.6.4.4}$$

$$0 \longrightarrow H^i(A^\bullet) \longrightarrow \operatorname{coker} f^{i-1} \longrightarrow \operatorname{im} f^i \longrightarrow 0$$

(These are somehow dual to (2.6.4.3). In fact in some mirror universe this might have been given as the standard definition of homology.)

2.6.B. EXERCISE. Suppose

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of finite-dimensional k-vector spaces (often called $A^\bullet$ for short). Show that $\sum(-1)^i \dim A^i = \sum(-1)^i h^i(A^\bullet)$. (Recall that $h^i(A^\bullet) = \dim \ker(d^i)/\operatorname{im}(d^{i-1})$.) In particular, if $A^\bullet$ is exact, then $\sum(-1)^i \dim A^i = 0$. (If you haven't dealt much with cohomology, this will give you some practice.)

2.6.C. IMPORTANT EXERCISE. Suppose $\mathcal{C}$ is an abelian category. Define the category $Com_{\mathcal{C}}$ as follows. The objects are infinite complexes

$$A^\bullet: \qquad \cdots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots$$

in $\mathcal{C}$, and the morphisms $A^\bullet \to B^\bullet$ are commuting diagrams

$$\begin{tikzcd} A^\bullet : \arrow[d] & \cdots \arrow[r] & A^{i-1} \arrow[r, "f^{i-1}"] \arrow[d] & A^i \arrow[r, "f^i"] \arrow[d] & A^{i+1} \arrow[r, "f^{i+1}"] \arrow[d] & \cdots \\ B^\bullet : & \cdots \arrow[r] & B^{i-1} \arrow[r, "g^{i-1}"] & B^i \arrow[r, "g^i"] & B^{i+1} \arrow[r, "g^{i+1}"] & \cdots \end{tikzcd} \tag{2.6.4.5}$$

Show that $Com_{\mathcal{C}}$ is an abelian category. (Feel free to deal with the special case Mod_A.)

Essentially the same argument shows that the functor category $\mathcal{C}^{\mathcal{I}}$ is an abelian category for any small category $\mathcal{I}$ and any abelian category $\mathcal{C}$. This immediately implies that the category of presheaves on at topological space X with values in an abelian category $\mathcal{C}$ is automatically an abelian category.

2.6.D. IMPORTANT EXERCISE. Show that (2.6.4.5) induces a map of homology $H^i(A^\bullet) \to H^i(B^\bullet)$. (Again, feel free to deal with the special case Mod_A.)

We will later define when two maps of complexes are *homotopic* (§24.1), and show that homotopic maps induce isomorphisms on cohomology (Exercise 24.1.A), but we won't need that any time soon.

2.6.5. Theorem (Long exact sequence). — *A short exact sequence of complexes*

$$\begin{tikzcd}
0^\bullet : \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
A^\bullet : \arrow[d] & \cdots \arrow[r] & A^{i-1} \arrow[r, "f^{i-1}"] \arrow[d] & A^i \arrow[r, "f^i"] \arrow[d] & A^{i+1} \arrow[r, "f^{i+1}"] \arrow[d] & \cdots \\
B^\bullet : \arrow[d] & \cdots \arrow[r] & B^{i-1} \arrow[r, "g^{i-1}"] \arrow[d] & B^i \arrow[r, "g^i"] \arrow[d] & B^{i+1} \arrow[r, "g^{i+1}"] \arrow[d] & \cdots \\
C^\bullet : \arrow[d] & \cdots \arrow[r] & C^{i-1} \arrow[r, "h^{i-1}"] \arrow[d] & C^i \arrow[r, "h^i"] \arrow[d] & C^{i+1} \arrow[r, "h^{i+1}"] \arrow[d] & \cdots \\
0^\bullet : & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

induces a **long exact sequence in cohomology**

$$\begin{tikzcd}
 & \cdots \arrow[r] & H^{i-1}(C^\bullet) \arrow[r] & {} \\
H^i(A^\bullet) \arrow[r] & H^i(B^\bullet) \arrow[r] & H^i(C^\bullet) \arrow[r] & {} \\
H^{i+1}(A^\bullet) \arrow[r] & \cdots & &
\end{tikzcd}$$

(This requires a definition of the *connecting homomorphism* $H^{i-1}(C^\bullet) \to H^i(A^\bullet)$, which is natural in an appropriate sense.) For a concise proof in the case of complexes of modules, and a discussion of how to show this in general, see [**W**, §1.3]. It will also come out of our discussion of spectral sequences as well (again, in the category of modules over a ring), see Exercise 2.7.F, but this is a somewhat perverse way of proving it. For a proof in general, see [**KS**, Theorem 12.3.3].

2.6.6. *Exactness of functors.* If $F : \mathcal{A} \to \mathcal{B}$ is a covariant additive functor from one abelian category to another, we say that F is **right-exact** if the exactness of

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

in $\mathcal{A}$ implies that

$$F(A') \longrightarrow F(A) \longrightarrow F(A'') \longrightarrow 0$$

is also exact. Dually, we say that F is **left-exact** if the exactness of

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \qquad \text{implies}$$

$$0 \longrightarrow F(A') \longrightarrow F(A) \longrightarrow F(A'') \qquad \text{is exact.}$$

A contravariant functor is **left-exact** if the exactness of

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow 0 \qquad \text{implies}$$

$$0 \longrightarrow F(A'') \longrightarrow F(A) \longrightarrow F(A') \qquad \text{is exact.}$$

The reader should be able to deduce what it means for a contravariant functor to be **right-exact**.

A covariant or contravariant functor is **exact** if it is both left-exact and right-exact.

2.6.E. EXERCISE. Suppose F is an exact functor. Show that applying F to an exact sequence preserves exactness. For example, if F is covariant, and $A' \to A \to A''$ is exact, then $FA' \to FA \to FA''$ is exact. (This will be generalized in Exercise 2.6.H(c).)

2.6.F. EXERCISE. Suppose A is a ring, $S \subset A$ is a multiplicative subset, and M is an A-module.
(a) Show that localization of A-modules $Mod_A \to Mod_{S^{-1}A}$ is an exact covariant functor.
(b) Show that $\cdot \otimes M$ is a right-exact covariant functor $Mod_A \to Mod_A$. (This is a repeat of Exercise 2.3.H.)
(c) Show that $\operatorname{Hom}(M, \cdot)$ is a left-exact covariant functor $Mod_A \to Mod_A$. If $\mathcal{C}$ is any abelian category, and $C \in \mathcal{C}$, show that $\operatorname{Hom}(C, \cdot)$ is a left-exact covariant functor $\mathcal{C} \to Ab$.
(d) Show that $\operatorname{Hom}(\cdot, M)$ is a left-exact contravariant functor $Mod_A \to Mod_A$. If $\mathcal{C}$ is any abelian category, and $C \in \mathcal{C}$, show that $\operatorname{Hom}(\cdot, C)$ is a left-exact covariant functor $\mathcal{C} \to Ab$.

2.6.G. EXERCISE. Suppose M is a **finitely presented A-module**: M has a finite number of generators, and with these generators it has a finite number of relations; or usefully equivalently, fits in an exact sequence

$$A^{\oplus q} \to A^{\oplus p} \to M \to 0 \tag{2.6.6.1}$$

Use (2.6.6.1) and the left-exactness of Hom to describe an isomorphism

$$S^{-1} \operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

(You might be able to interpret this in light of a variant of Exercise 2.6.H below, for left-exact contravariant functors rather than right-exact covariant functors.)

2.6.7. *Example:* Hom *doesn't always commute with localization.* In the language of Exercise 2.6.G, take $A = N = \mathbb{Z}$, $M = \mathbb{Q}$, and $S = \mathbb{Z} \setminus \{0\}$.

2.6.8. $\star$ **Two useful facts in homological algebra.**

We now come to two (sets of) facts I wish I had learned as a child, as they would have saved me lots of grief. They encapsulate what is best and worst of abstract nonsense. The statements are so general as to be nonintuitive. The proofs are very short. They generalize some specific behavior it is easy to prove in an ad hoc basis. Once they are second nature to you, many subtle facts will become obvious to you as special cases. And you will see that they will get used (implicitly or explicitly) repeatedly.

2.6.9. ⋆ *Interaction of homology and (right/left-)exact functors.*
You might wait to prove this until you learn about cohomology in Chapter 20, when it will first be used in a serious way.

2.6.H. IMPORTANT EXERCISE (THE FHHF THEOREM). This result can take you far, and perhaps for that reason it has sometimes been called the Fernbahnhof (FernbaHnHoF) Theorem. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a covariant functor of abelian categories. Suppose $C^\bullet$ is a complex in $\mathcal{A}$.

(a) *(F right-exact yields* $FH^\bullet \longrightarrow H^\bullet F$ *)* If F is right-exact, describe a natural morphism $FH^\bullet \to H^\bullet F$. (More precisely, for each i, the left side is F applied to the cohomology at piece i of $C^\bullet$, while the right side is the cohomology at piece i of $FC^\bullet$.)
(b) *(F left-exact yields* $FH^\bullet \longleftarrow H^\bullet F$ *)* If F is left-exact, describe a natural morphism $H^\bullet F \to FH^\bullet$.
(c) *(F exact yields* $FH^\bullet \longleftrightarrow H^\bullet F$ *)* If F is exact, show that the morphisms of (a) and (b) are inverses and thus isomorphisms.

Hint for (a): use $C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \operatorname{coker} d^i \longrightarrow 0$ to give an isomorphism $F \operatorname{coker} d^i \cong \operatorname{coker} Fd^i$. Then use the first line of (2.6.4.4) to give a epimorphism $F \operatorname{im} d^i \twoheadrightarrow \operatorname{im} Fd^i$. Then use the second line of (2.6.4.4) to give the desired map $FH^iC^\bullet \longrightarrow H^iFC^\bullet$. While you are at it, you may as well describe a map for the fourth member of the quartet $\{\ker, \operatorname{coker}, \operatorname{im}, H\}$: $F \ker d^i \longrightarrow \ker Fd^i$.

2.6.10. If this makes your head spin, you may prefer to think of it in the following specific case, where both $\mathcal{A}$ and $\mathcal{B}$ are the category of A-modules, and F is $\cdot \otimes N$ for some fixed N-module. Your argument in this case will translate without change to yield a solution to Exercise 2.6.H(a) and (c) in general. If $\otimes N$ is exact, then N is called a **flat** A-module. (The notion of flatness will turn out to be very important, and is discussed in detail in Chapter 25.)

For example, localization is exact (Exercise 2.6.F(a)), so $S^{-1}A$ is a *flat* A-algebra for all multiplicative sets S. Thus taking cohomology of a complex of A-modules commutes with localization — something you could verify directly.

2.6.11. ⋆ *Interaction of adjoints, (co)limits, and (left- and right-) exactness.*
A surprising number of arguments boil down to the statement:
Limits commute with limits and right-adjoints. In particular, because kernels are limits, both right-adjoints and limits are left exact.
as well as its dual:
Colimits commute with colimits and left-adjoints. In particular, because cokernels are colimits, both left-adjoints and colimits are right exact.
These statements were promised in §2.5.4. The latter has a useful extension:
In an abelian category, colimits over filtered index categories are exact.
("Filtered" was defined in §2.4.6.) If you want to use these statements (for example, later in these notes), you will have to prove them. Let's now make them precise.

2.6.I. EXERCISE (KERNELS COMMUTE WITH LIMITS). Suppose $\mathcal{C}$ is an abelian category, and $a : \mathcal{I} \to \mathcal{C}$ and $b : \mathcal{I} \to \mathcal{C}$ are two diagrams in $\mathcal{C}$ indexed by $\mathcal{I}$. For convenience, let $A_i = a(i)$ and $B_i = b(i)$ be the objects in those two diagrams. Let $h_i : A_i \to B_i$ be maps commuting with the maps in the diagram. (Translation: h is a natural transformation of functors $a \to b$, see §2.2.21.) Then the $\ker h_i$ form another diagram in $\mathcal{C}$ indexed by $\mathcal{I}$. Describe a canonical isomorphism $\varprojlim \ker h_i \cong \ker(\varprojlim A_i \to \varprojlim B_i)$.

2.6.J. EXERCISE. Make sense of the statement that "limits commute with limits" in a general category, and prove it. (Hint: recall that kernels are limits. The previous exercise should be a corollary of this one.)

2.6.12. Proposition (right-adjoints commute with limits). — *Suppose $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ is a pair of adjoint functors. If $A = \varprojlim A_i$ is a limit in $\mathcal{D}$ of a diagram indexed by I, then $GA = \varprojlim GA_i$ (with the corresponding maps $GA \to GA_i$) is a limit in $\mathcal{C}$.*

Proof. We must show that $GA \to GA_i$ satisfies the universal property of limits. Suppose we have maps $W \to GA_i$ commuting with the maps of $\mathcal{I}$. We wish to show that there exists a unique $W \to GA$ extending the $W \to GA_i$. By adjointness of F and G, we can restate this as: Suppose we have maps $FW \to A_i$ commuting with the maps of $\mathcal{I}$. We wish to show that there exists a unique $FW \to A$ extending the $FW \to A_i$. But this is precisely the universal property of the limit. $\square$

Of course, the dual statements to Exercise 2.6.J and Proposition 2.6.12 hold by the dual arguments.

If F and G are additive functors between abelian categories, and (F, G) is an adjoint pair, then (as kernels are limits and cokernels are colimits) G is left-exact and F is right-exact.

2.6.K. EXERCISE. Show that in Mod_A, colimits over filtered index categories are exact. (Your argument will apply without change to any abelian category whose objects can be interpreted as "sets with additional structure".) Right-exactness follows from the above discussion, so the issue is left-exactness. (Possible hint: After you show that localization is exact, Exercise 2.6.F(a), or sheafification is exact, Exercise 3.5.D, in a hands-on way, you will be easily able to prove this. Conversely, if you do this exercise, those two will be easy.)

2.6.L. EXERCISE. Show that filtered colimits commute with homology in Mod_A. Hint: use the FHHF Theorem (Exercise 2.6.H), and the previous Exercise.

In light of Exercise 2.6.L, you may want to think about how limits (and colimits) commute with homology in general, and which way maps go. The statement of the FHHF Theorem should suggest the answer. (Are limits analogous to left-exact functors, or right-exact functors?) We won't directly use this insight.

2.6.13. $\star$ **Dreaming of derived functors.** When you see a left-exact functor, you should always dream that you are seeing the end of a long exact sequence. If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence in abelian category $\mathcal{A}$, and $F : \mathcal{A} \to \mathcal{B}$ is a left-exact functor, then

$$0 \to FM' \to FM \to FM''$$

is exact, and you should always dream that it should continue in some natural way. For example, the next term should depend only on M', call it R^1FM', and if it is zero, then $FM \to FM''$ is an epimorphism. This remark holds true for left-exact and contravariant functors too. In good cases, such a continuation exists, and is incredibly useful. We will discuss this in Chapter 24.

2.7 ⋆ Spectral sequences

Spectral sequences are a powerful book-keeping tool for proving things involving complicated commutative diagrams. They were introduced by Leray in the 1940's at the same time as he introduced sheaves. They have a reputation for being abstruse and difficult. It has been suggested that the name 'spectral' was given because, like spectres, spectral sequences are terrifying, evil, and dangerous. I have heard no one disagree with this interpretation, which is perhaps not surprising since I just made it up.

Nonetheless, the goal of this section is to tell you enough that you can use spectral sequences without hesitation or fear, and why you shouldn't be frightened when they come up in a seminar. What is perhaps different in this presentation is that we will use spectral sequences to prove things that you may have already seen, and that you can prove easily in other ways. This will allow you to get some hands-on experience for how to use them. We will also see them only in the special case of double complexes (which is the version by far the most often used in algebraic geometry), and not in the general form usually presented (filtered complexes, exact couples, etc.). See [**W**, Ch. 5] for more detailed information if you wish.

You should *not* read this section when you are reading the rest of Chapter 2. Instead, you should read it just before you need it for the first time. When you finally *do* read this section, you *must* do the exercises.

For concreteness, we work in the category Mod_A of module over a ring A. However, everything we say will apply in any abelian category. (And if it helps you feel secure, work instead in the category Vec_k of vector spaces over a field k.)

2.7.1. Double complexes.

A **double complex** is a collection of A-modules $E^{p,q}$ ($p, q \in \mathbb{Z}$), and "rightward" morphisms $d_{\to}^{p,q} : E^{p,q} \to E^{p+1,q}$ and "upward" morphisms $d_{\uparrow}^{p,q} : E^{p,q} \to E^{p,q+1}$. In the superscript, the first entry denotes the column number (the "x-coordinate"), and the second entry denotes the column number (the "y-coordinate"). (Warning: this is opposite to the convention for matrices.) The subscript is meant to suggest the direction of the arrows. We will always write these as $d_{\to}$ and $d_{\uparrow}$ and ignore the superscripts. We require that $d_{\to}$ and $d_{\uparrow}$ satisfy (a) $d_{\to}^2 = 0$, (b) $d_{\uparrow}^2 = 0$, and one more condition: (c) either $d_{\to}d_{\uparrow} = d_{\uparrow}d_{\to}$ (all the squares commute) or $d_{\to}d_{\uparrow} + d_{\uparrow}d_{\to} = 0$ (they all anticommute). Both come up in nature, and you can switch from one to the other by replacing $d_{\uparrow}^{p,q}$ with $(-1)^q d_{\uparrow}^{p,q}$. So I will assume that all the squares anticommute, but that you know how to turn the commuting case into this one. (You will see that there is no difference in the recipe, basically because the image and kernel of a homomorphism f equal the image and kernel respectively of $-f$.)

$$\begin{tikzcd}
E^{p,q+1} \arrow[r, "d_{\rightarrow}^{p,q+1}"] & E^{p+1,q+1} \\
E^{p,q} \arrow[u, "d_{\uparrow}^{p,q}"] \arrow[r, "d_{\rightarrow}^{p,q}"'] \arrow[ur, phantom, "\text{anticommutes}"] & E^{p,q+1} \arrow[u, "d_{\uparrow}^{p,q+1}"']
\end{tikzcd}$$

There are variations on this definition, where for example the vertical arrows go downwards, or some different subset of the $E^{p,q}$ are required to be zero, but I will leave these straightforward variations to you.

From the double complex we construct a corresponding (single) complex $E^\bullet$ with $E^k = \oplus_i E^{i,k-i}$, with $d = d_\rightarrow + d_\uparrow$. In other words, when there is a *single* superscript k, we mean a sum of the kth antidiagonal of the double complex. The single complex is sometimes called the **total complex**. Note that $d^2 = (d_\rightarrow + d_\uparrow)^2 = d_\rightarrow^2 + (d_\rightarrow d_\uparrow + d_\uparrow d_\rightarrow) + d_\uparrow^2 = 0$, so $E^\bullet$ is indeed a complex.

The **cohomology** of the single complex is sometimes called the **hypercohomology** of the double complex. We will instead use the phrase "cohomology of the double complex".

Our initial goal will be to find the cohomology of the double complex. You will see later that we secretly also have other goals.

A spectral sequence is a recipe for computing some information about the cohomology of the double complex. I won't yet give the full recipe. Surprisingly, this fragmentary bit of information is sufficent to prove lots of things.

2.7.2. Approximate Definition. A **spectral sequence** with **rightward orientation** is a sequence of tables or **pages** ${}_\rightarrow E_0^{p,q}, {}_\rightarrow E_1^{p,q}, {}_\rightarrow E_2^{p,q}, \dots$ $(p, q \in \mathbb{Z})$, where ${}_\rightarrow E_0^{p,q} = E^{p,q}$, along with a differential

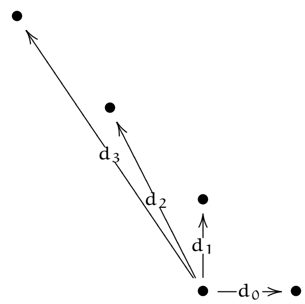
$${}_\rightarrow d_r^{p,q} : {}_\rightarrow E_r^{p,q} \rightarrow {}_\rightarrow E_r^{p-r+1,q+r}$$

with ${}_\rightarrow d_r^{p,q} \circ {}_\rightarrow d_r^{p-r,q+r-1} = 0$, and with an isomorphism of the cohomology of ${}_\rightarrow d_r$ at ${}_\rightarrow E_r^{p,q}$ (i.e. $\ker {}_\rightarrow d_r^{p,q} / \operatorname{im} {}_\rightarrow d_r^{p-r,q+r-1}$) with ${}_\rightarrow E_{r+1}^{p,q}$.

The orientation indicates that our 0th differential is the rightward one: $d_0 = d_\rightarrow$. The left subscript "$\rightarrow$" is usually omitted.

The order of the morphisms is best understood visually:

(2.7.2.1)



(the morphisms each apply to different pages). Notice that the map always is "degree 1" in terms of the grading of the single complex $E^\bullet$. (You should figure out what this informal statement really means.)

The actual definition describes what $E_r^{\bullet,\bullet}$ and $d_r^{\bullet,\bullet}$ really are, in terms of $E^{\bullet,\bullet}$. We will describe d_0, d_1, and d_2 below, and you should for now take on faith that this sequence continues in some natural way.

Note that $E_r^{p,q}$ is always a subquotient of the corresponding term on the 0th page $E_0^{p,q} = E^{p,q}$. In particular, if $E^{p,q} = 0$, then $E_r^{p,q} = 0$ for all r, so $E_r^{p,q} = 0$ unless $p, q \in \mathbb{Z}^{\geq 0}$.

Suppose now that $E^{\bullet,\bullet}$ is a **first quadrant double complex**, i.e. $E^{p,q} = 0$ for $p < 0$ or $q < 0$. Then for any fixed p, q, once r is sufficiently large, $E_{r+1}^{p,q}$ is computed from $(E_r^{\bullet,\bullet}, d_r)$ using the complex

$$0 \xrightarrow{d_r^{p-r+1,q+r}} E_r^{p,q} \xrightarrow{d_r^{p,q}} 0$$

and thus we have canonical isomorphisms

$$E_r^{p,q} \cong E_{r+1}^{p,q} \cong E_{r+2}^{p,q} \cong \cdots$$

We denote this module $E_\infty^{p,q}$. The same idea works in other circumstances, for example if the double complex is only nonzero in a finite number of rows — $E^{p,q} = 0$ unless $q_0 < q < q_1$. This will come up for example in the long exact sequence and mapping cone discussion (Exercises 2.7.F and 2.7.E below).

We now describe the first few pages of the spectral sequence explicitly. As stated above, the differential d_0 on $E_0^{\bullet,\bullet} = E^{\bullet,\bullet}$ is defined to be $d_\rightarrow$. The rows are complexes:

The 0th page E_0:
$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

and so E_1 is just the table of cohomologies of the rows. You should check that there are now vertical maps $d_1^{p,q} : E_1^{p,q} \to E_1^{p,q+1}$ of the row cohomology groups, induced by $d_\uparrow$, and that these make the columns into complexes. (This is essentially the fact that a map of complexes induces a map on homology.) We have "used up the horizontal morphisms", but "the vertical differentials live on".

The 1st page E_1:
$$\begin{tikzcd}
\bullet & \bullet & \bullet \\
\bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u]
\end{tikzcd}$$

We take cohomology of d_1 on E_1, giving us a new table, $E_2^{p,q}$. It turns out that there are natural morphisms from each entry to the entry two above and one to the left, and that the composition of these two is 0. (It is a very worthwhile exercise to work out how this natural morphism d_2 should be defined. Your argument may be reminiscent of the connecting homomorphism in the Snake Lemma 2.7.5 or in the long exact sequence in cohomology arising from a short exact sequence of complexes, Exercise 2.6.C. This is no coincidence.)

The 2nd page E_2:
$$\begin{tikzcd}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet \arrow[uul] & \bullet \arrow[uul]
\end{tikzcd}$$

This is the beginning of a pattern.

Then it is a theorem that there is a filtration of $H^k(E^\bullet)$ by $E_\infty^{p,q}$ where $p+q=k$. (We can't yet state it as an official **Theorem** because we haven't precisely defined the pages and differentials in the spectral sequence.) More precisely, there is a filtration

$$E_\infty^{0,k} \xhookrightarrow{E_\infty^{1,k-1}} \ ? \ \xhookrightarrow{E_\infty^{2,k-2}} \cdots \xhookrightarrow{E_\infty^{k,0}} H^k(E^\bullet) \tag{2.7.2.2}$$

where the quotients are displayed above each inclusion. (Here is a tip for remember which way the quotients are supposed to go. The later differentials point deeper and deeper into the filtration. Thus the entries in the direction of the later arrowheads are the subobjects, and the entries in the direction of the later "arrow-tails" are quotients. This tip has the advantage of being independent of the details of the spectral sequence, e.g. the "quadrant" or the orientation.)

We say that the spectral sequence ${}_{\rightarrow}E_{\bullet}^{\bullet,\bullet}$ **converges** to $H^{\bullet}(E^{\bullet})$. We often say that ${}_{\rightarrow}E_2^{\bullet,\bullet}$ (or any other page) abuts to $H^{\bullet}(E^{\bullet})$.

Although the filtration gives only partial information about $H^{\bullet}(E^{\bullet})$, sometimes one can find $H^{\bullet}(E^{\bullet})$ precisely. One example is if all $E_{\infty}^{i,k-i}$ are zero, or if all but one of them are zero (e.g. if $E_r^{\bullet,\bullet}$ has precisely one non-zero row or column, in which case one says that the spectral sequence **collapses** at the rth step, although we will not use this term). Another example is in the category of vector spaces over a field, in which case we can find the dimension of $H^k(E^{\bullet})$. Also, in lucky circumstances, E_2 (or some other small page) already equals E_{∞}.

2.7.A. EXERCISE: INFORMATION FROM THE SECOND PAGE. Show that $H^0(E^{\bullet}) = E_{\infty}^{0,0} = E_2^{0,0}$ and

$$0 \longrightarrow E_2^{0,1} \longrightarrow H^1(E^{\bullet}) \longrightarrow E_2^{1,0} \xrightarrow{d_2^{1,0}} E_2^{0,2} \longrightarrow H^2(E^{\bullet}).$$

2.7.3. *The other orientation.*

You may have observed that we could as well have done everything in the opposite direction, i.e. reversing the roles of horizontal and vertical morphisms. Then the sequences of arrows giving the spectral sequence would look like this (compare to (2.7.2.1)).

(2.7.3.1)
$$\begin{tikzcd}
\bullet & & & \\
\bullet \arrow[u] \arrow[r] \arrow[drr] \arrow[ddrrr] & \bullet & & \\
& & \bullet & \\
& & & \bullet
\end{tikzcd}$$

This spectral sequence is denoted ${}_{\uparrow}E_{\bullet}^{\bullet,\bullet}$ ("with the upwards orientation"). Then we would again get pieces of a filtration of $H^{\bullet}(E^{\bullet})$ (where we have to be a bit careful with the order with which ${}_{\uparrow}E_{\infty}^{p,q}$ corresponds to the subquotients — it in the opposite order to that of (2.7.2.2) for ${}_{\rightarrow}E_{\infty}^{p,q}$). Warning: in general there is no isomorphism between ${}_{\rightarrow}E_{\infty}^{p,q}$ and ${}_{\uparrow}E_{\infty}^{p,q}$.

In fact, this observation that we can start with either the horizontal or vertical maps was our secret goal all along. Both algorithms compute information about the same thing ($H^{\bullet}(E^{\bullet})$), and usually we don't care about the final answer — we often care about the answer we get in one way, and we get at it by doing the spectral sequence in the *other* way.

2.7.4. Examples.

We are now ready to see how this is useful. The moral of these examples is the following. In the past, you may have proved various facts involving various sorts of diagrams, by chasing elements around. Now, you will just plug them into a spectral sequence, and let the spectral sequence machinery do your chasing for you.

2.7.5. *Example: Proving the Snake Lemma.* Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

where the rows are exact in the middle (at B, C, D, G, H, I) and the squares commute. (Normally the Snake Lemma is described with the vertical arrows pointing downwards, but I want to fit this into my spectral sequence conventions.) We wish to show that there is an exact sequence

$$0 \to \ker \alpha \to \ker \beta \to \ker \gamma \to \operatorname{coker} \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \gamma \to 0. \tag{2.7.5.1}$$

We plug this into our spectral sequence machinery. We first compute the cohomology using the rightwards orientation, i.e. using the order (2.7.2.1). Then because the rows are exact, $E_1^{p,q} = 0$, so the spectral sequence has already converged: $E_\infty^{p,q} = 0$.

We next compute this "0" in another way, by computing the spectral sequence using the upwards orientation. Then ${}_{\uparrow}E_1^{\bullet,\bullet}$ (with its differentials) is:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker} \alpha \arrow[r] & \operatorname{coker} \beta \arrow[r] & \operatorname{coker} \gamma \arrow[r] & 0 \\
0 \arrow[r] & \ker \alpha \arrow[r] & \ker \beta \arrow[r] & \ker \gamma \arrow[r] & 0.
\end{tikzcd}$$

Then ${}_{\uparrow}E_2^{\bullet,\bullet}$ is of the form:

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & & & & & \\
& 0 \arrow[drr] & ?? \arrow[drr] & ? \arrow[drr] & ? & 0 & \\
& 0 & ? & ? \arrow[drr] & ?? \arrow[drr] & 0 & \\
& & & & & 0 & 0
\end{tikzcd}$$

We see that after ${}_{\uparrow}E_2$, all the terms will stabilize except for the double-question-marks — all maps to and from the single question marks are to and from 0-entries. And after ${}_{\uparrow}E_3$, even these two double-question-mark terms will stabilize. But in the end our complex must be the 0 complex. This means that in ${}_{\uparrow}E_2$, all the entries must be zero, except for the two double-question-marks, and these two must be isomorphic. This means that $0 \to \ker \alpha \to \ker \beta \to \ker \gamma$ and $\operatorname{coker} \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \gamma \to 0$ are both exact (that comes from the vanishing of the single-question-marks), and

$$\operatorname{coker}(\ker \beta \to \ker \gamma) \cong \ker(\operatorname{coker} \alpha \to \operatorname{coker} \beta)$$

is an isomorphism (that comes from the equality of the double-question-marks). Taken together, we have proved the exactness of (2.7.5.1), and hence the Snake

Lemma! (Notice: in the end we didn't really care about the double complex. We just used it as a prop to prove the snake lemma.)

Spectral sequences make it easy to see how to generalize results further. For example, if $A \to B$ is no longer assumed to be injective, how would the conclusion change?

2.7.B. UNIMPORTANT EXERCISE (GRAFTING EXACT SEQUENCES, A WEAKER VERSION OF THE SNAKE LEMMA). Extend the snake lemma as follows. Suppose we have a commuting diagram

$$\begin{tikzcd}
& 0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & A' \arrow[r] & \cdots \\
\cdots \arrow[r] & W \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "a"] & Y \arrow[r] \arrow[u, "b"] & Z \arrow[r] \arrow[u, "c"] & 0. \arrow[u]
\end{tikzcd}$$

where the top and bottom rows are exact. Show that the top and bottom rows can be "grafted together" to an exact sequence

$$\cdots \longrightarrow W \longrightarrow \ker a \longrightarrow \ker b \longrightarrow \ker c$$

$$\longrightarrow \operatorname{coker} a \longrightarrow \operatorname{coker} b \longrightarrow \operatorname{coker} c \longrightarrow A' \longrightarrow \cdots .$$

2.7.6. *Example: the Five Lemma.* Suppose

(2.7.6.1)
$$\begin{tikzcd}
F \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & J \\
A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & D \arrow[r] \arrow[u, "\delta"] & E \arrow[u, "\epsilon"]
\end{tikzcd}$$

where the rows are exact and the squares commute.

Suppose $\alpha, \beta, \delta, \epsilon$ are isomorphisms. We will show that γ is an isomorphism.

We first compute the cohomology of the total complex using the rightwards orientation (2.7.2.1). We choose this because we see that we will get lots of zeros. Then ${}_{\to}E_1^{\bullet,\bullet}$ looks like this:

$$\begin{tikzcd}
? & 0 & 0 & 0 & ? \\
? \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & ? \arrow[u]
\end{tikzcd}$$

Then ${}_{\to}E_2$ looks similar, and the sequence will converge by E_2, as we will never get any arrows between two non-zero entries in a table thereafter. We can't conclude that the cohomology of the total complex vanishes, but we can note that it vanishes in all but four degrees — and most important, it vanishes in the two degrees corresponding to the entries C and H (the source and target of γ).

We next compute this using the upwards orientation (2.7.3.1). Then ${}_{\uparrow}E_1$ looks like this:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

and the spectral sequence converges at this step. We wish to show that those two question marks are zero. But they are precisely the cohomology groups of the total complex that we just showed *were* zero — so we are done!

The best way to become comfortable with this sort of argument is to try it out yourself several times, and realize that it really is easy. So you should do the following exercises! Many can readily be done directly, but you should deliberately try to use this spectral sequence machinery in order to get practice and develop confidence.

2.7.C. EXERCISE: THE SUBTLE FIVE LEMMA. By looking at the spectral sequence proof of the Five Lemma above, prove a subtler version of the Five Lemma, where one of the isomorphisms can instead just be required to be an injection, and another can instead just be required to be a surjection. (I am deliberately not telling you which ones, so you can see how the spectral sequence is telling you how to improve the result.)

2.7.D. EXERCISE. If β and δ (in (2.7.6.1)) are injective, and α is surjective, show that γ is injective. Give the dual statement (whose proof is of course essentially the same).

2.7.E. EXERCISE (THE MAPPING CONE). Suppose $\mu : A^\bullet \to B^\bullet$ is a morphism of complexes. Suppose $C^\bullet$ is the single complex associated to the double complex $A^\bullet \to B^\bullet$. ($C^\bullet$ is called the *mapping cone* of μ.) Show that there is a long exact sequence of complexes:

$$\cdots \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots .$$

(There is a slight notational ambiguity here; depending on how you index your double complex, your long exact sequence might look slightly different.) In particular, we will use the fact that μ induces an isomorphism on cohomology if and only if the mapping cone is exact. (We won't use it until the proof of Theorem 20.2.4.)

2.7.F. EXERCISE. Use spectral sequences to show that a short exact sequence of complexes gives a long exact sequence in cohomology (Exercise 2.6.C). (This is a generalization of Exercise 2.7.E.)

The Grothendieck (or composition of functor) spectral sequence (Exercise 24.3.D) will be an important example of a spectral sequence that specializes in a number of useful ways.

You are now ready to go out into the world and use spectral sequences to your heart's content!

2.7.7. ⋆⋆ **Complete definition of the spectral sequence, and proof.**

You should most definitely not read this section any time soon after reading the introduction to spectral sequences above. Instead, flip quickly through it to convince yourself that nothing fancy is involved.

We consider the rightwards orientation. The upwards orientation is of course a trivial variation of this.

2.7.8. *Goals.* We wish to describe the pages and differentials of the spectral sequence explicitly, and prove that they behave the way we said they did. More precisely, we wish to:

(a) describe $E_r^{p,q}$ (and verify that $E_0^{p,q}$ is indeed $E^{p,q}$),
(b) verify that $H^k(E^\bullet)$ is filtered by $E_\infty^{p,k-p}$ as in (2.7.2.2),
(c) describe d_r and verify that $d_r^2 = 0$, and
(d) verify that $E_{r+1}^{p,q}$ is given by cohomology using d_r.

Before tackling these goals, you can impress your friends by giving this short description of the pages and differentials of the spectral sequence. We say that an element of $E^{\bullet,\bullet}$ is a (p,q)-*strip* if it is an element of $\oplus_{l \geq 0} E^{p-l,q+l}$ (see Fig. 2.1). Its non-zero entries lie on an "upper-leftwards" semi-infinite antidiagonal starting with position (p,q). We say that the (p,q)-entry (the projection to $E^{p,q}$) is the *leading term* of the (p,q)-strip. Let $\boxed{S^{p,q}} \subset E^{\bullet,\bullet}$ be the submodule of all the (p,q)-strips. Clearly $S^{p,q} \subset E^{p+q}$, and $S^{k,0} = E^k$.

$$\begin{array}{ccccc}
\ddots & 0 & 0 & 0 & 0 \\
0 & *^{p-2,q+2} & 0 & 0 & 0 \\
0 & 0 & *^{p-1,q+1} & 0 & 0 \\
0 & 0 & 0 & *^{p,q} & 0 \\
0 & 0 & 0 & 0 & 0^{p+1,q-1}
\end{array}$$

FIGURE 2.1. A (p,q)-strip (in $S^{p,q} \subset E^{p+q}$). Clearly $S^{0,k} = E^k$.

Note that the differential $d = d_\uparrow + d_\rightarrow$ sends a (p,q)-strip x to a $(p+1,q)$-strip dx. If dx is furthermore a $(p-r+1,q+r)$-strip ($r \in \mathbb{Z}^{\geq 0}$), we say that x is an r-*closed* (p,q)-*strip* — "the differential knocks x at least r terms deeper into the filtration". We denote the set of r-closed (p,q)-strips $\boxed{S_r^{p,q}}$ (so for example $S_0^{p,q} = S^{p,q}$, and $S_0^{k,0} = E^k$). An element of $S_r^{p,q}$ may be depicted as:

$$\begin{tikzcd}
\ddots \arrow[r] & ? & & & \\
& *^{p-2,q+2} \arrow[u] \arrow[r] & 0 & & \\
& & *^{p-1,q+1} \arrow[u] \arrow[r] & 0 & \\
& & & *^{p,q} \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

2.7.9. *Preliminary definition of* $E_r^{p,q}$. We are now ready to give a first definition of $E_r^{p,q}$, which by construction should be a subquotient of $E^{p,q} = E_0^{p,q}$. We describe it as such by describing two submodules $Y_r^{p,q} \subset X_r^{p,q} \subset E^{p,q}$, and defining $E_r^{p,q} = X_r^{p,q}/Y_r^{p,q}$. Let $X_r^{p,q}$ be those elements of $E^{p,q}$ that are the leading terms of r-closed (p,q)-strips. Note that by definition, d sends $(r-1)$-closed $(p+(r-1)-1, q-(r-1))$-strips to (p,q)-strips. Let $Y_r^{p,q}$ be the leading $((p,q))$-terms of the differential d of $(r-1)$-closed $(p+(r-1)-1, q-(r-1))$-strips (where the differential is considered as a (p,q)-strip).

2.7.G. EXERCISE (REALITY CHECK). Verify that $E_0^{p,q}$ is (canonically isomorphic to) $E^{p,q}$.

We next give the definition of the differential d_r of such an element $x \in X_r^{p,q}$. We take *any* r-closed (p,q)-strip with leading term x. Its differential d is a $(p-r+1, q+r)$-strip, and we take its leading term. The choice of the r-closed (p,q)-strip means that this is not a well-defined element of $E^{p,q}$. But it is well-defined modulo the differentials of the $(r-1)$-closed $(p+1, q+1)$-strips, and hence gives a map $E_r^{p,q} \to E_r^{p-r+1,q+r}$.

This definition is fairly short, but not much fun to work with, so we will forget it, and instead dive into a snakes' nest of subscripts and superscripts.

We begin with making some quick but important observations about (p,q)-strips.

2.7.H. EXERCISE (NOT HARD). Verify the following.

(a) $S^{p,q} = S^{p-1,q+1} \oplus E^{p,q}$.

(b) *(Any closed (p,q)-strip is r-closed for all r.)* Any element x of $S^{p,q} = S_0^{p,q}$ that is a cycle (i.e. $dx = 0$) is automatically in $S_r^{p,q}$ for all r. For example, this holds when x is a boundary (i.e. of the form dy).

(c) Show that for fixed p, q,

$$S_0^{p,q} \supset S_1^{p,q} \supset \cdots \supset S_r^{p,q} \supset \cdots$$

stabilizes for $r \gg 0$ (i.e. $S_r^{p,q} = S_{r+1}^{p,q} = \cdots$). Denote the stabilized module $S_\infty^{p,q}$. Show $S_\infty^{p,q}$ is the set of closed (p,q)-strips (those (p,q)-strips annihilated by d, i.e. the cycles). In particular, $S_\infty^{0,k}$ is the set of cycles in E^k.

2.7.10. *Defining* $E_r^{p,q}$.

Define $X_r^{p,q} := S_r^{p,q}/S_{r-1}^{p-1,q+1}$ and $Y_r^{p,q} := dS_{r-1}^{p+(r-1)-1,q-(r-1)}/S_{r-1}^{p-1,q+1}$. Then $Y_r^{p,q} \subset X_r^{p,q}$ by Exercise 2.7.H(b). We define

$$\boxed{E_r^{p,q} = \frac{X_r^{p,q}}{Y_r^{p,q}} = \frac{S_r^{p,q}}{dS_{r-1}^{p+(r-1)-1,q-(r-1)} + S_{r-1}^{p-1,q+1}}} \tag{2.7.10.1}$$

We have completed Goal 2.7.8(a).

You are welcome to verify that these definitions of $X_r^{p,q}$ and $Y_r^{p,q}$ and hence $E_r^{p,q}$ agree with the earlier ones of §2.7.9 (and in particular $X_r^{p,q}$ and $Y_r^{p,q}$ are both submodules of $E^{p,q}$), but we won't need this fact.

2.7.I. EXERCISE: $E_\infty^{p,k-p}$ GIVES SUBQUOTIENTS OF $H^k(E^\bullet)$. By Exercise 2.7.H(c), $E_r^{p,q}$ stabilizes as $r \to \infty$. For $r \gg 0$, interpret $S_r^{p,q}/dS_{r-1}^{p+(r-1)-1,q-(r-1)}$ as the cycles in $S^{p,q}_\infty \subset E^{p+q}$ modulo those boundary elements of dE^{p+q-1} contained in $S^{p,q}_\infty$. Finally, show that $H^k(E^\bullet)$ is indeed filtered as described in (2.7.2.2).

We have completed Goal 2.7.8(b).

2.7.11. *Definition of* d_r.

We shall see that the map $d_r : E^{p,q}_r \to E^{p-r+1,q+r}_r$ is just induced by our differential d. Notice that d sends r-closed (p,q)-strips $S^{p,q}_r$ to $(p-r+1, q+r)$-strips $S^{p-r+1,q+r}$, by the definition "r-closed". By Exercise 2.7.H(b), the image lies in $S^{p-r+1,q+r}_r$.

2.7.J. EXERCISE. Verify that d sends

$$dS^{p+(r-1)-1,q-(r-1)}_{r-1} + S^{p-1,q+1}_{r-1} \to dS^{(p-r+1)+(r-1)-1,(q+r)-(r-1)}_{r-1} + S^{(p-r+1)-1,(q+r)+1}_{r-1}.$$

(The first term on the left goes to 0 from $d^2 = 0$, and the second term on the left goes to the first term on the right.)

Thus we may define

$$\boxed{d_r : E^{p,q}_r = \frac{S^{p,q}_r}{dS^{p+(r-1)-1,q-(r-1)}_{r-1} + S^{p-1,q+1}_{r-1}} \to}$$

$$\boxed{\frac{S^{p-r+1,q+r}_r}{dS^{p-1,q+1}_{r-1} + S^{p-r,q+r+1}_{r-1}} = E^{p-r+1,q+r}_r}$$

and clearly $d_r^2 = 0$ (as we may interpret it as taking an element of $S^{p,q}_r$ and applying d twice).

We have accomplished Goal 2.7.8(c).

2.7.12. *Verifying that the cohomology of* d_r *at* $E^{p,q}_r$ *is* $E^{p,q}_{r+1}$. We are left with the unpleasant job of verifying that the cohomology of

$$\frac{S^{p+r-1,q-r}_r}{dS^{p+2r-3,q-2r+1}_{r-1} + S^{p+r-2,q-r+1}_{r-1}} \xrightarrow{d_r} \frac{S^{p,q}_r}{dS^{p+r-2,q-r+1}_{r-1} + S^{p-1,q+1}_{r-1}} \tag{2.7.12.1}$$

$$\xrightarrow{d_r} \frac{S^{p-r+1,q+r}_r}{dS^{p-1,q+1}_{r-1} + S^{p-r,q+r+1}_{r-1}}$$

is naturally identified with

$$\frac{S^{p,q}_{r+1}}{dS^{p+r-1,q-r}_r + S^{p-1,q+1}_r}$$

and this will conclude our final Goal 2.7.8(d).

We begin by understanding the kernel of the right map of (2.7.12.1). Suppose $a \in S^{p,q}_r$ is mapped to 0. This means that $da = db + c$, where $b \in S^{p-1,q+1}_{r-1}$. If $u = a - b$, then $u \in S^{p,q}$, while $du = c \in S^{p-r,q+r+1}_{r-1} \subset S^{p-r,q+r+1}$, from which u is $(r+1)$-closed, i.e. $u \in S^{p,q}_{r+1}$. Thus $a = b + u \in S^{p-1,q+1}_{r-1} + S^{p,q}_{r+1}$. Conversely, any $a \in S^{p-1,q+1}_{r-1} + S^{p,q}_{r+1}$ satisfies

$$da \in dS^{p-1,q+1}_{r-1} + dS^{p,q}_{r+1} \subset dS^{p-1,q+1}_{r-1} + S^{p-r,q+r+1}_{r-1}$$

(using $dS_{r+1}^{p,q} \subset S_0^{p-r,q+r+1}$ and Exercise 2.7.H(b)) so any such a is indeed in the kernel of

$$S_r^{p,q} \to \frac{S_r^{p-r+1,q+r}}{dS_{r-1}^{p-1,q+1} + S_{r-1}^{p-r,q+r+1}}.$$

Hence the kernel of the right map of (2.7.12.1) is

$$\ker = \frac{S_{r-1}^{p-1,q+1} + S_{r+1}^{p,q}}{dS_{r-1}^{p+r-2,q-r+1} + S_{r-1}^{p-1,q+1}}.$$

Next, the image of the left map of (2.7.12.1) is immediately

$$\mathrm{im} = \frac{dS_r^{p+r-1,q-r} + dS_{r-1}^{p+r-2,q-r+1} + S_{r-1}^{p-1,q+1}}{dS_{r-1}^{p+r-2,q-r+1} + S_{r-1}^{p-1,q+1}} = \frac{dS_r^{p+r-1,q-r} + S_{r-1}^{p-1,q+1}}{dS_{r-1}^{p+r-2,q-r+1} + S_{r-1}^{p-1,q+1}}$$

(as $S_r^{p+r-1,q-r}$ contains $S_{r-1}^{p+r-2,q-r+1}$).

Thus the cohomology of (2.7.12.1) is

$$\ker / \mathrm{im} = \frac{S_{r-1}^{p-1,q+1} + S_{r+1}^{p,q}}{dS_r^{p+r-1,q-r} + S_{r-1}^{p-1,q+1}} = \frac{S_{r+1}^{p,q}}{S_{r+1}^{p,q} \cap (dS_r^{p+r-1,q-r} + S_{r-1}^{p-1,q+1})}$$

where the equality on the right uses the fact that $dS_r^{p+r-1,q-r} \subset S_{r+1}^{p,q}$ and an isomorphism theorem. We thus must show

$$S_{r+1}^{p,q} \cap (dS_r^{p+r-1,q-r} + S_{r-1}^{p-1,q+1}) = dS_r^{p+r-1,q-r} + S_r^{p-1,q+1}.$$

However,

$$S_{r+1}^{p,q} \cap (dS_r^{p+r-1,q-r} + S_{r-1}^{p-1,q+1}) = dS_r^{p+r-1,q-r} + S_{r+1}^{p,q} \cap S_{r-1}^{p-1,q+1}$$

and $S_{r+1}^{p,q} \cap S_{r-1}^{p+1,q-1}$ consists of $(p-1,q+1)$-strips whose differential vanishes up to row $p+r$, from which $S_{r+1}^{p,q} \cap S_{r-1}^{p-1,q+1} = S_r^{p-1,q+1}$ as desired.

This completes the explanation of how spectral sequences work for a first-quadrant double complex. The argument applies without significant change to more general situations, including filtered complexes.

CHAPTER 3

Sheaves

It is perhaps surprising that geometric spaces are often best understood in terms of (nice) functions on them. For example, a differentiable manifold that is a subset of $\mathbb{R}^n$ can be studied in terms of its differentiable functions. Because "geometric spaces" can have few (everywhere-defined) functions, a more precise version of this insight is that the structure of the space can be well understood by considering all functions on all open subsets of the space. This information is encoded in something called a *sheaf*. Sheaves were introduced by Leray in the 1940's, and Serre introduced them to algebraic geometry. (The reason for the name will be somewhat explained in Remark 3.4.3.) We will define sheaves and describe useful facts about them. We will begin with a motivating example to convince you that the notion is not so foreign.

One reason sheaves are slippery to work with is that they keep track of a huge amount of information, and there are some subtle local-to-global issues. There are also three different ways of getting a hold of them.

- in terms of open sets (the definition §3.2) — intuitive but in some ways the least helpful
- in terms of stalks (see §3.4.1)
- in terms of a base of a topology (§3.7).

Knowing which to use requires experience, so it is essential to do a number of exercises on different aspects of sheaves in order to truly understand the concept.

3.1 Motivating example: The sheaf of differentiable functions.

Consider differentiable functions on the topological space $X = \mathbb{R}^n$ (or more generally on a smooth manifold X). The sheaf of differentiable functions on X is the data of all differentiable functions on all open subsets on X. We will see how to manage this data, and observe some of its properties. On each open set $U \subset X$, we have a ring of differentiable functions. We denote this ring of functions $\mathcal{O}(U)$.

Given a differentiable function on an open set, you can restrict it to a smaller open set, obtaining a differentiable function there. In other words, if $U \subset V$ is an inclusion of open sets, we have a "restriction map" $\operatorname{res}_{V,U} : \mathcal{O}(V) \to \mathcal{O}(U)$.

Take a differentiable function on a big open set, and restrict it to a medium open set, and then restrict that to a small open set. The result is the same as if you restrict the differentiable function on the big open set directly to the small open set.

In other words, if $U \hookrightarrow V \hookrightarrow W$, then the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}(W) \arrow[rr, "\mathrm{res}_{W,V}"] \arrow[dr, "\mathrm{res}_{W,U}"'] & & \mathcal{O}(V) \arrow[dl, "\mathrm{res}_{V,U}"] \\ & \mathcal{O}(U) & \end{tikzcd}$$

Next take two differentiable functions f_1 and f_2 on a big open set U, and an open cover of U by some $\{U_i\}$. Suppose that f_1 and f_2 agree on each of these U_i. Then they must have been the same function to begin with. In other words, if $\{U_i\}_{i \in I}$ is a cover of U, and $f_1, f_2 \in \mathcal{O}(U)$, and $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$, then $f_1 = f_2$. Thus we can *identify* functions on an open set by looking at them on a covering by small open sets.

Finally, given the same U and cover $\{U_i\}$, take a differentiable function on each of the U_i — a function f_1 on U_1, a function f_2 on U_2, and so on — and they agree on the pairwise overlaps. Then they can be "glued together" to make one differentiable function on all of U. In other words, given $f_i \in \mathcal{O}(U_i)$ for all i, such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all i and j, then there is some $f \in \mathcal{O}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all i.

The entire example above would have worked just as well with continuous functions, or smooth functions, or just plain functions. Thus all of these classes of "nice" functions share some common properties. We will soon formalize these properties in the notion of a sheaf.

3.1.1. The germ of a differentiable function. Before we do, we first give another definition, that of the germ of a differentiable function at a point $p \in X$. Intuitively, it is a "shred" of a differentiable function at p. Germs are objects of the form $\{(f, \text{open } U) \;:\; p \in U, f \in \mathcal{O}(U)\}$ modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ containing p where $f|_W = g|_W$ (i.e., $\mathrm{res}_{U,W} f = \mathrm{res}_{V,W} g$). In other words, two functions that are the same in a neighborhood of p (but may differ elsewhere) have the same germ. We call this set of germs the stalk at p, and denote it $\mathcal{O}_p$. Notice that the stalk is a ring: you can add two germs, and get another germ: if you have a function f defined on U, and a function g defined on V, then $f + g$ is defined on $U \cap V$. Moreover, $f + g$ is well-defined: if f' has the same germ as f, meaning that there is some open set W containing p on which they agree, and g' has the same germ as g, meaning they agree on some open W' containing p, then $f' + g'$ is the same function as $f + g$ on $U \cap V \cap W \cap W'$.

Notice also that if $p \in U$, you get a map $\mathcal{O}(U) \to \mathcal{O}_p$. Experts may already see that we are talking about germs as colimits.

We can see that $\mathcal{O}_p$ is a local ring as follows. Consider those germs vanishing at p, which we denote $\mathfrak{m}_p \subset \mathcal{O}_p$. They certainly form an ideal: $\mathfrak{m}_p$ is closed under addition, and when you multiply something vanishing at p by any function, the result also vanishes at p. We check that this ideal is maximal by showing that the quotient ring is a field:

$$0 \longrightarrow \mathfrak{m}_p := \text{ideal of germs vanishing at } p \longrightarrow \mathcal{O}_p \xrightarrow{f \mapsto f(p)} \mathbb{R} \longrightarrow 0 \tag{3.1.1.1}$$

3.1.A. Exercise. Show that this is the only maximal ideal of $\mathcal{O}_p$. (Hint: show that every element of $\mathcal{O}_p \setminus \mathfrak{m}$ is invertible.)

Note that we can interpret the value of a function at a point, or the value of a germ at a point, as an element of the local ring modulo the maximal ideal. (We will see that this doesn't work for more general sheaves, but *does* work for things behaving like sheaves of functions. This will be formalized in the notion of a *locally ringed space*, which we will see, briefly, in §7.3.)

3.1.2. *Aside.* Notice that $\mathfrak{m}/\mathfrak{m}^2$ is a module over $\mathcal{O}_p/\mathfrak{m} \cong \mathbb{R}$, i.e. it is a real vector space. It turns out to be naturally (whatever that means) the cotangent space to the manifold at p. This insight will prove handy later, when we define tangent and cotangent spaces of schemes.

3.1.B. EXERCISE FOR THOSE WITH DIFFERENTIAL GEOMETRIC BACKGROUND. Prove this.

3.2 Definition of sheaf and presheaf

We now formalize these notions, by defining presheaves and sheaves. Presheaves are simpler to define, and notions such as kernel and cokernel are straightforward. Sheaves are more complicated to define, and some notions such as cokernel require more thought. But sheaves are more useful because they are in some vague sense more geometric; you can get information about a sheaf locally.

3.2.1. Definition of sheaf and presheaf on a topological space X.

To be concrete, we will define sheaves of sets. However, in the definition the category *Sets* can be replaced by any category, and other important examples are abelian groups *Ab*, k-vector spaces Vec_k, rings *Rings*, modules over a ring Mod_A, and more. (You may have to think more when dealing with a category of objects that aren't "sets with additional structure", but there aren't any new complications. In any case, this won't be relevant for us, although people who want to do this should start by solving Exercise 3.2.C.) Sheaves (and presheaves) are often written in calligraphic font. The fact that $\mathcal{F}$ is a sheaf on a topological space X is often written as

$$\begin{array}{c}\mathcal{F}\\ \Big| \\ X\end{array}$$

3.2.2. Definition: Presheaf. A **presheaf** $\mathcal{F}$ on a topological space X is the following data.

• To each open set $U \subset X$, we have a set $\mathcal{F}(U)$ (e.g. the set of differentiable functions in our motivating example). (Notational warning: Several notations are in use, for various good reasons: $\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F})$. We will use them all.) The elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over** U.

• For each inclusion $U \hookrightarrow V$ of open sets, we have a **restriction map** $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ (just as we did for differentiable functions).

The data is required to satisfy the following two conditions.

• The map $\mathrm{res}_{U,U}$ is the identity: $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$.

• If $U \hookrightarrow V \hookrightarrow W$ are inclusions of open sets, then the restriction maps commute, i.e.

$$\begin{tikzcd} \mathcal{F}(W) \arrow[rr, "\mathrm{res}_{W,V}"] \arrow[dr, "\mathrm{res}_{W,U}"'] & & \mathcal{F}(V) \arrow[dl, "\mathrm{res}_{V,U}"] \\ & \mathcal{F}(U) & \end{tikzcd}$$

commutes.

3.2.A. EXERCISE FOR CATEGORY-LOVERS: "A PRESHEAF IS THE SAME AS A CONTRAVARIANT FUNCTOR". Given any topological space X, we have a "category of open sets" (Example 2.2.9), where the objects are the open sets and the morphisms are inclusions. Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of X to the category of sets. (This interpretation is surprisingly useful.)

3.2.3. Definition: Stalks and germs. We define the stalk of a presheaf at a point in two equivalent ways. One will be hands-on, and the other will be as a colimit.

3.2.4. Define the **stalk** of a presheaf $\mathcal{F}$ at a point p to be the set of **germs** of $\mathcal{F}$ at p, denoted $\mathcal{F}_p$, as in the example of §3.1.1. Germs correspond to sections over some open set containing p, and two of these sections are considered the same if they agree on some smaller open set. More precisely: the stalk is

$$\{(f, \text{open } U) \ : \ p \in U, f \in \mathcal{F}(U)\}$$

modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ where $p \in W$ and $\mathrm{res}_{U,W} f = \mathrm{res}_{V,W} g$.

3.2.5. A useful (and better) equivalent definition of a stalk is as a colimit of all $\mathcal{F}(U)$ over all open sets U containing p:

$$\mathcal{F}_p = \varinjlim \mathcal{F}(U).$$

The index category is a directed set (given any two such open sets, there is a third such set contained in both), so these two definitions are the same by Exercise 2.4.C. Hence we can define stalks for sheaves of sets, groups, rings, and other things for which colimits exist for directed sets.

If $p \in U$, and $f \in \mathcal{F}(U)$, then the image of f in $\mathcal{F}_p$ is called the **germ of** f **at** p. (Warning: unlike the example of §3.1.1, in general, the value of a section at a point doesn't make sense.)

3.2.6. Definition: Sheaf. A presheaf is a **sheaf** if it satisfies two more axioms. Notice that these axioms use the additional information of when some open sets cover another.

Identity axiom. If $\{U_i\}_{i \in I}$ is an open cover of U, and $f_1, f_2 \in \mathcal{F}(U)$, and $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$ for all i, then $f_1 = f_2$.

(A presheaf satisfying the identity axiom is called a **separated presheaf**, but we will not use that notation in any essential way.)

Gluability axiom. If $\{U_i\}_{i \in I}$ is a open cover of U, then given $f_i \in \mathcal{F}(U_i)$ for all i, such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all i.

In mathematics, definitions often come paired: "at most one" and "at least one". In this case, identity means there is at most one way to glue, and gluability means that there is at least one way to glue.

(For experts and scholars of the empty set only: an additional axiom sometimes included is that $F(\varnothing)$ is a one-element set, and in general, for a sheaf with values in a category, $F(\varnothing)$ is required to be the final object in the category. This actually follows from the above definitions, assuming that the empty product is appropriately defined as the final object.)

Example. If U and V are disjoint, then $\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V)$. Here we use the fact that $F(\varnothing)$ is the final object.

The **stalk of a sheaf** at a point is just its stalk as a presheaf — the same definition applies — and similarly for the **germs** of a section of a sheaf.

3.2.B. UNIMPORTANT EXERCISE: PRESHEAVES THAT ARE NOT SHEAVES. Show that the following are presheaves on $\mathbb{C}$ (with the classical topology), but not sheaves: (a) bounded functions, (b) holomorphic functions admitting a holomorphic square root.

Both of the presheaves in the previous Exercise satisfy the identity axiom. A "natural" example failing even the identity axiom is implicit in Remark 3.7.4.

We now make a couple of points intended only for category-lovers.

3.2.7. *Interpretation in terms of the equalizer exact sequence.* The two axioms for a presheaf to be a sheaf can be interpreted as "exactness" of the "equalizer exact sequence": $\cdot \longrightarrow \mathcal{F}(U) \longrightarrow \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)$. Identity is exactness at $\mathcal{F}(U)$, and gluability is exactness at $\prod \mathcal{F}(U_i)$. I won't make this precise, or even explain what the double right arrow means. (What is an exact sequence of sets?!) But you may be able to figure it out from the context.

3.2.C. EXERCISE. The identity and gluability axioms may be interpreted as saying that $\mathcal{F}(\cup_{i \in I} U_i)$ is a certain limit. What is that limit?

We now give a number of examples of sheaves.

3.2.D. EXERCISE. (a) Verify that the examples of §3.1 are indeed sheaves (of differentiable functions, or continuous functions, or smooth functions, or functions on a manifold or $\mathbb{R}^n$).
(b) Show that real-valued continuous functions on (open sets of) a topological space X form a sheaf.

3.2.8. *Important Example: Restriction of a sheaf.* Suppose $\mathcal{F}$ is a sheaf on X, and U is an open subset of X. Define the **restriction of $\mathcal{F}$ to** U, denoted $\mathcal{F}|_U$, to be the collection $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for all open subsets $V \subset U$. Clearly this is a sheaf on U. (Unimportant but fun fact: §3.6 will tell us how to restrict sheaves to arbitrary subsets.)

3.2.9. *Important Example: skyscraper sheaf.* Suppose X is a topological space, with $p \in X$, and S is a set. Let $i_p : p \to X$ be the inclusion. Then $i_{p,*}S$ defined by

$$i_{p,*}S(U) = \begin{cases} S & \text{if } p \in U \text{, and} \\ \{e\} & \text{if } p \notin U \end{cases}$$

forms a sheaf. Here $\{e\}$ is any one-element set. (Check this if it isn't clear to you — what are the restriction maps?) This is called a **skyscraper sheaf**, because the informal picture of it looks like a skyscraper at p. There is an analogous definition for sheaves of abelian groups, except $i_{p,*}(S)(U) = \{0\}$ if $p \notin U$; and for sheaves with values in a category more generally, $i_{p,*}S(U)$ should be a final object.

3.2.10. *Constant presheaves and constant sheaves.* Let X be a topological space, and S a set. Define $\underline{S}^{\mathrm{pre}}(U) = S$ for all open sets U. You will readily verify that $\underline{S}^{\mathrm{pre}}$ forms a presheaf (with restriction maps the identity). This is called the **constant presheaf associated to** S. This isn't (in general) a sheaf. (It may be distracting to say why. Lovers of the empty set will insist that the sheaf axioms force the sections over the empty set to be the final object in the category, i.e. a one-element set. But even if we patch the definition by setting $\underline{S}^{\mathrm{pre}}(\varnothing) = \{e\}$, if S has more than one element, and X is the two-point space with the discrete topology, you can check that $\underline{S}^{\mathrm{pre}}$ fails gluability.)

3.2.E. EXERCISE (CONSTANT SHEAVES). Now let $\mathcal{F}(U)$ be the maps to S that are *locally constant*, i.e. for any point x in U, there is a neighborhood of x where the function is constant. Show that this is a *sheaf*. (A better description is this: endow S with the discrete topology, and let $\mathcal{F}(U)$ be the continuous maps $U \to S$.) This is called the **constant sheaf** (associated to S); do not confuse it with the constant presheaf. We denote this sheaf $\underline{S}$.

3.2.F. EXERCISE ("MORPHISMS GLUE"). Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps of U to Y. Show that this forms a sheaf. (Exercise 3.2.D(b), with $Y = \mathbb{R}$, and Exercise 3.2.E, with $Y = S$ with the discrete topology, are both special cases.)

3.2.G. EXERCISE. This is a fancier version of the previous exercise.
(a) (sheaf of sections of a map) Suppose we are given a continuous map $f : Y \to X$. Show that "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps $s : U \to Y$ such that $f \circ s = \mathrm{id}|_U$. Show that this forms a sheaf. (For those who have heard of vector bundles, these are a good example.) This is motivation for the phrase "section of a sheaf".
(b) (This exercise is for those who know what a topological group is. If you don't know what a topological group is, you might be able to guess.) Suppose that Y is a topological group. Show that continuous maps to Y form a sheaf of *groups*. (Example 3.2.D(b), with $Y = \mathbb{R}$, is a special case.)

3.2.11. $\star$ *The space of sections (espace étalé) of a (pre)sheaf.* Depending on your background, you may prefer the following perspective on sheaves, which we will not discuss further. Suppose $\mathcal{F}$ is a presheaf (e.g. a sheaf) on a topological space X. Construct a topological space Y along with a continuous map $\pi : Y \to X$ as follows: as a set, Y is the disjoint union of all the stalks of X. This also describes a natural set map $\pi : Y \to X$. We topologize Y as follows. Each section s of $\mathcal{F}$ over an open set U determines a subset $\{(x, s_x) : x \in U\}$ of Y. The topology on Y is the weakest topology such that these subsets are open. (These subsets form a base of the topology. For each $y \in Y$, there is a neighborhood V of y and a neighborhood U of X such that $\pi|_V$ is a homeomorphism from V to U. Do you see why these facts are true?) The topological space is could be thought of as the "space of sections" of $\mathcal{F}$ (and in french is called the **espace étalé** of $\mathcal{F}$). The reader may wish to show that (a) if $\mathcal{F}$ is a sheaf, then the sheaf of sections of $Y \to X$ (see the previous exercise 3.2.G(a)) can be naturally identified with the sheaf $\mathcal{F}$ itself. (b) Moreover, if $\mathcal{F}$ is a presheaf, the sheaf of sections of $Y \to X$ is the *sheafification* of $\mathcal{F}$, to be defined in Definition 3.4.5 (see Remark 3.4.7). Example 3.2.E may be interpreted as an example of this construction.

3.2.H. IMPORTANT EXERCISE: THE PUSHFORWARD SHEAF OR DIRECT IMAGE SHEAF. Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a presheaf on X. Then define $f_*\mathcal{F}$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, where V is an open subset of Y. Show that $f_*\mathcal{F}$ is a presheaf on Y, and is a sheaf if $\mathcal{F}$ is. This is called the **direct image** or **pushforward** of $\mathcal{F}$. More precisely, $f_*\mathcal{F}$ is called the **pushforward of $\mathcal{F}$ by f**.

As the notation suggests, the skyscraper sheaf (Example 3.2.9) can be interpreted as the pushforward of the constant sheaf $\underline{S}$ on a one-point space p, under the inclusion morphism $i : \{p\} \to X$.

Once we realize that sheaves form a category, we will see that the pushforward is a functor from sheaves on X to sheaves on Y (Exercise 3.3.B).

3.2.I. EXERCISE (PUSHFORWARD INDUCES MAPS OF STALKS). Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf of sets (or rings or A-modules) on X. If $f(x) = y$, describe the natural morphism of stalks $(f_*\mathcal{F})_y \to \mathcal{F}_x$. (You can use the explicit definition of stalk using representatives, §3.2.4, or the universal property, §3.2.5. If you prefer one way, you should try the other.) Once we define the category of sheaves of sets on a topological space in §3.3.1, you will see that your construction will make the following diagram commute:

$$\begin{tikzcd} Sets_X \arrow[r, "f_*"] \arrow[d] & Sets_Y \arrow[d] \\ Sets \arrow[r] & Sets \end{tikzcd}$$

3.2.12. Important Example: Ringed spaces, and $\mathcal{O}_X$-modules. Suppose $\mathcal{O}_X$ is a sheaf of rings on a topological space X (i.e. a sheaf on X with values in the category of *Rings*). Then $(X, \mathcal{O}_X)$ is called a **ringed space**. The sheaf of rings is often denoted by $\mathcal{O}_X$, pronounced "oh-X". This sheaf is called the **structure sheaf** of the ringed space. (Note: the stalk of $\mathcal{O}_X$ at a point is written "$\mathcal{O}_{X,x}$", because this looks less hideous than "$\mathcal{O}_{X_x}$".) We now define the notion of an **$\mathcal{O}_X$-module**. The notion is analogous to one we have seen before: just as we have modules over a ring, we have $\mathcal{O}_X$-modules over the structure sheaf (of rings) $\mathcal{O}_X$.

There is only one possible definition that could go with this name. An $\mathcal{O}_X$-module is a sheaf of abelian groups $\mathcal{F}$ with the following additional structure. For each U, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module. Furthermore, this structure should behave well with respect to restriction maps: if $U \subset V$, then

$$\begin{tikzcd} \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r, "\text{action}"] \arrow[d, "\mathrm{res}_{V,U} \times \mathrm{res}_{V,U}"'] & \mathcal{F}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\text{action}"] & \mathcal{F}(U) \end{tikzcd} \tag{3.2.12.1}$$

commutes. (You should convince yourself that I haven't forgotten anything.)

Recall that the notion of A-module generalizes the notion of abelian group, because an abelian group is the same thing as a $\mathbb{Z}$-module. Similarly, the notion of $\mathcal{O}_X$-module generalizes the notion of sheaf of abelian groups, because the latter is the same thing as a $\underline{\mathbb{Z}}$-module, where $\underline{\mathbb{Z}}$ is the constant sheaf associated to $\mathbb{Z}$. Hence when we are proving things about $\mathcal{O}_X$-modules, we are also proving things about sheaves of abelian groups.

3.2.J. EXERCISE. If $(X, \mathcal{O}_X)$ is a ringed space, and $\mathcal{F}$ is an $\mathcal{O}_X$-module, describe how for each $x \in X$, $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module.

3.2.13. *For those who know about vector bundles.* The motivating example of $\mathcal{O}_X$-modules is the sheaf of sections of a vector bundle. If $(X, \mathcal{O}_X)$ is a differentiable manifold (so $\mathcal{O}_X$ is the sheaf of differentiable functions), and $\pi : V \to X$ is a vector bundle over X, then the sheaf of differentiable sections $\phi : X \to V$ is an $\mathcal{O}_X$-module. Indeed, given a section s of π over an open subset $U \subset X$, and a function f on U, we can multiply s by f to get a new section fs of π over U. Moreover, if V is a smaller subset, then we could multiply f by s and then restrict to V, or we could restrict both f and s to V and then multiply, and we would get the same answer. That is precisely the commutativity of (3.2.12.1).

3.3 Morphisms of presheaves and sheaves

3.3.1. Whenever one defines a new mathematical object, category theory teaches to try to understand maps between them. We now define morphisms of presheaves, and similarly for sheaves. In other words, we will descibe the *category of presheaves* (of sets, abelian groups, etc.) and the *category of sheaves*.

A **morphism of presheaves** of sets (or indeed of sheaves with values in any category) on X, $\phi : \mathcal{F} \to \mathcal{G}$, is the data of maps $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all U behaving well with respect to restriction: if $U \hookrightarrow V$ then

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "\mathrm{res}_{V,U}"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ \mathcal{F}(U) \arrow[r, "\phi(U)"] & \mathcal{G}(U) \end{tikzcd}$$

commutes. (Notice: the underlying space of both $\mathcal{F}$ and $\mathcal{G}$ is X.)

Morphisms of sheaves are defined identically: the morphisms from a sheaf $\mathcal{F}$ to a sheaf $\mathcal{G}$ are precisely the morphisms from $\mathcal{F}$ to $\mathcal{G}$ as presheaves. (Translation: The category of sheaves on X is a full subcategory of the category of presheaves on

X.) If $(X, \mathcal{O}_X)$ is a ringed space, then morphisms of $\mathcal{O}_X$-modules have the obvious definition. (Can you write it down?)

An example of a morphism of sheaves is the map from the sheaf of differentiable functions on $\mathbb{R}$ to the sheaf of continuous functions. This is a "forgetful map": we are forgetting that these functions are differentiable, and remembering only that they are continuous.

We may as well set some notation: let $Sets_X$, Ab_X, etc. denote the category of sheaves of sets, abelian groups, etc. on a topological space X. Let $Mod_{\mathcal{O}_X}$ denote the category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$. Let $Sets_X^{\mathrm{pre}}$, etc. denote the category of presheaves of sets, etc. on X.

3.3.2. *Aside for category-lovers.* If you interpret a presheaf on X as a contravariant functor (from the category of open sets), a morphism of presheaves on X is a natural transformation of functors (§2.2.21).

3.3.A. EXERCISE: MORPHSMS OF (PRE)SHEAVES INDUCE MORPHISMS OF STALKS. If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X, and $x \in X$, describe the induced morphism of stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$. (Your proof will extend in obvious ways. For example, if ϕ is a morphism of $\mathcal{O}_X$-modules, then ϕ_x is a map of $\mathcal{O}_{X,x}$-modules.)

3.3.B. EXERCISE. Suppose $f : X \to Y$ is a continuous map of topological spaces (i.e. a morphism in the category of topological spaces). Show that pushforward gives a functor $Sets_X \to Sets_Y$. Here *Sets* can be replaced by many other categories. (Watch out for some possible confusion: a presheaf is a functor, and presheaves form a category. It may be best to forget that presheaves are functors for now.)

3.3.C. IMPORTANT EXERCISE AND DEFINITION: "SHEAF Hom". Suppose $\mathcal{F}$ and $\mathcal{G}$ are two sheaves of sets on X. (In fact, it will suffice that $\mathcal{F}$ is a presheaf.) Let $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Mor}(\mathcal{F}|_U, \mathcal{G}|_U).$$

(Recall the notation $\mathcal{F}|_U$, the restriction of the sheaf to the open set U, Example 3.2.8.) Show that this is a sheaf of sets on X. This is called the "sheaf $\mathcal{H}om$". (Strictly speaking, we should reserve Hom for when we are in additive category, so this should possibly be called "sheaf Mor". But the terminology sheaf $\mathcal{H}om$ is too established to uproot.) Show that if $\mathcal{G}$ is a sheaf of abelian groups, then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups. Implicit in this fact is that $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is an abelian group. (This exercise is somewhat tedious, but in the end very rewarding.) The same construction will "obviously" work for sheaves with values in any category. Also, it will be clear from your construction that, like Hom, $\mathcal{H}om$ is a contravariant functor in its first argument and a covariant functor in its second argument.

Warning: $\mathcal{H}om$ does not commute with taking stalks. More precisely: it is not true that $\mathcal{H}om(\mathcal{F}, \mathcal{G})_p$ is isomorphic to $\mathrm{Hom}(\mathcal{F}_p, \mathcal{G}_p)$. (Can you think of a counterexample? Does there at least exist a map from one of these to the other?)

We will use many variants of the definition of $\mathcal{H}om$. For example, if $\mathcal{F}$ and $\mathcal{G}$ are sheaves of abelian groups on X, then $\mathcal{H}om_{Ab_X}(\mathcal{F}, \mathcal{G})$ is defined by taking $\mathcal{H}om_{Ab_X}(\mathcal{F}, \mathcal{G})(U)$ to be the maps *as sheaves of abelian groups* $\mathcal{F}|_U \to \mathcal{G}|_U$. (Note that $\mathcal{H}om_{Ab_X}(\mathcal{F}, \mathcal{G})$ has the structure of a sheaf of abelian groups in a natural way.) Similarly, if $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, we define $\mathcal{H}om_{Mod_{\mathcal{O}_X}}(\mathcal{F}, \mathcal{G})$ in the analogous way (and it is an $\mathcal{O}_X$-module). Obnoxiously, the subscripts Ab_X and $Mod_{\mathcal{O}_X}$ are always dropped (here and in the literature), so be careful which category you are working in! We call $\mathcal{H}om_{Mod_{\mathcal{O}_X}}(\mathcal{F}, \mathcal{O}_X)$ the *dual* of the $\mathcal{O}_X$-module $\mathcal{F}$, and denoted it $\mathcal{F}^\vee$.

3.3.D. UNIMPORTANT EXERCISE (REALITY CHECK).

(a) If $\mathcal{F}$ is a sheaf of sets on X, then show that $\mathcal{H}om(\underline{\{p\}}, \mathcal{F}) \cong \mathcal{F}$, where $\underline{\{p\}}$ is the constant sheaf associated to the one element set $\{p\}$.
(b) If $\mathcal{F}$ is a sheaf of abelian groups on X, then show that $\mathcal{H}om_{Ab_X}(\underline{\mathbb{Z}}, \mathcal{F}) \cong \mathcal{F}$ (an isomorphism of sheaves of abelian groups).
(c) If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then show that $\mathcal{H}om_{Mod_{\mathcal{O}_X}}(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}$ (an isomorphism of $\mathcal{O}_X$-modules).

A key idea in (b) and (c) is that 1 "generates" (in some sense) $\underline{\mathbb{Z}}$ (in (b)) and $\mathcal{O}_X$ (in (c)).

3.3.3. Presheaves of abelian groups (and even "presheaf $\mathcal{O}_X$-modules") form an abelian category.

We can make module-like constructions using presheaves of abelian groups on a topological space X. (In this section, all (pre)sheaves are of abelian groups.) For example, we can clearly add maps of presheaves and get another map of presheaves: if $\phi, \psi : \mathcal{F} \to \mathcal{G}$, then we define the map $f + g$ by $(\phi + \psi)(V) = \phi(V) + \psi(V)$. (There is something small to check here: that the result is indeed a map of presheaves.) In this way, presheaves of abelian groups form an additive category (Definition 2.6.1). For exactly the same reasons, sheaves of abelian groups also form an additive category.

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, define the **presheaf kernel** $\ker_{\text{pre}} \phi$ by $(\ker_{\text{pre}} \phi)(U) = \ker \phi(U)$.

3.3.E. EXERCISE. Show that $\ker_{\text{pre}} \phi$ is a presheaf. (Hint: if $U \hookrightarrow V$, define the restriction map by chasing the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker_{\text{pre}} \phi(V) \arrow[r] \arrow[d, dotted, "\exists!"] & \mathcal{F}(V) \arrow[r] \arrow[d, "\text{res}_{V,U}"] & \mathcal{G}(V) \arrow[d, "\text{res}_{V,U}"] \\
0 \arrow[r] & \ker_{\text{pre}} \phi(U) \arrow[r] & \mathcal{F}(U) \arrow[r] & \mathcal{G}(U)
\end{tikzcd}$$

You should check that the restriction maps compose as desired.)

Define the **presheaf cokernel** $\text{coker}_{\text{pre}} \phi$ similarly. It is a presheaf by essentially the same argument.

3.3.F. EXERCISE: THE COKERNEL DESERVES ITS NAME. Show that the presheaf cokernel satisfies the universal property of cokernels (Definition 2.6.3) in the category of presheaves.

Similarly, $\ker_{\text{pre}} \phi \to \mathcal{F}$ satisfies the universal property for kernels in the category of presheaves.

It is not too tedious to verify that presheaves of abelian groups form an abelian category, and the reader is free to do so. The key idea is that all abelian-categorical notions may be defined and verified "open set by open set". We needn't worry about restriction maps — they "come along for the ride". Hence we can define terms such as **subpresheaf**, **image presheaf**, **quotient presheaf**, **cokernel presheaf**, and they behave the way one expect. You construct kernels, quotients, cokernels, and images open set by open set. Homological algebra (exact sequences and so forth) works, and also "works open set by open set". In particular:

3.3.G. EASY EXERCISE. Show (or observe) that for a topological space X with open set U, $\mathcal{F} \mapsto \mathcal{F}(U)$ gives a functor from presheaves of abelian groups on X, Ab_X^{pre}, to abelian groups, Ab. Then show that this functor is exact.

3.3.H. EXERCISE. Show that $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is exact if and only if $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is exact for all U.

The above discussion essentially carries over without change to presheaves with values in any abelian category. (Think this through if you wish.)

However, we are interested in more geometric objects, sheaves, where things can be understood in terms of their local behavior, thanks to the identity and gluing axioms. We will soon see that sheaves of abelian groups also form an abelian category, but a complication will arise that will force the notion of *sheafification* on us. Sheafification will be the answer to many of our prayers. We just haven't yet realized what we should be praying for.

To begin with, sheaves Ab_X may be easily seen to form an additive category (essentially because presheaves Ab_X^{pre} already do, and sheaves form a full subcategory).

Kernels work just as with presheaves:

3.3.I. IMPORTANT EXERCISE. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of *sheaves*. Show that the presheaf kernel $\ker_{\text{pre}} \phi$ is in fact a sheaf. Show that it satisfies the universal property of kernels (Definition 2.6.3). (Hint: the second question follows immediately from the fact that $\ker_{\text{pre}} \phi$ satisfies the universal property in the category of *presheaves*.)

Thus if ϕ is a morphism of sheaves, we define

$$\ker \phi := \ker_{\text{pre}} \phi.$$

The problem arises with the cokernel.

3.3.J. IMPORTANT EXERCISE. Let X be $\mathbb{C}$ with the classical topology, let $\underline{\mathbb{Z}}$ be the constant sheaf on X associated to $\mathbb{Z}$, $\mathcal{O}_X$ the sheaf of holomorphic functions, and $\mathcal{F}$ the *presheaf* of functions admitting a holomorphic logarithm. Describe an exact sequence of presheaves on X:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\underline{\mathbb{Z}} \to \mathcal{O}_X$ is the natural inclusion and $\mathcal{O}_X \to \mathcal{F}$ is given by $f \mapsto \exp 2\pi i f$. (Be sure to verify exactness.) Show that $\mathcal{F}$ is *not* a sheaf. (Hint: $\mathcal{F}$ does not satisfy the gluability axiom. The problem is that there are functions that don't have a logarithm but locally have a logarithm.) This will come up again in Example 3.4.9.

We will have to put our hopes for understanding cokernels of sheaves on hold for a while. We will first learn to understand sheaves using stalks.

3.4 Properties determined at the level of stalks, and sheafification

3.4.1. Properties determined by stalks. In this section, we will see that lots of facts about sheaves can be checked "at the level of stalks". This isn't true for presheaves, and reflects the local nature of sheaves. We will see that sections and morphisms are determined "by their stalks", and the property of a morphism being an isomorphism may be checked at stalks. (The last one is the trickiest.)

3.4.A. IMPORTANT EXERCISE **(sections are determined by germs)**. Prove that a section of a sheaf of sets is determined by its germs, i.e. the natural map

$$\mathcal{F}(U) \to \prod_{p \in U} \mathcal{F}_p \tag{3.4.1.1}$$

is injective. Hint 1: you won't use the gluability axiom, so this is true for separated presheaves. Hint 2: it is false for presheaves in general, see Exercise 3.4.F, so you *will* use the identity axiom. (Your proof will also apply to sheaves of groups, rings, etc. — informally speaking, to categories of "sets with additional structure". The same is true of many exercises in this section.)

This exercise suggests an important question: which elements of the right side of (3.4.1.1) are in the image of the left side?

3.4.2. Important definition. We say that an element $\prod_{p \in U} s_p$ of the right side $\prod_{p \in U} \mathcal{F}_p$ of (3.4.1.1) consists of **compatible germs** if for all $p \in U$, there is some representative $(U_p, s'_p \in \mathcal{F}(U_p))$ for s_p (where $p \in U_p \subset U$) such that the germ of s'_p at all $y \in U_p$ is s_y.You will have to think about this a little. Clearly any section s of $\mathcal{F}$ over U gives a choice of compatible germs for U — take $(U_p, s'_p) = (U, s)$.

3.4.B. IMPORTANT EXERCISE. Prove that any choice of compatible germs for a sheaf of sets $\mathcal{F}$ over U is the image of a section of $\mathcal{F}$ over U. (Hint: you will use gluability.)

We have thus completely described the image of (3.4.1.1), in a way that we will find useful.

3.4.3. *Remark.* This perspective is part of the motivation for the agricultural terminology "sheaf": it is (the data of) a bunch of stalks, bundled together appropriately.

Now we throw morphisms into the mix.

3.4.C. EXERCISE. Show a morphism of (pre)sheaves (of sets, or rings, or abelian groups, or $\mathcal{O}_X$-modules) induces a morphism of stalks. More precisely, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of (pre)sheaves on X, and $p \in X$, describe a natural map $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$. (You may wish to state this in the language of functors.)

3.4.D. EXERCISE **(morphisms are determined by stalks)**. If ϕ_1 and ϕ_2 are morphisms from a presheaf of sets $\mathcal{F}$ to a sheaf of sets $\mathcal{G}$ that induce the same maps on each stalk, show that $\phi_1 = \phi_2$. Hint: consider the following diagram.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d, hook] \\ \prod_{p \in U} \mathcal{F}_p \arrow[r] & \prod_{p \in U} \mathcal{G}_p \end{tikzcd} \tag{3.4.3.1}$$

3.4.E. TRICKY EXERCISE **(isomorphisms are determined by stalks)**. Show that a morphism of sheaves of sets is an isomorphism if and only if it induces an isomorphism of all stalks. Hint: Use (3.4.3.1). Injectivity of maps of stalks uses the previous exercise 3.4.D. Once you have injectivity, show surjectivity, perhaps using Exercise 3.4.B, or gluability in some other way; this is more subtle.

3.4.F. EXERCISE. (a) Show that Exercise 3.4.A is false for general presheaves.
(b) Show that Exercise 3.4.D is false for general presheaves.
(c) Show that Exercise 3.4.E is false for general presheaves.
(General hint for finding counterexamples of this sort: consider a 2-point space with the discrete topology, i.e. every subset is open.)

3.4.4. Sheafification.

Every sheaf is a presheaf (and indeed by definition sheaves on X form a full subcategory of the category of presheaves on X). Just as groupification (§2.5.3) gives a group that best approximates a semigroup, sheafification gives the sheaf that best approximates a presheaf, with an analogous universal property. (One possible example to keep in mind is the sheafification of the presheaf of holomorphic functions admitting a square root on $\mathbb{C}$ with the classical topology.)

3.4.5. *Definition.* If $\mathcal{F}$ is a presheaf on X, then a morphism of presheaves $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ on X is a **sheafification of** $\mathcal{F}$ if $\mathcal{F}^{\mathrm{sh}}$ is a sheaf, and for any sheaf $\mathcal{G}$, and any presheaf morphism $g : \mathcal{F} \to \mathcal{G}$, there *exists* a *unique* morphism of sheaves $f : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}$ making the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathrm{sh}"] \arrow[dr, "g"'] & \mathcal{F}^{\mathrm{sh}} \arrow[d, "f"] \\ & \mathcal{G} \end{tikzcd}$$

commute.

3.4.G. EXERCISE. Show that sheafification is unique up to unique isomorphism. Show that if $\mathcal{F}$ is a sheaf, then the sheafification is $\mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F}$. (This should be second nature by now.)

3.4.6. Construction. We next show that any presheaf of sets (or groups, rings, etc.) has a sheafification. Suppose $\mathcal{F}$ is a *presheaf*. Define $\mathcal{F}^{\mathrm{sh}}$ by defining $\mathcal{F}^{\mathrm{sh}}(U)$ as the set of compatible germs of the presheaf $\mathcal{F}$ over U. Explicitly:

$$\begin{aligned} \mathcal{F}^{\mathrm{sh}}(U) \quad := \quad & \{(f_x \in \mathcal{F}_x)_{x \in U} : \text{for all } x \in U, \text{ there exists } x \in V \subset U \text{ and } s \in \mathcal{F}(V) \\ & \text{with } s_y = f_y \text{ for all } y \in V\}. \end{aligned}$$

(Those who want to worry about the empty set are welcome to.)

3.4.H. EASY EXERCISE. Show that $\mathcal{F}^{sh}$ (using the tautological restriction maps) forms a sheaf.

3.4.I. EASY EXERCISE. Describe a natural map of presheaves $sh : \mathcal{F} \to \mathcal{F}^{sh}$.

3.4.J. EXERCISE. Show that the map sh satisfies the universal property of sheafification (Definition 3.4.5). (This is easier than you might fear.)

3.4.K. EASY EXERCISE. Use the universal property to show that for any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$. Show that sheafification is a functor from presheaves on X to sheaves on X.

3.4.L. USEFUL EXERCISE, NOT JUST FOR CATEGORY-LOVERS. Show that the sheafification functor is left-adjoint to the forgetful functor from sheaves on X to presheaves on X. This is not difficult — it is largely a restatement of the universal property. But it lets you use results from §2.6.11, and can "explain" why you don't need to sheafify when taking kernel (why the presheaf kernel is already the sheaf kernel), and why you need to sheafify when taking cokernel and (soon, in Exercise 3.5.I) $\otimes$.

3.4.M. EXERCISE. Show $\mathcal{F} \to \mathcal{F}^{sh}$ induces an isomorphism of stalks. (Possible hint: Use the concrete description of the stalks. Another possibility once you read Remark 3.6.3: judicious use of adjoints.)

As a reality check, you may want to verify that "the sheafification of a constant presheaf is the corresponding constant sheaf" (see §3.2.10): if X is a topological space and S is a set, then $(\underline{S}^{pre})^{sh}$ may be naturally identified with $\underline{S}$.

3.4.7. $\star$ *Remark.* The total space of sections (*espace étalé*) construction (§3.2.11) yields a different-sounding description of sheafification which may be preferred by some readers. The main idea is identical. This is essentially the same construction as the one given here. Another construction is described in **[EH]**.

3.4.8. Subsheaves and quotient sheaves.

3.4.N. EXERCISE. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of sets on a topological space X. Show that the following are equivalent.

(a) ϕ is a monomorphism in the category of sheaves.
(b) ϕ is injective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$.
(c) ϕ is injective on the level of open sets: $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open $U \subset X$.

(Possible hints: for (b) implies (a), recall that morphisms are determined by stalks, Exercise 3.4.D. For (a) implies (c), use the "indicator sheaf" with one section over every open set contained in U, and no section over any other open set.)

If these conditions hold, we say that $\mathcal{F}$ is a **subsheaf** of $\mathcal{G}$ (where the "inclusion" ϕ is sometimes left implicit).

(You may later wish to extend your solution to Exercise 3.4.N to show that for any morphism of *presheaves*, if all maps of sections are injective, then all stalk maps are injective. And furthermore, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism from a separated presheaf to an arbitrary presheaf, then injectivity on the level of stalks implies that ϕ is a monomorphism in the category of presheaves. This is useful in some approaches to Exercise 3.5.C.)

3.4.O. EXERCISE. Continuing the notation of the previous exercise, show that the following are equivalent.

(a) ϕ is an epimorphism in the category of sheaves.
(b) ϕ is surjective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$.

If these conditions hold, we say that $\mathcal{G}$ is a **quotient sheaf** of $\mathcal{F}$.

Thus *monomorphisms and epimorphisms — subsheafiness and quotient sheafiness — can be checked at the level of stalks.*

Both exercises generalize readily to sheaves with values in any reasonable category, where "injective" is replaced by "monomorphism" and "surjective" is replaced by "epimorphism".

Notice that there was no part (c) to Exercise 3.4.O, and Example 3.4.9 shows why. (But there is a version of (c) that *implies* (a) and (b): surjectivity on all open sets in the base of a topology implies surjectivity of the map of sheaves, Exercise 3.7.E.)

3.4.9. *Example (cf. Exercise 3.3.J).* Let $X = \mathbb{C}$ with the classical topology, and define $\mathcal{O}_X$ to be the sheaf of holomorphic functions, and $\mathcal{O}_X^*$ to be the sheaf of invertible (nowhere zero) holomorphic functions. This is a sheaf of abelian groups under multiplication. We have maps of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1 \tag{3.4.9.1}$$

where $\underline{\mathbb{Z}}$ is the constant sheaf associated to $\mathbb{Z}$. (You can figure out what the sheaves 0 and 1 mean; they are isomorphic, and are written in this way for reasons that may be clear.) We will soon interpret this as an exact sequence of sheaves of abelian groups (the *exponential exact sequence*), although we don't yet have the language to do so.

3.4.P. ENLIGHTENING EXERCISE. Show that $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$ describes $\mathcal{O}_X^*$ as a quotient sheaf of $\mathcal{O}_X$. Show that it is not surjective on all open sets.

This is a great example to get a sense of what "surjectivity" means for sheaves: nowhere vanishing holomorphic functions have logarithms locally, but they need not globally.

3.5 Sheaves of abelian groups, and $\mathcal{O}_X$-modules, form abelian categories

We are now ready to see that sheaves of abelian groups, and their cousins, $\mathcal{O}_X$-modules, form abelian categories. In other words, we may treat them similarly to vector spaces, and modules over a ring. In the process of doing this, we will see that this is much stronger than an analogy; kernels, cokernels, exactness, and so forth can be understood at the level of germs (which are just abelian groups), and the compatibility of the germs will come for free.

The category of sheaves of abelian groups is clearly an additive category (Definition 2.6.1). In order to show that it is an abelian category, we must show that any morphism $\phi : \mathcal{F} \to \mathcal{G}$ has a kernel and a cokernel. We have already seen that ϕ has a kernel (Exercise 3.3.I): the presheaf kernel is a sheaf, and is a kernel in the category of sheaves.

3.5.A. EXERCISE. Show that the stalk of the kernel is the kernel of the stalks: there is a natural isomorphism

$$(\ker(\mathcal{F} \to \mathcal{G}))_x \cong \ker(\mathcal{F}_x \to \mathcal{G}_x).$$

We next address the issue of the cokernel. Now $\phi : \mathcal{F} \to \mathcal{G}$ has a cokernel in the category of presheaves; call it $\mathcal{H}^{\text{pre}}$ (where the superscript is meant to remind us that this is a presheaf). Let $\mathcal{H}^{\text{pre}} \xrightarrow{\text{sh}} \mathcal{H}$ be its sheafification. Recall that the cokernel is defined using a universal property: it is the colimit of the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[d] & \mathcal{G} \\ 0 & \end{tikzcd}$$

in the category of presheaves. We claim that $\mathcal{H}$ is the cokernel of ϕ in the category of sheaves, and show this by proving the universal property. Given any sheaf $\mathcal{E}$ and a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[d] & \mathcal{G} \arrow[d] \\ 0 \arrow[r] & \mathcal{E} \end{tikzcd}$$

We construct

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[d] & \mathcal{G} \arrow[d] \arrow[rrd, bend left] & & \\ 0 \arrow[r] \arrow[rrrd] & \mathcal{H}^{\text{pre}} \arrow[r, "\text{sh}"] & \mathcal{H} & \\ & & & \mathcal{E} \end{tikzcd}$$

We show that there is a unique morphism $\mathcal{H} \to \mathcal{E}$ making the diagram commute. As $\mathcal{H}^{\text{pre}}$ is the cokernel in the category of presheaves, there is a unique morphism of presheaves $\mathcal{H}^{\text{pre}} \to \mathcal{E}$ making the diagram commute. But then by the universal property of sheafification (Definition 3.4.5), there is a unique morphism of *sheaves* $\mathcal{H} \to \mathcal{E}$ making the diagram commute.

3.5.B. EXERCISE. Show that the stalk of the cokernel is naturally isomorphic to the cokernel of the stalk.

We have now defined the notions of kernel and cokernel, and verified that they may be checked at the level of stalks. We have also verified that the properties of a morphism being a monomorphism or epimorphism are also determined at the level of stalks (Exercises 3.4.N and 3.4.O). Hence sheaves of abelian groups on X form an abelian category.

We see more: all structures coming from the abelian nature of this category may be checked at the level of stalks. For example:

3.5.C. EXERCISE. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups. Show that the image sheaf $\operatorname{im} \phi$ is the sheafification of the image presheaf. (You must use the definition of image in an abelian category. In fact, this gives the accepted definition of image sheaf for a morphism of sheaves of sets.) Show that the stalk of the image is the image of the stalk.

As a consequence, **exactness of a sequence of sheaves may be checked at the level of stalks**. In particular:

3.5.D. IMPORTANT EXERCISE. Show that taking the stalk of a sheaf of abelian groups is an exact functor. More precisely, if X is a topological space and $p \in X$ is a point, show that taking the stalk at p defines an exact functor $Ab_X \to Ab$.

3.5.E. EXERCISE: LEFT-EXACTNESS OF THE FUNCTOR OF "SECTIONS OVER U". Suppose $U \subset X$ is an open set, and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups. Show that

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. (You should do this "by hand", even if you realize there is a very fast proof using the left-exactness of the "forgetful" right-adjoint to the sheafification functor.) Show that the section functor need not be exact: show that if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of sheaves of abelian groups, then

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

need not be exact. (Hint: the exponential exact sequence (3.4.9.1).)

3.5.F. EXERCISE: LEFT-EXACTNESS OF PUSHFORWARD. Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups on X. If $f : X \to Y$ is a continuous map, show that

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H}$$

is exact. (The previous exercise, dealing with the left-exactness of the global section functor can be interpreted as a special case of this, in the case where Y is a point.)

3.5.G. EXERCISE: LEFT-EXACTNESS OF $\mathcal{H}om$. Suppose Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups on X, and $\mathcal{E}$ is a sheaf of abelian groups. Show that

$$0 \to \mathcal{H}om(\mathcal{E}, \mathcal{F}) \to \mathcal{H}om(\mathcal{G}, \mathcal{F}) \to \mathcal{H}om(\mathcal{H}, \mathcal{F})$$

and

$$0 \to \mathcal{H}om(\mathcal{H}, \mathcal{E}) \to \mathcal{H}om(\mathcal{G}, \mathcal{E}) \to \mathcal{H}om(\mathcal{F}, \mathcal{E}) \tag{3.5.0.2}$$

are both exact, where the maps come from the fact that $\mathcal{H}om$ is a functor (covariant or contravariant) in both its arguments.

3.5.H. EXERCISE. Show that if $(X, \mathcal{O}_X)$ is a ringed space, then $\mathcal{O}_X$-modules form an abelian category. (There is a fair bit to check, but there aren't many new ideas.)

3.5.1. Many facts about sheaves of abelian groups carry over to $\mathcal{O}_X$-modules without change. For example, $\mathcal{H}om_{\mathcal{O}_X}$ is a left-exact contravariant functor in its first argument and a left-exact covariant functor in its second argument.

We end with a useful construction using some of the ideas in this section.

3.5.I. IMPORTANT EXERCISE: TENSOR PRODUCTS OF $\mathcal{O}_X$-MODULES. (a) Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Define (categorically) what we should mean by **tensor product of two $\mathcal{O}_X$-modules**. Give an explicit construction, and show that it satisfies your categorical definition. *Hint:* take the "presheaf tensor product" — which needs to be defined — and sheafify. Note: $\otimes_{\mathcal{O}_X}$ is often written $\otimes$ when the subscript is clear from the context. (An example showing sheafification is necessary will arise in Example 15.1.1.))
(b) Show that the tensor product of stalks is the stalk of tensor product. (If you can show this, you may be able to make sense of the phrase "colimits commute with tensor products".)

3.5.2. *Conclusion.* Just as presheaves are abelian categories because all abelian-categorical notions make sense open set by open set, sheaves are abelian categories because all abelian-categorical notions make sense stalk by stalk.

3.6 The inverse image sheaf

We next describe a notion that is fundamental, but rather intricate. We will not need it for some time, so this may be best left for a second reading. Suppose we have a continuous map $f : X \to Y$. If $\mathcal{F}$ is a sheaf on X, we have defined the pushforward or direct image sheaf $f_*\mathcal{F}$, which is a sheaf on Y. There is also a notion of inverse image sheaf. (We will not call it the pullback sheaf, reserving that name for a later construction for quasicoherent sheaves, §17.3.) This is a covariant functor f^{-1} from sheaves on Y to sheaves on X. If the sheaves on Y have some additional structure (e.g. group or ring), then this structure is respected by f^{-1}.

3.6.1. *Definition by adjoint: elegant but abstract.* We define f^{-1} as the left-adjoint to f_*.

This isn't really a definition; we need a construction to show that the adjoint exists. Note that we then get canonical maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ (associated to the identity in $\mathrm{Mor}_Y(f_*\mathcal{F}, f_*\mathcal{F})$) and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ (associated to the identity in $\mathrm{Mor}_X(f^{-1}\mathcal{G}, f^{-1}\mathcal{G})$).

3.6.2. *Construction: concrete but ugly.* Define the temporary notation $f^{-1}\mathcal{G}^{\mathrm{pre}}(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V)$. (Recall the explicit description of colimit: sections are sections on open sets containing $f(U)$, with an equivalence relation. Note that $f(U)$ won't be an open set in general.)

3.6.A. EXERCISE. Show that this defines a presheaf on X.

Now define the **inverse image of** $\mathcal{G}$ by $f^{-1}\mathcal{G} := (f^{-1}\mathcal{G}^{\text{pre}})^{\text{sh}}$. Note that f^{-1} is a functor from sheaves on Y to sheaves on X. The next exercise shows that this satisfies the universal property. But you may wish to try the later exercises first, and come back to Exercise 3.6.B later. (For the later exercises, try to give two proofs, one using the universal property, and the other using the explicit description.)

3.6.B. IMPORTANT TRICKY EXERCISE. If $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X and $\mathcal{G}$ is a sheaf on Y, describe a bijection

$$\text{Mor}_X(f^{-1}\mathcal{G}, \mathcal{F}) \leftrightarrow \text{Mor}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Observe that your bijection is "natural" in the sense of the definition of adjoints (i.e. functorial in both $\mathcal{F}$ and $\mathcal{G}$). Thus Construction 3.6.2 satisfies the universal property of Definition 3.6.1. Possible hint: Show that both sides agree with the following third construction, which we denote $\text{Mor}_{XY}(\mathcal{G}, \mathcal{F})$. A collection of maps $\phi_{UV} : \mathcal{G}(V) \to \mathcal{F}(U)$ (as U runs through all open sets of X, and V runs through all open sets of Y containing $f(U)$) is said to be *compatible* if for all open $U' \subset U \subset X$ and all open $V' \subset V \subset Y$ with $f(U) \subset V$, $f(U') \subset V'$, the diagram

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r, "\phi_{VU}"] \arrow[d, "\text{res}_{V,V'}"'] & \mathcal{F}(U) \arrow[d, "\text{res}_{U,U'}"] \\ \mathcal{G}(V') \arrow[r, "\phi_{V'U'}"] & \mathcal{F}(U') \end{tikzcd}$$

commutes. Define $\text{Mor}_{XY}(\mathcal{G}, \mathcal{F})$ to be the set of all compatible collections $\phi = \{\phi_{UV}\}$.

3.6.3. *Remark.* As a special case, if X is a point $p \in Y$, we see that $f^{-1}\mathcal{G}$ is the stalk $\mathcal{G}_p$ of $\mathcal{G}$, and maps from the stalk $\mathcal{G}_p$ to a set S are the same as maps of sheaves on Y from $\mathcal{G}$ to the skyscraper sheaf with set S supported at p. You may prefer to prove this special case by hand directly before solving Exercise 3.6.B, as it is enlightening. (It can also be useful — can you use it to solve Exercises 3.4.M and 3.4.O?)

3.6.C. EXERCISE. Show that the stalks of $f^{-1}\mathcal{G}$ are the same as the stalks of $\mathcal{G}$. More precisely, if $f(p) = q$, describe a natural isomorphism $\mathcal{G}_q \cong (f^{-1}\mathcal{G})_p$. (Possible hint: use the concrete description of the stalk, as a colimit. Recall that stalks are preserved by sheafification, Exercise 3.4.M. Alternatively, use adjointness.) This, along with the notion of compatible stalks, may give you a way of thinking about inverse image sheaves.

3.6.D. EXERCISE (EASY BUT USEFUL). If U is an open subset of Y, $i : U \to Y$ is the inclusion, and $\mathcal{G}$ is a sheaf on Y, show that $i^{-1}\mathcal{G}$ is naturally isomorphic to $\mathcal{G}|_U$.

3.6.E. EXERCISE. Show that f^{-1} is an exact functor from sheaves of abelian groups on Y to sheaves of abelian groups on X (cf. Exercise 3.5.D). (Hint: exactness can be checked on stalks, and by Exercise 3.6.C, the stalks are the same.) Essentially the same argument will show that f^{-1} is an exact functor from $\mathcal{O}_Y$-modules (on Y) to $f^{-1}\mathcal{O}_Y$-modules (on X), but don't bother writing that down. (Remark for experts:

f^{-1} is a left-adjoint, hence right-exact by abstract nonsense, as discussed in §2.6.11. Left-exactness holds because colimits over directed systems are exact.)

3.6.F. EXERCISE. (a) Suppose $Z \subset Y$ is a closed subset, and $i : Z \hookrightarrow Y$ is the inclusion. If $\mathcal{F}$ is a sheaf on Z, then show that the stalk $(i_*\mathcal{F})_y$ is a one element-set if $y \notin Z$, and $\mathcal{F}_y$ if $y \in Z$.
(b) *Definition:* Define the **support** of a sheaf $\mathcal{G}$ of sets, denoted $\operatorname{Supp} \mathcal{G}$, as the locus where the stalks are not the one-element set:

$$\operatorname{Supp} \mathcal{G} := \{x \in X \ : \ |\mathcal{G}_x| \neq 1\}.$$

(More generally, if the sheaf has value in some category, the support consists of points where the stalk is not the final object. For sheaves of abelian groups, the support consists of points with non-zero stalks.) Suppose $\operatorname{Supp} \mathcal{G} \subset Z$ where Z is closed. Show that the natural map $\mathcal{G} \to i_* i^{-1} \mathcal{G}$ is an isomorphism. Thus a sheaf supported on a closed subset can be considered a sheaf on that closed subset. ("Support" is a useful notion, and will arise again in §14.7.C.)

3.6.G. EXERCISE (EXTENSION BY ZERO $f_!$: AN OCCASIONAL *left-adjoint* TO f^{-1}). In addition to always being a left-adjoint, f^{-1} can sometimes be a right-adjoint. Suppose $i : U \hookrightarrow Y$ is an inclusion of an open set into Y. We denote the restriction of the sheaf $\mathcal{O}_Y$ to U by $\mathcal{O}_U$. (We will later call $i : (U, \mathcal{O}_U) \to (Y, \mathcal{O}_Y)$ an *open immersion* of ringed spaces in Definition 7.2.1.) Define **extension by zero** $i_! : \mathit{Mod}_{\mathcal{O}_U} \to \mathit{Mod}_{\mathcal{O}_Y}$ as follows. Suppose $\mathcal{F}$ is an $\mathcal{O}_U$-module. For open $W \subset Y$, let $i_!^{\mathrm{pre}} \mathcal{F}(W) = \mathcal{F}(W)$ if $W \subset U$, and 0 otherwise (with the obvious restriction maps). This is clearly a presheaf $\mathcal{O}_Y$-module. Define $i_!$ as $(i_!^{\mathrm{pre}})^{\mathrm{sh}}$. Note that $i_! \mathcal{F}$ is an $\mathcal{O}_Y$-module, and that this defines a functor. (The symbol "!" is read as "shriek". I have no idea why. Thus "$i_!$" is read as "i-lower-shriek".)
(a) For $y \in Y$, show that $(i_! \mathcal{F})_y = \mathcal{F}_y$ if $y \in U$, and 0 otherwise.
(b) Show that $i_!$ is an exact functor.
(c) If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, describe an inclusion $i_! i^{-1} \mathcal{G} \hookrightarrow \mathcal{G}$.
(d) Show that $(i_!, i^{-1})$ is an adjoint pair, so there is a natural bijection $\operatorname{Hom}_{\mathcal{O}_Y}(i_! \mathcal{F}, \mathcal{G}) \leftrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}, \mathcal{G}|_U)$ for any $\mathcal{O}_X$-module $\mathcal{F}$ and $\mathcal{O}_Y$-module $\mathcal{G}$. (In particular, the sections of $\mathcal{G}$ over U can be identified with $\operatorname{Hom}_{\mathcal{O}_Y}(i_! \mathcal{O}_U, \mathcal{G})$.)

3.7 Recovering sheaves from a "sheaf on a base"

Sheaves are natural things to want to think about, but hard to get our hands on. We like the identity and gluability axioms, but they make proving things trickier than for presheaves. We have discussed how we can understand sheaves using stalks (using "compatible germs"). We now introduce a second way of getting a hold of sheaves, by introducing the notion of a *sheaf on a base*. Warning: this way of understanding an entire sheaf from limited information is confusing. It may help to keep sight of the central insight that this partial information is enough to understand germs, and the notion of when they are compatible (with nearby germs).

First, we define the notion of a **base of a topology**. Suppose we have a topological space X, i.e. we know which subsets U_i of X are open. Then a base of a topology is a subcollection of the open sets $\{B_j\} \subset \{U_i\}$, such that each U_i is a union of the B_j. Here is one example that you have seen early in your mathematical life. Suppose $X = \mathbb{R}^n$. Then the way the usual topology is often first defined is by defining *open balls* $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, and declaring that any union of open balls is open. So the balls form a base of the classical topology — we say they *generate* the classical topology. As an application of how we use them, to check continuity of some map $f : X \to \mathbb{R}^n$, you need only think about the pullback of balls on $\mathbb{R}^n$ — part of the traditional δ-ϵ definition of continuity.

Now suppose we have a sheaf $\mathcal{F}$ on a topological space X, and a base $\{B_i\}$ of open sets on X. Then consider the information $(\{\mathcal{F}(B_i)\}, \{\mathrm{res}_{B_i,B_j} : \mathcal{F}(B_i) \to \mathcal{F}(B_j)\})$, which is a subset of the information contained in the sheaf — we are only paying attention to the information involving elements of the base, not all open sets.

We can recover the entire sheaf from this information. This is because we can determine the stalks from this information, and we can determine when germs are compatible.

3.7.A. EXERCISE. Make this precise. How can you recover a sheaf $\mathcal{F}$ from this partial information?

This suggests a notion, called a **sheaf on a base**. A sheaf of sets (or abelian groups, rings, ...) on a base $\{B_i\}$ is the following. For each B_i in the base, we have a set $F(B_i)$. If $B_i \subset B_j$, we have maps $\mathrm{res}_{B_j,B_i} : F(B_j) \to F(B_i)$, with $\mathrm{res}_{B_i,B_i} = \mathrm{id}_{F(B_i)}$. (Things called B are always assumed to be in the base.) If $B_i \subset B_j \subset B_k$, then $\mathrm{res}_{B_k,B_i} = \mathrm{res}_{B_j,B_i} \circ \mathrm{res}_{B_k,B_j}$. So far we have defined a **presheaf on a base** $\{B_i\}$.

We also require the **base identity** axiom: If $B = \cup B_i$, then if $f, g \in F(B)$ are such that $\mathrm{res}_{B,B_i} f = \mathrm{res}_{B,B_i} g$ for all i, then $f = g$.

We require the **base gluability** axiom too: If $B = \cup B_i$, and we have $f_i \in F(B_i)$ such that f_i agrees with f_j on any basic open set contained in $B_i \cap B_j$ (i.e. $\mathrm{res}_{B_i,B_k} f_i = \mathrm{res}_{B_j,B_k} f_j$ for all $B_k \subset B_i \cap B_j$) then there exists $f \in F(B)$ such that $\mathrm{res}_{B,B_i} f = f_i$ for all i.

3.7.1. Theorem. — *Suppose $\{B_i\}$ is a base on X, and F is a sheaf of sets on this base. Then there is a sheaf $\mathcal{F}$ extending F (with isomorphisms $\mathcal{F}(B_i) \cong F(B_i)$ agreeing with the restriction maps). This sheaf $\mathcal{F}$ is unique up to unique isomorphism*

Proof. We will define $\mathcal{F}$ as the sheaf of compatible germs of F.

Define the **stalk** of a base presheaf F at $p \in X$ by

$$F_p = \varinjlim F(B_i)$$

where the colimit is over all B_i (in the base) containing p.

We will say a family of germs in an open set U is compatible near p if there is a section s of F over some B_i containing p such that the germs over B_i are precisely the germs of s. More formally, define

$$\begin{aligned}\mathcal{F}(U) \quad := \quad & \{(f_p \in F_p)_{p \in U} : \text{ for all } p \in U, \text{ there exists } B \text{ with } p \subset B \subset U, s \in F(B), \\ & \text{with } s_q = f_q \text{ for all } q \in B\}\end{aligned}$$

where each B is in our base.

This is a sheaf (for the same reasons as the sheaf of compatible germs was earlier, cf. Exercise 3.4.H).

I next claim that if B is in our base, the natural map $F(B) \to \mathcal{F}(B)$ is an isomorphism.

3.7.B. TRICKY EXERCISE. Verify that $F(B) \to \mathcal{F}(B)$ is an isomorphism, likely by describing the inverse map $\mathcal{F}(B) \to F(B)$, and verifying that it is indeed inverse. Possible hint: elements of $\mathcal{F}(B)$ are determined by stalks, as are elements of $F(B)$.

It will be clear from your solution to the Tricky Exercise 3.7.B that the restriction maps for F are the same as the restriction maps of $\mathcal{F}$ (for elements of the base). $\square$

Theorem 3.7.1 shows that sheaves on X can be recovered from their "restriction to a base". It is clear from the argument (and in particular the solution to the Tricky Exercise 3.7.B) that if $\mathcal{F}$ is a sheaf and F is the corresponding sheaf on the base B, then for any x, $\mathcal{F}_x$ is naturally isomorphic to F_x.

Theorem 3.7.1 is a statement about *objects* in a category, so we should hope for a similar statement about *morphisms*.

3.7.C. IMPORTANT EXERCISE: MORPHISMS OF SHEAVES CORRESPOND TO MORPHISMS OF SHEAVES ON A BASE. Suppose $\{B_i\}$ is a base for the topology of X. A morphism $F \to G$ on sheaves on the base is a collection of maps $F(B_k) \to G(B_k)$ such that the diagram

$$\begin{tikzcd} F(B_i) \arrow[r] \arrow[d, "\mathrm{res}_{B_i,B_j}"'] & G(B_i) \arrow[d, "\mathrm{res}_{B_i,B_j}"] \\ F(B_j) \arrow[r] & G(B_j) \end{tikzcd}$$

commutes for all $B_j \hookrightarrow B_i$.
(a) Verify that a morphism of sheaves is determined by the induced morphism of sheaves on the base
(b) Show that a morphism of sheaves on the base gives a morphism of the induced sheaves. (Possible hint: compatible stalks.)

3.7.2. *Remark.* The above constructions and arguments describe an equivalence of categories (§2.2.21) between sheaves on X and sheaves on a given base of X. There is no new content to this statement, but you may wish to think through what it means. What are the functors in each direction? Why aren't their compositions the identity?

3.7.3. *Remark.* It will be useful to extend these notions to $\mathcal{O}_X$-modules (see for example Exercise 14.3.C). You will readily be able to verify that there is a correspondence (really, equivalence of categories) between $\mathcal{O}_X$-modules and $\mathcal{O}_X$-modules on a base. Rather than working out the details, you should just informally think through the main points: what is an "$\mathcal{O}_X$-module on a base"? Given an $\mathcal{O}_X$-module on a base, why is the corresponding sheaf naturally an $\mathcal{O}_X$-module? Later, if you are forced at gunpoint to fill in details, you will be able to.

3.7.D. IMPORTANT EXERCISE. Suppose $X = \cup U_i$ is an open cover of X, and we have sheaves $\mathcal{F}_i$ on U_i along with isomorphisms $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ (with ϕ_{ii} the identity) that agree on triple overlaps, i.e. $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on $U_i \cap U_j \cap U_k$ (this is called the **cocycle condition**, for reasons we ignore). Show that these sheaves can be glued together into a sheaf $\mathcal{F}$ on X (unique up to unique isomorphism), such that $\mathcal{F}_i \cong \mathcal{F}|_{U_i}$, and the isomorphisms over $U_i \cap U_j$ are the obvious ones. (Thus we can "glue sheaves together", using limited patching information.) Warning: we are not assuming this is a finite cover, so you cannot use induction. For this reason this exercise can be perplexing. (You can use the ideas of this section to solve this problem, but you don't necessarily need to. Hint: As the base, take those open sets contained in *some* U_i. Small observation: the hypothesis on ϕ_{ii} is extraneous, as it follows from the cocycle condition.)

3.7.4. *Remark for experts.* Exercise 3.7.D almost says that the "set" of sheaves forms a sheaf itself, but not quite. Making this precise leads one to the notion of a *stack*.

3.7.E. UNIMPORTANT EXERCISE. Suppose a morphism of sheaves $F \to G$ on a base B_i is surjective for all B_i (i.e. $F(B_i) \to G(B_i)$ is surjective for all i). Show that the morphism of sheaves (*not* on the base) is surjective. The converse is not true, unlike the case for injectivity. This gives a useful criterion for surjectivity: a morphism of sheaves is surjective if it is surjective for sections on a base. You may enjoy trying this out with Example 3.4.9 (dealing with holomorphic functions in the classical topology on $X = \mathbb{C}$), showing that the exponential map $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ is surjective, using the base of contractible open sets.

3.7.5. *Observation.* In the proof of Theorem 3.7.1, we need even less information than given in the hypotheses. What we are really using is that the opens in the base, and their inclusions, form a filtered set. You will appreciate this observation much later, in the proof of Theorem 14.3.2.

Part II

Schemes

CHAPTER 4

Toward affine schemes: the underlying set, and topological space

The very idea of scheme is of infantile simplicity — so simple, so humble, that no one before me thought of stooping so low. So childish, in short, that for years, despite all the evidence, for many of my erudite colleagues, it was really "not serious"! — Grothendieck

4.1 Toward schemes

We are now ready to consider the notion of a *scheme*, which is the type of geometric space central to algebraic geometry. We should first think through what we mean by "geometric space". You have likely seen the notion of a manifold, and we wish to abstract this notion so that it can be generalized to other settings, notably so that we can deal with non-smooth and arithmetic objects.

The key insight behind this generalization is the following: we can understand a geometric space (such as a manifold) well by understanding the functions on this space. More precisely, we will understand it through the sheaf of functions on the space. If we are interested in differentiable manifolds, we will consider differentiable functions; if we are interested in smooth manifolds, we will consider smooth functions; and so on.

Thus we will define a scheme to be the following data

- *The set:* the points of the scheme
- *The topology:* the open sets of the scheme
- *The structure sheaf:* the sheaf of "algebraic functions" (a sheaf of rings) on the scheme.

Recall that a topological space with a sheaf of rings is called a *ringed space* (§3.2.12).

We will try to draw pictures throughout. Pictures can help develop geometric intuition, which can guide the algebraic development (and, eventually, vice versa). Some people find pictures very helpful, while others are repulsed or nonplussed or confused.

We will try to make all three notions as intuitive as possible. For the set, in the key example of complex (affine) varieties (roughly, things cut out in $\mathbb{C}^n$ by polynomials), we will see that the points are the "traditional points" (n-tuples of complex numbers), plus some extra points that will be handy to have around. For the topology, we will require that "algebraic functions vanish on closed sets", and require nothing else. For the sheaf of algebraic functions (the structure sheaf), we will expect that in the complex plane, $(3x^2 + y^2)/(2x + 4xy + 1)$ should be an algebraic function on the open set consisting of points where the denominator doesn't vanish, and this will largely motivate our definition.

4.1.1. Example: Differentiable manifolds. As motivation, we return to our example of differentiable manifolds, reinterpreting them in this light. We will be quite informal in this discussion. Suppose X is a manifold. It is a topological space, and has a *sheaf of differentiable functions* $\mathcal{O}_X$ (see §3.1). This gives X the structure of a ringed space. We have observed that evaluation at a point $p \in X$ gives a surjective map from the stalk to $\mathbb{R}$

$$\mathcal{O}_{X,p} \twoheadrightarrow \mathbb{R},$$

so the kernel, the (germs of) functions vanishing at p, is a maximal ideal $\mathfrak{m}_X$ (see §3.1.1).

We could *define* a differentiable real manifold as a topological space X with a sheaf of rings. We would require that there is a cover of X by open sets such that on each open set the ringed space is isomorphic to a ball around the origin in $\mathbb{R}^n$ (with the sheaf of differentiable functions on that ball). With this definition, the ball is the basic patch, and a general manifold is obtained by gluing these patches together. (Admittedly, a great deal of geometry comes from how one chooses to patch the balls together!) In the algebraic setting, the basic patch is the notion of an *affine scheme*, which we will discuss soon. (In the definition of manifold, there is an additional requirement that the topological space be Hausdorff, to avoid pathologies. Schemes are often required to be "separated" to avoid essentially the same pathologies. Separatedness will be discussed in Chapter 11.)

Functions are determined by their values at points. This is an obvious statement, but won't be true for schemes in general. We will see an example in Exercise 4.2.A(a), and discuss this behavior further in §4.2.9.

Morphisms of manifolds. How can we describe differentiable maps of manifolds $X \to Y$? They are certainly continuous maps — but which ones? We can pull back functions along continuous maps. Differentiable functions pull back to differentiable functions. More formally, we have a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. (The inverse image sheaf f^{-1} was defined in §3.6.) Inverse image is left-adjoint to pushforward, so we also get a map $f^\sharp \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Certainly given a differentiable map of manifolds, differentiable functions pull back to differentiable functions. It is less obvious that *this is a sufficient condition for a continuous function to be differentiable*.

4.1.A. IMPORTANT EXERCISE FOR THOSE WITH A LITTLE EXPERIENCE WITH MANIFOLDS. Prove that a continuous function of differentiable manifolds $f : X \to Y$ is differentiable if differentiable functions pull back to differentiable functions, i.e. if pullback by f gives a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. (Hint: check this on small patches. Once you figure out what you are trying to show, you will realize that the result is immediate.)

4.1.B. EXERCISE. Show that a morphism of differentiable manifolds $f : X \to Y$ with $f(p) = q$ induces a morphism of stalks $f^\sharp : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$. Show that $f^\sharp(\mathfrak{m}_{Y,q}) \subset \mathfrak{m}_{X,p}$. In other words, if you pull back a function that vanishes at q, you get a function that vanishes at p — not a huge surprise. (In §7.3, we formalize this by saying that maps of differentiable manifolds are maps of locally ringed spaces.)

4.1.2. *Aside.* Here is a little more for experts: Notice that this induces a map on tangent spaces (see Aside 3.1.2)

$$(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2)^\vee \to (\mathfrak{m}_{Y,q}/\mathfrak{m}_{Y,q}^2)^\vee.$$

This is the tangent map you would geometrically expect. Again, it is interesting that the cotangent map $\mathfrak{m}_{Y,q}/\mathfrak{m}_{Y,q}^2 \to \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$ is algebraically more natural than the tangent map (there are no "duals").

Experts are now free to try to interpret other differential-geometric information using only the map of topological spaces and map of sheaves. For example: how can one check if f is a smooth map? How can one check if f is an immersion? (We will see that the algebro-geometric version of these notions are *smooth morphism* and *locally closed immersion*, see Chapter 26 and §9.1.3 respectively.)

4.1.3. *Side Remark.* Manifolds are covered by disks that are all isomorphic. This isn't true for schemes (even for "smooth complex varieties"). There are examples of two "smooth complex curves" (the algebraic version of Riemann surfaces) X and Y so that no non-empty open subset of X is isomorphic to a non-empty open subset of Y. And there is an example of a Riemann surface X such that no two open subsets of X are isomorphic. Informally, this is because in the Zariski topology on schemes, all non-empty open sets are "huge" and have more "structure".

4.1.4. Other examples. If you are interested in differential geometry, you will be interested in differentiable manifolds, on which the functions under consideration are differentiable functions. Similarly, if you are interested in topology, you will be interested in topological spaces, on which you will consider the continuous function. If you are interested in complex geometry, you will be interested in complex manifolds (or possibly "complex analytic varieties"), on which the functions are holomorphic functions. In each of these cases of interesting "geometric spaces", the topological space and sheaf of functions is clear. The notion of scheme fits naturally into this family.

4.2 The underlying set of affine schemes

For any ring A, we are going to define something called $\operatorname{Spec} A$, the **spectrum of** A. In this section, we will define it as a set, but we will soon endow it with a topology, and later we will define a sheaf of rings on it (the structure sheaf). Such an object is called an *affine scheme*. Later $\operatorname{Spec} A$ will denote the set along with the topology, and a sheaf of functions. But for now, as there is no possibility of confusion, $\operatorname{Spec} A$ will just be the set.

4.2.1. The set $\operatorname{Spec} A$ is the set of prime ideals of A. The prime ideal $\mathfrak{p}$ of A when considered as an element of $\operatorname{Spec} A$ will be denoted $[\mathfrak{p}]$, to avoid confusion. Elements $a \in A$ will be called **functions on** $\operatorname{Spec} A$, and their **value** at the point $[\mathfrak{p}]$ will be $a \pmod{\mathfrak{p}}$. *This is weird: a function can take values in different rings at different points — the function* 5 *on* $\operatorname{Spec} \mathbb{Z}$ *takes the value* $1 \pmod 2$ *at* $[(2)]$ *and* $2 \pmod 3$ *at* $[(3)]$. "An element a of the ring lying in a prime ideal $\mathfrak{p}$" translates to "a function a that is 0 at the point $[\mathfrak{p}]$" or "a function a vanishing at the point $[\mathfrak{p}]$", and we will use these phrases interchangeably. Notice that if you add or multiply two functions, you add or multiply their values at all points; this is a translation of the fact that $A \to A/\mathfrak{p}$ is a ring homomorphism. These translations are important — make sure you are very comfortable with them! They should become second nature.

We now give some examples.

Example 1 (the complex affine line): $\mathbb{A}^1_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x]$**.** Let's find the prime ideals of $\mathbb{C}[x]$. As $\mathbb{C}[x]$ is an integral domain, 0 is prime. Also, $(x-a)$ is prime, for any $a \in \mathbb{C}$: it is even a maximal ideal, as the quotient by this ideal is a field:

$$0 \longrightarrow (x-a) \longrightarrow \mathbb{C}[x] \xrightarrow{f \mapsto f(a)} \mathbb{C} \longrightarrow 0$$

(This exact sequence may remind you of (3.1.1.1) in our motivating example of manifolds.)

We now show that there are no other prime ideals. We use the fact that $\mathbb{C}[x]$ has a division algorithm, and is a unique factorization domain. Suppose $\mathfrak{p}$ is a prime ideal. If $\mathfrak{p} \neq (0)$, then suppose $f(x) \in \mathfrak{p}$ is a non-zero element of smallest degree. It is not constant, as prime ideals can't contain 1. If $f(x)$ is not linear, then factor $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have positive degree. (Here we use that $\mathbb{C}$ is algebraically closed.) Then $g(x) \in \mathfrak{p}$ or $h(x) \in \mathfrak{p}$, contradicting the minimality of the degree of f. Hence there is a linear element $x-a$ of $\mathfrak{p}$. Then I claim that $\mathfrak{p} = (x-a)$. Suppose $f(x) \in \mathfrak{p}$. Then the division algorithm would give $f(x) = g(x)(x-a) + m$ where $m \in \mathbb{C}$. Then $m = f(x) - g(x)(x-a) \in \mathfrak{p}$. If $m \neq 0$, then $1 \in \mathfrak{p}$, giving a contradiction.

Thus we have a picture of $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$ (see Figure 4.1). There is one point for each complex number, plus one extra point $[(0)]$. We can mostly picture $\mathbb{A}^1_{\mathbb{C}}$ as $\mathbb{C}$: the point $[(x-a)]$ we will reasonably associate to $a \in \mathbb{C}$. Where should we picture the point $[(0)]$? The best way of thinking about it is somewhat zen. It is somewhere on the complex line, but nowhere in particular. Because (0) is contained in all of these primes, we will somehow associate it with this line passing through all the other points. $[(0)]$ is called the "generic point" of the line; it is "generically on the line" but you can't pin it down any further than that. (We will formally define "generic point" in §4.6.) We will place it far to the right for lack of anywhere better to put it. You will notice that we sketch $\mathbb{A}^1_{\mathbb{C}}$ as one-(real-)dimensional (even though we picture it as an enhanced version of $\mathbb{C}$); this is to later remind ourselves that this will be a one-dimensional space, where dimensions are defined in an algebraic (or complex-geometric) sense. (Dimension will be defined in Chapter 12.)

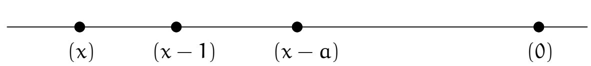


FIGURE 4.1. A picture of $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$

To give you some feeling for this space, we make some statements that are currently undefined, but suggestive. The functions on $\mathbb{A}^1_{\mathbb{C}}$ are the polynomials. So $f(x) = x^2 - 3x + 1$ is a function. What is its value at $[(x-1)]$, which we think of as the point $1 \in \mathbb{C}$? Answer: $f(1)$! Or equivalently, we can evalute $f(x)$ modulo $x-1$

— this is the same thing by the division algorithm. (What is its value at (0)? It is $f(x) \pmod 0$, which is just $f(x)$.)

Here is a more complicated example: $g(x) = (x-3)^3/(x-2)$ is a "rational function". It is defined everywhere but $x = 2$. (When we know what the structure sheaf is, we will be able to say that it is an element of the structure sheaf on the open set $\mathbb{A}^1_{\mathbb{C}} \setminus \{2\}$.) We want to say that $g(x)$ has a triple zero at 3, and a single pole at 2, and we will be able to after §13.4.

Example 2 (the affine line over $k = \overline{k}$): $\mathbb{A}^1_k := \operatorname{Spec} k[x]$ where k is an algebraically closed field. This is called the affine line over k. All of our discussion in the previous example carries over without change. We will use the same picture, which is after all intended to just be a metaphor.

Example 3: $\operatorname{Spec} \mathbb{Z}$. An amazing fact is that from our perspective, this will look a lot like the affine line $\mathbb{A}^1_{\overline{k}}$. The integers, like $\overline{k}[x]$, form a unique factorization domain, with a division algorithm. The prime ideals are: (0), and (p) where p is prime. Thus everything from Example 1 carries over without change, even the picture. Our picture of $\operatorname{Spec} \mathbb{Z}$ is shown in Figure 4.2.

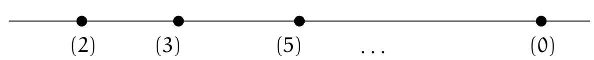


FIGURE 4.2. A "picture" of $\operatorname{Spec} \mathbb{Z}$, which looks suspiciously like Figure 4.1

Let's blithely carry over our discussion of functions to this space. 100 is a function on $\operatorname{Spec} \mathbb{Z}$. Its value at (3) is "$1 \pmod 3$". Its value at (2) is "$0 \pmod 2$", and in fact it has a double zero. 27/4 is a rational function on $\operatorname{Spec} \mathbb{Z}$, defined away from (2). We want to say that it has a double pole at (2), and a triple zero at (3). Its value at (5) is

$$27 \times 4^{-1} \equiv 2 \times (-1) \equiv 3 \pmod 5.$$

Example 4: silly but important examples, and the German word for bacon. The set $\operatorname{Spec} k$ where k is any field is boring: one point. $\operatorname{Spec} 0$, where 0 is the zero-ring, is the empty set, as 0 has no prime ideals.

4.2.A. A SMALL EXERCISE ABOUT SMALL SCHEMES. (a) Describe the set $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$. The ring $k[\epsilon]/(\epsilon^2)$ is called the ring of **dual numbers**, and will turn out to be quite useful. You should think of ϵ as a very small number, so small that its square is 0 (although it itself is not 0). It is a non-zero function whose value at all points is zero, thus giving our first example of functions not being determined by their values at points. We will discuss this phenomenon further in §4.2.9.
(b) Describe the set $\operatorname{Spec} k[x]_{(x)}$ (see §2.3.3 for discussion of localization). We will see this scheme again repeatedly, starting with §4.2.6 and Exercise 4.4.J. You might later think of it as a shred of a particularly nice "smooth curve".

In Example 2, we restricted to the case of algebraically closed fields for a reason: things are more subtle if the field is not algebraically closed.

Example 5 (the affine line over $\mathbb{R}$): $\mathbb{R}[x]$. Using the fact that $\mathbb{R}[x]$ is a unique factorization domain, similar arguments to those of Examples 1–3 show that the primes are (0), $(x-a)$ where $a \in \mathbb{R}$, and (x^2+ax+b) where x^2+ax+b is an irreducible quadratic. The latter two are maximal ideals, i.e. their quotients are fields. For example: $\mathbb{R}[x]/(x-3) \cong \mathbb{R}$, $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.

4.2.B. UNIMPORTANT EXERCISE. Show that for the last type of prime, of the form (x^2+ax+b), the quotient is *always* isomorphic to $\mathbb{C}$.

So we have the points that we would normally expect to see on the real line, corresponding to real numbers; the generic point 0; and new points which we may interpret as *conjugate pairs* of complex numbers (the roots of the quadratic). This last type of point should be seen as more akin to the real numbers than to the generic point. You can picture $\mathbb{A}^1_{\mathbb{R}}$ as the complex plane, folded along the real axis. But the key point is that Galois-conjugate points (such as i and $-i$) are considered glued.

Let's explore functions on this space. Consider the function $f(x) = x^3 - 1$. Its value at the point $[(x-2)]$ is $f(x) = 7$, or perhaps better, $7 \pmod{x-2}$. How about at (x^2+1)? We get

$$x^3 - 1 \equiv -x - 1 \pmod{x^2+1},$$

which may be profitably interpreted as $-i-1$.

One moral of this example is that we can work over a non-algebraically closed field if we wish. It is more complicated, but we can recover much of the information we care about.

4.2.C. IMPORTANT EXERCISE. Describe the set $\mathbb{A}^1_{\mathbb{Q}}$. (This is harder to picture in a way analogous to $\mathbb{A}^1_{\mathbb{R}}$. But the rough cartoon of points on a line, as in Figure 4.1, remains a reasonable sketch.)

Example 6 (the affine line over $\mathbb{F}_p$): $\mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[x]$. As in the previous examples, $\mathbb{F}_p[x]$ is a Euclidean domain, so the prime ideals are of the form (0) or $(f(x))$ where $f(x) \in \mathbb{F}_p[x]$ is an irreducible polynomial, which can be of any degree. Irreducible polynomials correspond to sets of Galois conjugates in $\overline{\mathbb{F}}_p$.

Note that $\operatorname{Spec} \mathbb{F}_p[x]$ has p points corresponding to the elements of $\mathbb{F}_p$, but also (infinitely) many more. This makes this space much richer than simply p points. For example, a polynomial $f(x)$ is not determined by its values at the p elements of $\mathbb{F}_p$, but it *is* determined by its values at the points of $\operatorname{Spec} \mathbb{F}_p[x]$. (As we have mentioned before, this is not true for all schemes.)

You should think about this, even if you are a geometric person — this intuition will later turn up in geometric situations. Even if you think you are interested only in working over an algebraically closed field (such as $\mathbb{C}$), you will have non-algebraically closed fields (such as $\mathbb{C}(x)$) forced upon you.

Example 7 (the complex affine plane): $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$. (As with Examples 1 and 2, our discussion will apply with $\mathbb{C}$ replaced by *any* algebraically closed field.) Sadly, $\mathbb{C}[x,y]$ is not a principal ideal domain: (x,y) is not a principal ideal. We can quickly name *some* prime ideals. One is (0), which has the same flavor as the (0) ideals in the previous examples. $(x-2, y-3)$ is prime, and indeed maximal, because $\mathbb{C}[x,y]/(x-2,y-3) \cong \mathbb{C}$, where this isomorphism is via $f(x,y) \mapsto f(2,3)$. More generally, $(x-a, y-b)$ is prime for any $(a,b) \in \mathbb{C}^2$. Also, if $f(x,y)$ is an irreducible polynomial (e.g. $y - x^2$ or $y^2 - x^3$) then $(f(x,y))$ is prime.

4.2.D. EXERCISE. (We will see a different proof of this in §12.2.3.) Show that we have identified all the prime ideals of $\mathbb{C}[x,y]$. Hint: Suppose $\mathfrak{p}$ is a prime ideal that is not principal. Show you can find $f(x,y), g(x,y) \in \mathfrak{p}$ with no common factor. By considering the Euclidean algorithm in the Euclidean domain $\mathbb{C}(x)[y]$, show that you can find a nonzero $h(x) \in (f(x,y), g(x,y)) \subset \mathfrak{p}$. Using primality, show that one of the linear factors of $h(x)$, say $(x-a)$, is in $\mathfrak{p}$. Similarly show there is some $(y-b) \in \mathfrak{p}$.

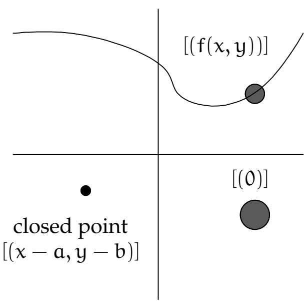


FIGURE 4.3. Picturing $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$

We now attempt to draw a picture of $\mathbb{A}^2_{\mathbb{C}}$ (see Figure 4.3). The maximal primes of $\mathbb{C}[x,y]$ correspond to the traditional points in $\mathbb{C}^2$: $[(x-a, y-b)]$ corresponds to $(a,b) \in \mathbb{C}^2$. We now have to visualize the "bonus points". $[(0)]$ somehow lives behind all of the traditional points; it is somewhere on the plane, but nowhere in particular. So for example, it does not lie on the parabola $y = x^2$. The point $[(y-x^2)]$ lies on the parabola $y = x^2$, but nowhere in particular on it. (Figure 4.3 is a bit misleading. For example, the point $[(0)]$ isn't in the fourth quadrant; it is somehow near every other point, which is why it is depicted as a somewhat diffuse large dot.) You can see from this picture that we already are implicitly thinking about "dimension". The primes $(x-a, y-b)$ are somehow of dimension 0, the primes $(f(x,y))$ are of dimension 1, and (0) is of dimension 2. (All of our dimensions here are *complex* or *algebraic* dimensions. The complex plane $\mathbb{C}^2$ has real dimension 4, but complex dimension 2. Complex dimensions are in general half of real dimensions.) We won't define dimension precisely until Chapter 12, but you should feel free to keep it in mind before then.

Note too that maximal ideals correspond to the "smallest" points. Smaller ideals correspond to "bigger" points. "One prime ideal contains another" means that the points "have the opposite containment." All of this will be made precise once we have a topology. This order-reversal is a little confusing, and will remain so even once we have made the notions precise.

We now come to the obvious generalization of Example 7.

Example 8 (complex affine n-space — important!): Let $\mathbb{A}^n_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$. (More generally, $\mathbb{A}^n_A$ is defined to be $\operatorname{Spec} A[x_1, \dots, x_n]$, where A is an arbitrary ring. When the base ring is clear from context, the subscript A is often omitted.)

For concreteness, let's consider $n = 3$. We now have an interesting question in what at first appears to be pure algebra: What are the prime ideals of $\mathbb{C}[x,y,z]$?

Analogously to before, $(x-a, y-b, z-c)$ is a prime ideal. This is a maximal ideal, with residue field $\mathbb{C}$; we think of these as "0-dimensional points". We will often write (a,b,c) for $[(x-a, y-b, z-c)]$ because of our geometric interpretation of these ideals. There are no more maximal ideals, by Hilbert's Weak Nullstellensatz.

4.2.2. Hilbert's Weak Nullstellensatz. — *If* k *is an algebraically closed field, then the maximal ideals* $k[x_1, \dots, x_n]$*, are precisely those of the form* $(x_1 - a_1, \dots, x_n - a_n)$*, where* $a_i \in k$*.*

We may as well state a slightly stronger version now.

4.2.3. Hilbert's Nullstellensatz. — *If* k *is any field, the maximal ideals of* $k[x_1, \dots, x_n]$ *are precisely those with residue field a finite extension of* k*.*

The Nullstellensatz 4.2.3 clearly implies the Weak Nullstellensatz 4.2.2. You will prove the Nullstellensatz in Exercise 12.2.B.

There are other prime ideals of $\mathbb{C}[x,y,z]$ too. We have (0), which is corresponds to a "3-dimensional point". We have $(f(x,y,z))$, where f is irreducible. To this we associate the hypersurface $f = 0$, so this is "2-dimensional" in nature. But we have not found them all! One clue: we have prime ideals of "dimension" 0, 2, and 3 — we are missing "dimension 1". Here is one such prime ideal: (x,y). We picture this as the locus where $x = y = 0$, which is the z-axis. This is a prime ideal, as the corresponding quotient $\mathbb{C}[x,y,z]/(x,y) \cong \mathbb{C}[z]$ is an integral domain (and should be interpreted as the functions on the z-axis). There are lots of one-dimensional primes, and it is not possible to classify them in a reasonable way. It will turn out that they correspond to things that we think of as irreducible curves. Thus remarkably the answer to the purely algebraic question ("what are the primes of $\mathbb{C}[x,y,z]$") is fundamentally geometric!

The fact that the closed points $\mathbb{A}^1_{\mathbb{Q}}$ can be interpreted as points of $\overline{\mathbb{Q}}$ where Galois-conjugates are glued together (Exercise 4.2.C) extends to $\mathbb{A}^n_{\mathbb{Q}}$. For example, in $\mathbb{A}^2_{\mathbb{Q}}$, $(\sqrt{2}, \sqrt{2})$ is glued to $(-\sqrt{2}, -\sqrt{2})$ but not to $(\sqrt{2}, -\sqrt{2})$. The following exercise will give you some idea of how this works.

4.2.E. EXERCISE. Describe the maximal ideal of $\mathbb{Q}[x,y]$ corresponding to $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Describe the maximal ideal of $\mathbb{Q}[x,y]$ corresponding to $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$. What are the residue fields in both cases?

The description of closed points of $\mathbb{A}^2_{\mathbb{Q}}$ (and its generalizations) as Galois-orbits can even be extended to non-closed points, as follows.

4.2.F. UNIMPORTANT AND TRICKY BUT FUN EXERCISE. Consider the map of sets $\phi : \mathbb{C}^2 \to \mathbb{A}^2_{\mathbb{Q}}$ defined as follows. (z_1, z_2) is sent to the prime ideal of $\mathbb{Q}[x,y]$ consisting of polynomials vanishing at (z_1, z_2).
(a) What is the image of (π, π^2)?
$\star$ (b) Show that ϕ is surjective. (Warning: You will need some ideas we haven't discussed in order to solve this. Once we define the Zariski topology on $\mathbb{A}^2_{\mathbb{Q}}$, you will be able to check that ϕ is continuous, where we give $\mathbb{C}^2$ the classical topology. This example generalizes.)

4.2.4. Quotients and localizations. Two natural ways of getting new rings from old — quotients and localizations — have interpretations in terms of spectra.

4.2.5. *Quotients:* $\operatorname{Spec} A/I$ *as a subset of* $\operatorname{Spec} A$. It is an important fact that the primes of A/I are in bijection with the primes of A containing I.

4.2.G. ESSENTIAL ALGEBRA EXERCISE (MANDATORY IF YOU HAVEN'T SEEN IT BEFORE). Suppose A is a ring, and I an ideal of A. Let $\phi : A \to A/I$. Show that ϕ^{-1} gives an inclusion-preserving bijection between primes of A/I and primes of A containing I. Thus we can picture $\operatorname{Spec} A/I$ as a subset of $\operatorname{Spec} A$.

As an important motivational special case, you now have a picture of *complex affine varieties*. Suppose A is a finitely generated $\mathbb{C}$-algebra, generated by $x_1, \dots, x_n$, with relations $f_1(x_1, \dots, x_n) = \cdots = f_r(x_1, \dots, x_n) = 0$. Then this description in terms of generators and relations naturally gives us an interpretation of $\operatorname{Spec} A$ as a subset of $\mathbb{A}^n_{\mathbb{C}}$, which we think of as "traditional points" (n-tuples of complex numbers) along with some "bonus" points we haven't yet fully described. To see which of the traditional points are in $\operatorname{Spec} A$, we simply solve the equations $f_1 = \cdots = f_r = 0$. For example, $\operatorname{Spec} \mathbb{C}[x, y, z]/(x^2+y^2-z^2)$ may be pictured as shown in Figure 4.4. (Admittedly this is just a "sketch of the $\mathbb{R}$-points", but we will still find it helpful later.) This entire picture carries over (along with the Nullstellensatz) with $\mathbb{C}$ replaced by any algebraically closed field. Indeed, the picture of Figure 4.4 can be said to depict $k[x, y, z]/(x^2 + y^2 - z^2)$ for most algebraically closed fields k (although it is misleading in characteristic 2, because of the coincidence $x^2 + y^2 - z^2 = (x + y + z)^2$).

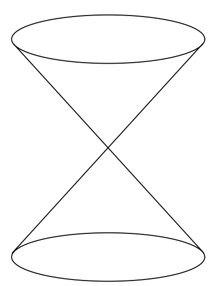


FIGURE 4.4. A "picture" of $\operatorname{Spec} \mathbb{C}[x, y, z]/(x^2 + y^2 - z^2)$

4.2.6. *Localizations:* $\operatorname{Spec} S^{-1}A$ *as a subset of* $\operatorname{Spec} A$. The following exercise shows how prime ideals behave under localization.

4.2.H. ESSENTIAL ALGEBRA EXERCISE (MANDATORY IF YOU HAVEN'T SEEN IT BEFORE). Suppose S is a multiplicative subset of A. Show that the map $\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ gives an order-preserving bijection of the primes of $S^{-1}A$ with the primes of A that *don't meet* the multiplicative set S.

Recall from §2.3.3 that there are two important flavors of localization. The first is $A_f = \{1, f, f^2, \dots\}^{-1}A$ where $f \in A$. A motivating example is $A = \mathbb{C}[x, y]$,

$f = y - x^2$. The second is $A_{\mathfrak{p}} = (A - \mathfrak{p})^{-1}A$, where $\mathfrak{p}$ is a prime ideal. A motivating example is $A = \mathbb{C}[x, y]$, $S = A - (x, y)$.

If $S = \{1, f, f^2, \dots\}$, the primes of $S^{-1}A$ are just those primes not containing f — the points where "f doesn't vanish". (In §4.5, we will call this a *distinguished open set*, once we know what open sets are.) So to picture $\operatorname{Spec} \mathbb{C}[x, y]_{y-x^2}$, we picture the affine plane, and throw out those points on the parabola $y - x^2$ — the points (a, a^2) for $a \in \mathbb{C}$ (by which we mean $[(x - a, y - a^2)]$), as well as the "new kind of point" $[(y - x^2)]$.

It can be initially confusing to think about localization in the case where zerodivisors are inverted, because localization $A \to S^{-1}A$ is not injective (Exercise 2.3.C). Geometric intuition can help. Consider the case $A = \mathbb{C}[x, y]/(xy)$ and $f = x$. What is the localization A_f? The space $\operatorname{Spec} \mathbb{C}[x, y]/(xy)$ "is" the union of the two axes in the plane. Localizing means throwing out the locus where x vanishes. So we are left with the x-axis, minus the origin, so we expect $\operatorname{Spec} \mathbb{C}[x]_x$. So there should be some natural isomorphism $(\mathbb{C}[x, y]/(xy))_x \cong \mathbb{C}[x]_x$.

4.2.I. EXERCISE. Show that these two rings are isomorphic. (You will see that y on the left goes to 0 on the right.)

If $S = A - \mathfrak{p}$, the primes of $S^{-1}A$ are just the primes of A contained in $\mathfrak{p}$. In our example $A = \mathbb{C}[x, y]$, $\mathfrak{p} = (x, y)$, we keep all those points corresponding to "things through the origin", i.e. the 0-dimensional point (x, y), the 2-dimensional point (0), and those 1-dimensional points $(f(x, y))$ where $f(0, 0) = 0$, i.e. those "irreducible curves through the origin". You can think of this being a shred of the plane near the origin; anything not actually "visible" at the origin is discarded (see Figure 4.5).

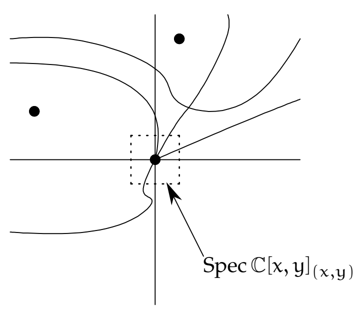


FIGURE 4.5. Picturing $\operatorname{Spec} \mathbb{C}[x, y]_{(x,y)}$ as a "shred of $\mathbb{A}^2_{\mathbb{C}}$". Only those points near the origin remain.

Another example is when $A = \operatorname{Spec} k[x]$, and $\mathfrak{p} = (x)$ (or more generally when $\mathfrak{p}$ is any maximal ideal). Then $A_{\mathfrak{p}}$ has only two prime ideals (Exercise 4.2.A(b)). You should see this as the germ of a "smooth curve", where one point is the "classical point", and the other is the "generic point of the curve". This is an example of a discrete valuation ring, and indeed all discrete valuation rings should be visualized in such a way. We will discuss discrete valuation rings in §13.4. By then we will have justified the use of the words "smooth" and "curve". (Reality check: try to picture Spec of $\mathbb{Z}$ localized at (2) and at (0). How do the two pictures differ?)

4.2.7. Important fact: Maps of rings induce maps of spectra (as sets). We now make an observation that will later grow up to be the notion of morphisms of schemes.

4.2.J. IMPORTANT EASY EXERCISE. If $\phi : B \to A$ is a map of rings, and $\mathfrak{p}$ is a prime ideal of A, show that $\phi^{-1}(\mathfrak{p})$ is a prime ideal of B.

Hence a map of rings $\phi : B \to A$ induces a map of sets $\operatorname{Spec} A \to \operatorname{Spec} B$ "in the opposite direction". This gives a contravariant functor from the category of rings to the category of sets: the composition of two maps of rings induces the composition of the corresponding maps of spectra.

4.2.K. EASY EXERCISE. Let B be a ring.
(a) Suppose $I \subset B$ is an ideal. Show that the map $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is the inclusion of §4.2.5.
(b) Suppose $S \subset B$ is a multiplicative set. Show that the map $\operatorname{Spec} S^{-1}B \to \operatorname{Spec} B$ is the inclusion of §4.2.6.

4.2.8. *An explicit example.* In the case of affine complex varieties (or indeed affine varieties over any algebraically closed field), the translation between maps given by explicit formulas and maps of rings is quite direct. For example, consider a map from the parabola in $\mathbb{C}^2$ (with coordinates a and b) given by $b = a^2$, to the "curve" in $\mathbb{C}^3$ (with coordinates x, y, and z) cut out by the equations $y = x^2$ and $z = y^2$. Suppose the map sends the point $(a, b) \in \mathbb{C}^2$ to the point $(a, b, b^2) \in \mathbb{C}^3$. In our new language, we have map

$$\operatorname{Spec} \mathbb{C}[a, b]/(b - a^2) \longrightarrow \operatorname{Spec} \mathbb{C}[x, y, z]/(y - x^2, z - y^2)$$

given by

$$\mathbb{C}[a, b]/(b - a^2) \longleftarrow \mathbb{C}[x, y, z]/(y - x^2, z - y^2)$$

$$(a, b, b^2) \longleftarrow\!\shortmid (x, y, z),$$

i.e. $x \mapsto a$, $y \mapsto b$, and $z \mapsto b^2$. If the idea is not yet clear, the following two exercises may help.

4.2.L. EXERCISE (SPECIAL CASE). Consider the map of complex manifolds sending $\mathbb{C} \to \mathbb{C}$ via $y \mapsto y^2$; you can picture it as the projection of the parabola $x = y^2$ in the plane to the x-axis (see Figure 4.6). Interpret the corresponding map of rings as given by $\mathbb{C}[x] \mapsto \mathbb{C}[y]$ by $x \mapsto y^2$. Verify that the preimage (the fiber) above the point $a \in \mathbb{C}$ is the point(s) $\pm\sqrt{a} \in \mathbb{C}$, using the definition given above. (A more sophisticated version of this example appears in Example 10.3.3.)

4.2.M. EXERCISE (GENERAL CASE). (a) Show that the map

$$\phi : (y_1, y_2, \dots, y_n) \mapsto (f_1(x_1, \dots, x_m), f_2(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m))$$

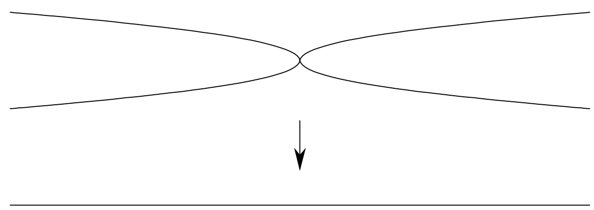


FIGURE 4.6. The map $\mathbb{C} \to \mathbb{C}$ given by $y \mapsto y^2$

determines a map

$$\operatorname{Spec} \mathbb{C}[x_1, \dots, x_m]/I \to \operatorname{Spec} \mathbb{C}[y_1, \dots, y_n]/J$$

if $\phi(J) \subset I$.
(b) Via the identification of the Nullstellensatz, interpret the map of (a) as a map $\mathbb{C}^m \to \mathbb{C}^n$ given by

$$(x_1, \dots, x_m) \mapsto (f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)).$$

The converse to (a) isn't quite true. Once you have more experience and intuition, you can figure out when it is true, and when it can be false. The failure of the converse to hold has to do with nilpotents, which we come to very shortly (§4.2.9).

4.2.N. IMPORTANT EXERCISE. Consider the map of sets $f : \mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$, given by the ring map $\mathbb{Z} \to \mathbb{Z}[x_1, \dots, x_n]$. If p is prime, describe a bijection between the fiber $f^{-1}([(p)])$ and $\mathbb{A}^n_{\mathbb{F}_p}$. (You won't need to describe either set! Which is good because you can't.) This exercise may give you a sense of how to picture maps (see Figure 4.7), and in particular why you can think of $\mathbb{A}^n_{\mathbb{Z}}$ as an "$\mathbb{A}^n$-bundle" over $\operatorname{Spec} \mathbb{Z}$. (Can you interpret the fiber over $[(0)]$ as $\mathbb{A}^n_k$ for some field k?)

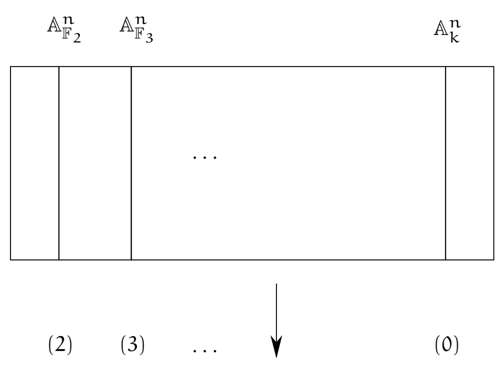


FIGURE 4.7. A picture of $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ as a "family of $\mathbb{A}^n$'s", or an "$\mathbb{A}^n$-bundle over $\operatorname{Spec} \mathbb{Z}$". What is k?

4.2.9. Functions are not determined by their values at points: the fault of nilpotents. We conclude this section by describing some strange behavior. We are developing machinery that will let us bring our geometric intuition to algebra. There is one serious serious point where your intuition will be false, so you should know now, and adjust your intuition appropriately. As noted by Mumford ([**M-CAS**, p. 12]), "it is this aspect of schemes which was most scandalous when Grothendieck defined them."

Suppose we have a function (ring element) vanishing at all points. Then it is not necessarily the zero function! The translation of this question is: is the intersection of all prime ideals necessarily just 0? The answer is no, as is shown by the example of the ring of dual numbers $k[\epsilon]/(\epsilon^2)$: $\epsilon \neq 0$, but $\epsilon^2 = 0$. (We saw this ring in Exercise 4.2.A(a).) Any function whose power is zero certainly lies in the intersection of all prime ideals.

4.2.O. EXERCISE. Ring elements that have a power that is 0 are called **nilpotents**.
(a) If I is an ideal of nilpotents, show that the inclusion $\operatorname{Spec} B/I \to \operatorname{Spec} B$ of Exercise 4.2.G is a bijection. Thus nilpotents don't affect the underlying set. (We will soon see in §4.4.5 that they won't affect the topology either — the difference will be in the structure sheaf.)
(b) Show that the nilpotents of a ring B form an ideal. This ideal is called the **nilradical**, and is denoted $\mathfrak{N} = \mathfrak{N}(B)$.

Thus the nilradical is contained in the intersection of all the prime ideals. The converse is also true:

4.2.10. Theorem. — *The nilradical $\mathfrak{N}(A)$ is the intersection of all the primes of A.*

4.2.P. EXERCISE. If you don't know this theorem, then look it up, or even better, prove it yourself. (Hint: Use the fact that any proper ideal of A is contained in a maximal ideal, which requires Zorn's lemma. Possible further hint: Suppose $x \notin \mathfrak{N}(A)$. We wish to show that there is a prime ideal not containing x. Show that A_x is not the 0-ring, by showing that $1 \neq 0$.)

4.2.11. In particular, although it is upsetting that functions are not determined by their values at points, we have precisely specified what the failure of this intuition is: two functions have the same values at points if and only if they differ by a nilpotent. You should think of this geometrically: a function vanishes at every point of the spectrum of a ring if and only if it has a power that is zero. And if there are no non-zero nilpotents — if $\mathfrak{N} = (0)$ — then functions *are* determined by their values at points. If a ring has no non-zero nilpotents, we say that it is **reduced**.

4.2.Q. EXERCISE (CONNECTION TO VARIETIES.) If k is an algebraically closed field, and $A = k[x_1, \dots, x_n]/I$ is a finitely generated k-algebra with $\mathfrak{N}(A) = \{0\}$. By Exercise 4.2.G, we can consider the set $\operatorname{Spec} A$ as a subset of $\mathbb{A}^n_k$. The space $\mathbb{A}^n_k$ considers the "classical" (old-fashioned) points k^n. Show that functions on A are determined by their values on the closed points, which by the Weak Nullstellensatz 4.2.2 are identified with the "classical" points $k^n \cap \operatorname{Spec} A$ of $\operatorname{Spec} A$. Hint: if f and g are 2 functions on X, then they differ on an open subset of X, so show that any nonempty open subset of X contains a closed point. (Remark: Before the advent of scheme theory, functions on varieties (over algebraically closed fields) were thought of as functions on "classical" points, and this exercise basically shows that there is no harm in thinking of "traditional" varieties as a particular kind of schemes.)

4.2.R. FUN UNIMPORTANT EXERCISE: DERIVATIVES WITHOUT DELTAS AND EPSILONS (OR AT LEAST WITHOUT DELTAS). Suppose we have a polynomial $f(x) \in k[x]$. Instead, we work in $k[x, \epsilon]/(\epsilon^2)$. What then is $f(x + \epsilon)$? (Do a couple of examples, then prove the pattern you observe.) This is a hint that nilpotents will be important in defining differential information (Chapter 23).

4.3 Visualizing schemes I: generic points

For years, you have been able to picture $x^2 + y^2 = 1$ in the plane, and you now have an idea of how to picture $\operatorname{Spec} \mathbb{Z}$. If we are claiming to understand rings as geometric objects (through the Spec functor), then we should wish to develop geometric insight into them. To develop geometric intuition about schemes, it is helpful to have pictures in your mind, extending your intuition about geometric spaces you are already familiar with. As we go along, we will empirically develop some idea of what schemes should look like. This section summarizes what we have gleaned so far.

Some mathematicians prefer to think completely algebraically, and never think in terms of pictures. Others will be disturbed by the fact that this is an art, not a science. And finally, this hand-waving will necessarily never be used in the rigorous development of the theory. For these reasons, you may wish to skip these sections. However, having the right picture in your mind can greatly help understanding what facts should be true, and how to prove them.

Our starting point is the example of "affine complex varieties" (things cut out by equations involving a finite number variables over $\mathbb{C}$), and more generally similar examples over arbitrary algebraically closed fields. We begin with notions that are intuitive ("traditional" points behaving the way you expect them to), and then add in the two features which are new and disturbing, generic points and nonreduced behavior. You can then extend this notion to seemingly different spaces, such as $\operatorname{Spec} \mathbb{Z}$.

Hilbert's Weak Nullstellensatz 4.2.2 shows that the "traditional points" are present as points of the scheme, and this carries over to any algebraically closed field. If the field is not algebraically closed, the traditional points are glued together into clumps by Galois conjugation, as in Examples 5 (the real affine line) and 6 (the affine line over $\mathbb{F}_p$) in §4.2 above. This is a geometric interpretation of Hilbert's Nullstellensatz 4.2.3.

But we have some additional points to add to the picture. You should remember that they "correspond" to "irreducible" "closed" (algebraic) subsets. As motivation, consider the case of the complex affine plane (Example 7): we had one for each irreducible polynomial, plus one corresponding to the entire plane. We will make "closed" precise when we define the Zariski topology (in the next section). You may already have an idea of what "irreducible" should mean; we make that precise at the start of §4.6. By "correspond" we mean that each closed irreducible subset has a corresponding point sitting on it, called its *generic point* (defined in §4.6). It is a new point, distinct from all the other points in the subset. The correspondence is described in Exercise 4.7.E for $\operatorname{Spec} A$, and in Exercise 6.1.B for schemes in general. We don't know precisely where to draw the generic point, so we may stick it arbitrarily anywhere, but you should think of it as being "almost everywhere", and in particular, near every other point in the subset.

In §4.2.5, we saw how the points of $\operatorname{Spec} A/I$ should be interpreted as a subset of $\operatorname{Spec} A$. So for example, when you see $\operatorname{Spec} \mathbb{C}[x, y]/(x+y)$, you should picture this not just as a line, but as a line in the xy-plane; the choice of generators x and y of the algebra $\mathbb{C}[x, y]$ implies an inclusion into affine space.

In §4.2.6, we saw how the points of $\operatorname{Spec} S^{-1}A$ should be interpreted as subsets of $\operatorname{Spec} A$. The two most important cases were discussed. The points of $\operatorname{Spec} A_f$ correspond to the points of $\operatorname{Spec} A$ where f doesn't vanish; we will later (§4.5) interpret this as a distinguished open set.

If $\mathfrak{p}$ is a prime ideal, then $\operatorname{Spec} A_{\mathfrak{p}}$ should be seen as a "shred of the space $\operatorname{Spec} A$ near the subset corresponding to $\mathfrak{p}$". The simplest nontrivial case of this is $\mathfrak{p} = (x) \subset \operatorname{Spec} k[x] = A$ (see Exercise 4.2.A, which we discuss again in Exercise 4.4.J).

4.4 The underlying topological space of an affine scheme

We next introduce the *Zariski topology* on the spectrum of a ring. For example, consider $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, the complex plane (with a few extra points). In algebraic geometry, we will only be allowed to consider algebraic functions, i.e. polynomials in x and y. The locus where a polynomial vanishes should reasonably be a closed set, and the Zariski topology is defined by saying that the only sets we should consider closed should be these sets, and other sets forced to be closed by these. In other words, it is the coarsest topology where these sets are closed.

In particular, although topologies are often described using open subsets, it will be more convenient for us to define this topology in terms of closed subsets. If S is a subset of a ring A, define the V**anishing set** of S by

$$V(S) := \{[\mathfrak{p}] \in \operatorname{Spec} A \ : \ S \subset \mathfrak{p}\}.$$

It is the set of points on which all elements of S are zero. (It should now be second nature to equate "vanishing at a point" with "contained in a prime".) We declare that these — and no other — are the closed subsets.

For example, consider $V(xy, yz) \subset \mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y, z]$. Which points are contained in this locus? We think of this as solving $xy = yz = 0$. Of the "traditional" points (interpreted as ordered triples of complex numbers, thanks to the Hilbert's Nullstellensatz 4.2.2), we have the points where $y = 0$ or $x = z = 0$: the xz-plane and the y-axis respectively. Of the "new" points, we have the generic point of the xz-plane (also known as the point $[(y)]$), and the generic point of the y-axis (also known as the point $[(x, z)]$). You might imagine that we also have a number of "one-dimensional" points contained in the xz-plane.

4.4.A. EASY EXERCISE. Check that the x-axis is contained in $V(xy, yz)$. (The x**-axis** is defined by $y = z = 0$, and the y-axis and z-axis are defined analogously.)

Let's return to the general situation. The following exercise lets us restrict attention to vanishing sets of *ideals*.

4.4.B. EASY EXERCISE. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$.

We define the **Zariski topology** by declaring that $V(S)$ is closed for all S. Let's check that this is a topology:

4.4.C. EXERCISE. (a) Show that $\varnothing$ and $\operatorname{Spec} A$ are both open.
(b) If I_i is a collection of ideals (as i runs over some index set), show that $\cap_i V(I_i) = V(\sum_i I_i)$. Hence the union of any collection of open sets is open.
(c) Show that $V(I_1) \cup V(I_2) = V(I_1 I_2)$. (The **product of two ideals** I_1 and I_2 of A are finite A-linear combinations of products of elements of I_1 and I_2, i.e. elements of the form $\sum_{j=1}^n i_{1,j} i_{2,j}$, where $i_{k,j} \in I_k$. Equivalently, it is the ideal generated by products of elements of I_1 and I_2. You should quickly check that this is an ideal, and that products are associative, i.e. $(I_1 I_2) I_3) = I_1(I_2 I_3)$.) Hence the intersection of any finite number of open sets is open.

4.4.1. Properties of the "vanishing set" function $V(\cdot)$. The function $V(\cdot)$ is obviously inclusion-reversing: If $S_1 \subset S_2$, then $V(S_2) \subset V(S_1)$. Warning: We could have equality in the second inclusion without equality in the first, as the next exercise shows.

4.4.D. EXERCISE/DEFINITION. If $I \subset A$ is an ideal, then define its **radical** by

$$\sqrt{I} := \{r \in A \ : \ r^n \in I \text{ for some } n \in \mathbb{Z}^{>0}\}.$$

For example, the nilradical $\mathfrak{N}$ (§4.2.O) is $\sqrt{(0)}$. Show that $\sqrt{I}$ is an ideal (cf. Exercise 4.2.O(b)). Show that $V(\sqrt{I}) = V(I)$. We say *an ideal is* **radical** if it equals its own radical. You should verify that $\sqrt{\sqrt{I}} = \sqrt{I}$, and that prime ideals are radical.

Here are two useful consequences. As $(I \cap J)^2 \subset IJ \subset I \cap J$ (products of ideals were defined in Exercise 4.4.C), we have that $V(IJ) = V(I \cap J)$ ($= V(I) \cup V(J)$ by Exercise 4.4.C(c)). Also, combining this with Exercise 4.4.B, we see $V(S) = V((S)) = V(\sqrt{(S)})$.

4.4.E. EXERCISE (RADICALS COMMUTE WITH FINITE INTERSECTION). If $I_1, \dots, I_n$ are ideals of a ring A, show that $\sqrt{\cap_{i=1}^n I_i} = \cap_{i=1}^n \sqrt{I_i}$. We will use this property without referring back to this exercise.

4.4.F. EXERCISE FOR LATER USE. Show that $\sqrt{I}$ is the intersection of all the prime ideals containing I. (Hint: Use Theorem 4.2.10 on an appropriate ring.)

4.4.2. Examples. Let's see how this meshes with our examples from the previous section.

Recall that $\mathbb{A}^1_{\mathbb{C}}$, as a set, was just the "traditional" points (corresponding to maximal ideals, in bijection with $a \in \mathbb{C}$), and one "new" point (0). The Zariski topology on $\mathbb{A}^1_{\mathbb{C}}$ is not that exciting: the open sets are the empty set, and $\mathbb{A}^1_{\mathbb{C}}$ minus a finite number of maximal ideals. (It "almost" has the cofinite topology. Notice that the open sets are determined by their intersections with the "traditional points". The "new" point (0) comes along for the ride, which is a good sign that it is harmless. Ignoring the "new" point, observe that the topology on $\mathbb{A}^1_{\mathbb{C}}$ is a coarser topology than the classical topology on $\mathbb{C}$.)

The case $\operatorname{Spec} \mathbb{Z}$ is similar. The topology is "almost" the cofinite topology in the same way. The open sets are the empty set, and $\operatorname{Spec} \mathbb{Z}$ minus a finite number of "ordinary" ((p) where p is prime) primes.

4.4.3. *Closed subsets of $\mathbb{A}^2_{\mathbb{C}}$.* The case $\mathbb{A}^2_{\mathbb{C}}$ is more interesting. You should think through where the "one-dimensional primes" fit into the picture. In Exercise 4.2.D, we identified all the primes of $\mathbb{C}[x, y]$ (i.e. the points of $\mathbb{A}^2_{\mathbb{C}}$) as the maximal ideals $(x-a, y-b)$ (where $a, b \in \mathbb{C}$), the "one-dimensional points" $(f(x, y))$ (where $f(x, y)$ is irreducible), and the "two-dimensional point" (0).

Then the closed subsets are of the following form:

(a) the entire space, and

(b) a finite number (possibly zero) of "curves" (each of which is the closure of a "one-dimensional point") and a finite number (possibly zero) of closed points.

4.4.4. Important fact: Maps of rings induce continuous maps of topological spaces. We saw in §4.2.7 that a map of rings $\phi : B \to A$ induces a map of sets $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$.

4.4.G. IMPORTANT EXERCISE. By showing that closed sets pull back to closed sets, show that π is a *continuous map*.

Not all continuous maps arise in this way. Consider for example the continuous map on $\mathbb{A}^1_{\mathbb{C}}$ that is the identity except 0 and 1 (i.e. $[(x)]$ and $[(x-1)]$) are swapped; no polynomial can manage this marvellous feat.

In §4.2.7, we saw that $\operatorname{Spec} B/I$ and $\operatorname{Spec} S^{-1}B$ are naturally *subsets* of $\operatorname{Spec} B$. It is natural to ask if the Zariski topology behaves well with respect to these inclusions, and indeed it does.

4.4.H. IMPORTANT EXERCISE (CF. EXERCISE 4.2.K). Suppose that $I, S \subset B$ are an ideal and multiplicative subset respectively.
(a) Show that $\operatorname{Spec} B/I$ is naturally a *closed* subset of $\operatorname{Spec} B$. If $S = \{1, f, f^2, \dots\}$ ($f \in B$), show that $\operatorname{Spec} S^{-1}B$ is naturally an *open* subset of $\operatorname{Spec} B$. Show that for arbitrary S, $\operatorname{Spec} S^{-1}B$ need not be open or closed. (Hint: Figure 4.5.)
(b) Show that the Zariski topology on $\operatorname{Spec} B/I$ (resp. $\operatorname{Spec} S^{-1}B$) is the subspace topology induced by inclusion in $\operatorname{Spec} B$. (Hint: compare closed subsets.)

4.4.5. In particular, if $I \subset \mathfrak{N}$ is an ideal of nilpotents, the bijection $\operatorname{Spec} B/I \to \operatorname{Spec} B$ (Exercise 4.2.O) is a homeomorphism. Thus nilpotents don't affect the topological space. (The difference will be in the structure sheaf.)

4.4.I. USEFUL EXERCISE FOR LATER. Suppose $I \subset B$ is an ideal. Show that f vanishes on $V(I)$ if and only if $f \in \sqrt{I}$ (i.e. $f^n \in I$ for some $n \geq 1$). (If you are stuck, you will get a hint when you see Exercise 4.5.E.)

4.4.J. EASY EXERCISE (CF. EXERCISE 4.2.A). Describe the topological space $\operatorname{Spec} k[x]_{(x)}$.

4.5 A base of the Zariski topology on $\operatorname{Spec} A$: Distinguished open sets

If $f \in A$, define the **distinguished open set** $D(f) = \{[\mathfrak{p}] \in \operatorname{Spec} A \ : \ f \notin \mathfrak{p}\}$. It is the locus where f doesn't vanish. (I often privately write this as $D(f \neq 0)$ to remind myself of this. I also privately call this a "Doesn't-vanish set" in analogy with $V(f)$ being the Vanishing set.) We have already seen this set when discussing $\operatorname{Spec} A_f$ as a subset of $\operatorname{Spec} A$. For example, we have observed that the Zariski-topology on the distinguished open set $D(f) \subset \operatorname{Spec} A$ coincides with the Zariski topology on $\operatorname{Spec} A_f$ (Exercise 4.4.H).

The reason these sets are important is that they form a particularly nice base for the (Zariski) topology:

4.5.A. EASY EXERCISE. Show that the distinguished open sets form a base for the (Zariski) topology. (Hint: Given a subset $S \subset A$, show that the complement of $V(S)$ is $\cup_{f \in S} D(f)$.)

Here are some important but not difficult exercises to give you a feel for this concept.

4.5.B. EXERCISE. Suppose $f_i \in A$ as i runs over some index set J. Show that $\cup_{i \in J} D(f_i) = \operatorname{Spec} A$ if and only if $(f_i) = A$, or equivalently and very usefully, there are a_i ($i \in J$), all but finitely many 0, such that $\sum_{i \in J} a_i f_i = 1$. (One of the directions will use the fact that any proper ideal of A is contained in some maximal ideal.)

4.5.C. EXERCISE. Show that if $\operatorname{Spec} A$ is an infinite union of distinguished open sets $\cup_{j \in J} D(f_j)$, then in fact it is a union of a finite number of these, i.e. there is a finite subset J' so that $\operatorname{Spec} A = \cup_{j \in J'} D(f_j)$. (Hint: exercise 4.5.B.)

4.5.D. EASY EXERCISE. Show that $D(f) \cap D(g) = D(fg)$.

4.5.E. IMPORTANT EXERCISE (CF. EXERCISE 4.4.I). Show that $D(f) \subset D(g)$ if and only if $f^n \in (g)$ for some $n \geq 1$, if and only if g is a unit in A_f.

We will use Exercise 4.5.E often. You can solve it thinking purely algebraically, but the following geometric interpretation may be helpful. Inside $\operatorname{Spec} A$, we have the closed subset $V(g) = \operatorname{Spec} A/(g)$, where g vanishes, and its complement $D(g)$, where g doesn't vanish. Then f is a function on this closed subset $V(g)$ (or more precisely, on $\operatorname{Spec} A/(g)$), and by assumption it vanishes at all points of the closed subset. Now any function vanishing at every point of the spectrum of a ring must be nilpotent (Theorem 4.2.10). In other words, there is some n such that $f^n = 0$ in $A/(g)$, i.e. $f^n \equiv 0 \pmod g$ in A, i.e. $f^n \in (g)$.

4.5.F. EASY EXERCISE. Show that $D(f) = \varnothing$ if and only if $f \in \mathfrak{N}$.

4.6 Topological definitions

A topological space is said to be **irreducible** if it is nonempty, and it is not the union of two proper closed subsets. In other words, X is irreducible if whenever $X = Y \cup Z$ with Y and Z closed, we have $Y = X$ or $Z = X$.

4.6.A. EASY EXERCISE. Show that in an irreducible topological space, any nonempty open set is dense. (The moral: unlike in the classical topology, in the Zariski topology, non-empty open sets are all "huge".)

4.6.B. EASY EXERCISE. If A is an integral domain, show that $\operatorname{Spec} A$ is irreducible. (Hint: pay attention to the generic point $[(0)]$.) We will generalize this in Exercise 4.6.O.

A point of a topological space $x \in X$ is said to be **closed** if $\{x\}$ is a closed subset. In the classical topology on $\mathbb{C}^n$, all points are closed.

4.6.C. EXERCISE. Show that the closed points of $\operatorname{Spec} A$ correspond to the maximal ideals.

Thus Hilbert's Nullstellensatz lets us interpret the closed points of $\mathbb{A}^n_{\mathbb{C}}$ as the n-tuples of complex numbers. Hence from now on we will say "closed point" instead of "traditional point" and "non-closed point" instead of "bonus" or "newfangled" point when discussing subsets of $\mathbb{A}^n_{\mathbb{C}}$.

4.6.1. Quasicompactness. A topological space X is **quasicompact** if given any cover $X = \cup_{i \in I} U_i$ by open sets, there is a finite subset S of the index set I such that $X = \cup_{i \in S} U_i$. Informally: every cover has a finite subcover. Depending on your definition of "compactness", this is the definition of compactness, minus possibly a Hausdorff condition. We will like this condition, because we are afraid of infinity.

4.6.D. EXERCISE. (a) Show that $\operatorname{Spec} A$ is quasicompact. (Hint: Exercise 4.5.C.)
$\star$ (b) (less important) Show that in general $\operatorname{Spec} A$ can have nonquasicompact open sets. (Possible hint: let $A = k[x_1, x_2, x_3, \dots]$ and $\mathfrak{m} = (x_1, x_2, \dots) \subset A$, and consider the complement of $V(\mathfrak{m})$. This example will be useful to construct other "counterexamples" later, e.g. Exercises 8.1.B and 8.3.E. In Exercise 4.6.M, we see that such weird behavior doesn't happen for "suitably nice" (Noetherian) rings.)

4.6.E. EXERCISE. (a) If X is a topological space that is a finite union of quasicompact spaces, show that X is quasicompact.
(b) Show that every closed subset of a quasicompact topological space is quasicompact.

4.6.2. Specialization and generization. Given two points x, y of a topological space X, we say that x is a **specialization** of y, and y is a **generization** of x, if $x \in \overline{\{y\}}$. This now makes precise our hand-waving about "one point containing another". It is of course nonsense for a point to contain another. But it is not nonsense to say that the closure of a point contains another. For example, in $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, $[(y - x^2)]$ is a generization of $(2, 4) = [(x - 2, y - 4)]$, and $(2, 4)$ is a specialization of $[(y - x^2)]$.

4.6.F. EXERCISE. If $X = \operatorname{Spec} A$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$. Hence show that $V(\mathfrak{p}) = \overline{\{[\mathfrak{p}]\}}$.

We say that a point $x \in X$ is a **generic point** for a closed subset K if $\overline{\{x\}} = K$. (Recall that if S is a subset of a topological space, then $\overline{S}$ denotes its closure.)

4.6.G. EXERCISE. Verify that $[(y - x^2)] \in \mathbb{A}^2$ is a generic point for $V(y - x^2)$.

We will soon see (Exercise 4.7.E) that there is a natural bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$. You know enough to prove this now, although we will wait until we have developed some convenient terminology.

4.6.H. TRICKY EXERCISE. (a) Suppose $I = (wz-xy, wy-x^2, xz-y^2) \subset k[w, x, y, z]$. Show that $\operatorname{Spec} k[w, x, y, z]/I$ is irreducible, by showing that I is prime. (This is hard, so here is one of several possible hints: Show that the quotient ring is an integral domain, by showing that it is isomorphic to the subring of $k[a, b]$ generated by monomials of degree divisible by 3. There are other approaches as well, some of which we will see later. This is an example of a hard question: how do you tell if an ideal is prime?) We will later see this as the cone over the *twisted cubic curve* (the twisted cubic curve is defined in Exercise 9.2.A, and is a special case of a Veronese embedding, §9.2.6).
(b) Note that the generators of the ideal of part (a) may be rewritten as the equations ensuring that

$$\operatorname{rank} \begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix} \leq 1,$$

i.e., as the determinants of the 2×2 submatrices. Generalize this to the ideal of rank one $2 \times n$ matrices. This notion will correspond to the cone (§9.2.11) over the *degree* n *rational normal curve* (Exercise 9.2.J).

4.6.3. Noetherian conditions.

In the examples we have considered (except in starred Exercise 4.6.D(b)), the spaces have naturally broken up into some obvious pieces. Let's make that a bit more precise.

A topological space X is called **Noetherian** if it satisfies the **descending chain condition** for closed subsets: any sequence $Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots$ of closed subsets eventually stabilizes: there is an r such that $Z_r = Z_{r+1} = \cdots$.

The following exercise may be enlightening.

4.6.I. EXERCISE. Show that any decreasing sequence of closed subsets of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ must eventually stabilize. Note that it can take arbitrarily long to stabilize. (The closed subsets of $\mathbb{A}^2_{\mathbb{C}}$ were described in §4.4.3.)

4.6.4. Noetherian rings. It turns out that all of the spectra we have considered have this property, but that isn't true of the spectra of all rings. The key characteristic all of our examples have had in common is that the rings were *Noetherian*. A ring is **Noetherian** if every ascending sequence $I_1 \subset I_2 \subset \cdots$ of ideals eventually stabilizes: there is an r such that $I_r = I_{r+1} = \cdots$. (This is called the **ascending chain condition** on ideals.)

Here are some quick facts about Noetherian rings. You should be able to prove them all.

- Fields are Noetherian. $\mathbb{Z}$ is Noetherian.

- If A is Noetherian, and $\phi : A \to B$ is any ring homomorphism, then $\phi(A)$ is Noetherian. Equivalently, quotients of Noetherian rings are Noetherian.
- If A is Noetherian, and S is any multiplicative set, then $S^{-1}A$ is Noetherian.
- Any submodule of a finitely generated module over a Noetherian ring is finitely generated. (Three possible hints, in no order: prove it for $A^{\oplus n}$; induct on the number of generators; use the next exercise.)

(The notion of a Noetherian *module* will come up in §14.6.)

4.6.J. IMPORTANT EXERCISE. Show that a ring A is Noetherian if and only if every ideal of A is finitely generated.

The next fact is non-trivial.

4.6.5. The Hilbert basis theorem. — *If A is Noetherian, then so is $A[x]$.*

Hilbert proved this in an epochal paper **[Hil]** where he also proved the Hilbert syzygy theorem (§16.3.2), and defined Hilbert functions and showed that they are eventually polynomial (§20.5).

By the results described above, any polynomial ring over any field, or over the integers, is Noetherian — and also any quotient or localization thereof. Hence for example any finitely-generated algebra over k or $\mathbb{Z}$, or any localization thereof, is Noetherian. Most "nice" rings are Noetherian, but not all rings are Noetherian: $k[x_1, x_2, \dots]$ is not, because $(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset \cdots$ is a strictly ascending chain of ideals (cf. Exercise 4.6.D(b)).

Proof of the Hilbert Basis Theorem 4.6.5. We show that any ideal $I \subset A[x]$ is finitely-generated. We inductively produce a set of generators $f_1, \dots$ as follows. For $n > 0$, if $I \neq (f_1, \dots, f_{n-1})$, let f_n be any non-zero element of $I - (f_1, \dots, f_{n-1})$ of lowest degree. Thus f_1 is any element of I of lowest degree, assuming $I \neq (0)$. If this procedure terminates, we are done. Otherwise, let $a_n \in A$ be the initial coefficient of f_n for $n > 0$. Then as A is Noetherian, $(a_1, a_2, \dots) = (a_1, \dots, a_N)$ for some N. Say $a_{N+1} = \sum_{i=1}^N b_i a_i$. Then

$$f_{N+1} - \sum_{i=1}^{N} b_i f_i x^{\deg f_{N+1} - \deg f_i}$$

is an element of I that is nonzero (as $f_{N+1} \notin (f_1, \dots, f_N)$) of lower degree than f_{N+1}, yielding a contradiction. $\square$

4.6.K. UNIMPORTANT EXERCISE. Show that if A is Noetherian, then so is $A[[x]] := \varprojlim A[x]/x^n$, the ring of power series in x. (Possible hint: Suppose $I \subset A[[x]]$ is an ideal. Let $I_n \subset A$ be the coefficients of x^n that appear in the elements of I. Show that I_n is an ideal. Show that $I_n \subset I_{n+1}$, and that I is determined by $(I_0, I_1, I_2, \dots)$.)

4.6.L. EXERCISE. If A is Noetherian, show that $\operatorname{Spec} A$ is a Noetherian topological space. Describe a ring A such that $\operatorname{Spec} A$ is not a Noetherian topological space. (Aside: if $\operatorname{Spec} A$ is a Noetherian topological space, A need not be Noetherian. One example is $A = k[x_1, x_2, x_3, \dots]/(x_1, x_2^2, x_3^3, \dots)$. Then $\operatorname{Spec} A$ has one point, so is

Noetherian. But A is not Noetherian as $([x_1]) \subsetneq ([x_1],[x_2]) \subsetneq ([x_1],[x_2],[x_3]) \subsetneq \cdots$ in A.

4.6.M. EXERCISE (PROMISED IN EXERCISE 4.6.D). Show that every open subset of a Noetherian topological space is quasicompact. Hence if A is Noetherian, every open subset of $\operatorname{Spec} A$ is quasicompact.

If X is a topological space, and Z is a maximal irreducible subset (an irreducible closed subset not contained in any larger irreducible closed subset), Z is said to be an **irreducible component** of X. We think of these as the "pieces of X" (see Figure 4.8).

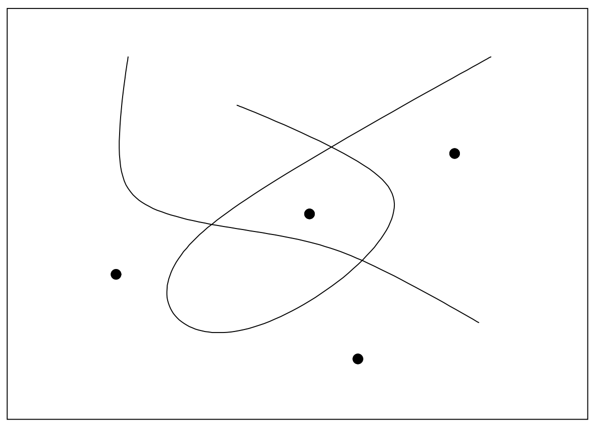


FIGURE 4.8. This closed subset of $\mathbb{A}^2_{\mathbb{C}}$ has six irreducible components

4.6.N. EXERCISE/DEFINITION. A prime of a ring A is a **minimal prime** if it is minimal with respect to inclusion. (For example, the only minimal prime of $k[x,y]$ is (0).) If A is any ring, show that the irreducible components of $\operatorname{Spec} A$ are in bijection with the minimal primes of A.

4.6.O. EXERCISE (GENERALIZING EXERCISE 4.6.B). Show that $\operatorname{Spec} A$ is irreducible if and only if A has only one **minimal prime** ideal.

4.6.P. EXERCISE. What are the minimal primes of $k[x,y]/(xy)$ (where k is a field)?

4.6.6. Proposition. — *Suppose X is a Noetherian topological space. Then every non-empty closed subset Z can be expressed uniquely as a finite union $Z = Z_1 \cup \cdots \cup Z_n$ of irreducible closed subsets, none contained in any other.*

Translation: any non-empty closed subset Z has a finite number of pieces. As a corollary, this implies that a Noetherian ring A has only finitely many minimal primes.

Proof. The following technique is called **Noetherian induction**, for reasons that will become clear.

Consider the collection of nonempty closed subsets of X that *cannot* be expressed as a finite union of irreducible closed subsets. We will show that it is empty. Otherwise, let Y_1 be one such. If it properly contains another such, then choose one, and call it Y_2. If this one contains another such, then choose one, and call it Y_3, and so on. By the descending chain condition, this must eventually stop, and we must have some Y_r that cannot be written as a finite union of irreducible closed subsets, but every closed subset properly contained in it can be so written. But then Y_r is not itself irreducible, so we can write $Y_r = Y' \cup Y''$ where Y' and Y'' are both proper closed subsets. Both of these by hypothesis can be written as the union of a finite number of irreducible subsets, and hence so can Y_r, yielding a contradiction. Thus each closed subset can be written as a finite union of irreducible closed subsets. We can assume that none of these irreducible closed subsets contain any others, by discarding some of them.

We now show uniqueness. Suppose

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_r = Z_1' \cup Z_2' \cup \cdots \cup Z_s'$$

are two such representations. Then $Z_1' \subset Z_1 \cup Z_2 \cup \cdots \cup Z_r$, so $Z_1' = (Z_1 \cap Z_1') \cup \cdots \cup (Z_r \cap Z_1')$. Now Z_1' is irreducible, so one of these is Z_1' itself, say (without loss of generality) $Z_1 \cap Z_1'$. Thus $Z_1' \subset Z_1$. Similarly, $Z_1 \subset Z_a'$ for some a; but because $Z_1' \subset Z_1 \subset Z_a'$, and Z_1' is contained in no other Z_i', we must have $a = 1$, and $Z_1' = Z_1$. Thus each element of the list of Z's is in the list of Z''s, and vice versa, so they must be the same list. $\square$

4.6.7. Definition. A topological space X is **connected** if it cannot be written as the disjoint union of two non-empty open sets. A subset Y of X is a **connected component** if it is a maximal connected subset.

4.6.Q. EXERCISE. Show that an irreducible topological space is connected.

4.6.R. EXERCISE. Give (with proof!) an example of a ring A where $\operatorname{Spec} A$ is connected but reducible. (Possible hint: a picture may help. The symbol "$\times$" has two "pieces" yet is connected.)

4.6.S. EXERCISE. If A is a Noetherian ring, show that the connected components of $\operatorname{Spec} A$ are unions of the irreducible components. Show that the subsets of $\operatorname{Spec} A$ that are simultaneously open and closed are precisely the unions of the connected components of $\operatorname{Spec} A$.

4.6.T. EXERCISE. If $A = A_1 \times A_2 \times \cdots \times A_n$, describe a homeomorphism $\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2 \coprod \cdots \coprod \operatorname{Spec} A_n \to \operatorname{Spec} A$ for which each $\operatorname{Spec} A_i$ is mapped onto a distinguished open subset $D(f_i)$ of $\operatorname{Spec} A$. Thus, $\operatorname{Spec} \prod_{i=1}^n A_i = \coprod_{i=1}^n \operatorname{Spec} A_i$. (Hint: let $f_i = (0, \dots, 0, 1, 0, \dots, 0)$ where the 1 is in the ith component.)

An extension of the previous exercise (that you can prove if you wish) is that $\operatorname{Spec} A$ is not connected if and only if A is isomorphic to the product of nonzero rings A_1 and A_2. The key idea is to show that both conditions are equivalent to there existing nonzero $a_1, a_2 \in A$ for which $a_1^2 = a_2^2 = 1$, $a_1 + a_2 = 1$, and hence $a_1 a_2 = 0$. An element $a \in A$ satisfying $a^2 = a$ is called an *idempotent*.

4.6.8. $\star$ *Fun but irrelevant remark.* The previous exercise shows that $\coprod_{i=1}^n \operatorname{Spec} A_i \cong \operatorname{Spec} \prod_{i=1}^n A_i$, but this can't hold if "n is infinite" and all A_i are nonzero, as Spec of any ring is quasicompact (Exercise 4.6.D(a)). This leads to an interesting phenomenon. We show that $\operatorname{Spec} \prod_{i=1}^\infty A_i$ is "strictly bigger" than $\coprod_{i=1}^\infty \operatorname{Spec} A_i$ where each A_i is isomorphic to the field k. First, we have an inclusion of sets $\coprod_{i=1}^\infty \operatorname{Spec} A_i \hookrightarrow \operatorname{Spec} \prod_{i=1}^\infty A_i$, as there is a maximal ideal of $\prod A_i$ corresponding to each i (precisely those elements 0 in the ith component.) But there are other maximal ideals of $\prod A_i$. Hint: describe a proper ideal not contained in any of these maximal ideals. (One idea: consider elements $\prod a_i$ that are "eventually zero", i.e. $a_i = 0$ for $i \gg 0$.) This leads to the notion of *ultrafilters*, which are very useful, but irrelevant to our current discussion.

4.6.9. *Remark.* We could define constructible and locally closed subsets now, but we wait until Exercise 8.4.A.

4.7 The function $I(\cdot)$, taking subsets of $\operatorname{Spec} A$ to ideals of A

We now introduce a notion that is in some sense "inverse" to the vanishing set function $V(\cdot)$. Given a subset $S \subset \operatorname{Spec} A$, $I(S)$ is the set of functions vanishing on S. In other words, $I(S) = \bigcap_{[\mathfrak{p}] \in S} \mathfrak{p} \subset A$ (at least when S is nonempty).

We make three quick observations:

- $I(S)$ is clearly an ideal of A.
- $I(\cdot)$ is inclusion-reversing: if $S_1 \subset S_2$, then $I(S_2) \subset I(S_1)$.
- $I(\overline{S}) = I(S)$.

4.7.A. EXERCISE. Let $A = k[x, y]$. If $S = \{[(x)], [(x-1, y)]\}$ (see Figure 4.9), then $I(S)$ consists of those polynomials vanishing on the y-axis, and at the point $(1, 0)$. Give generators for this ideal.

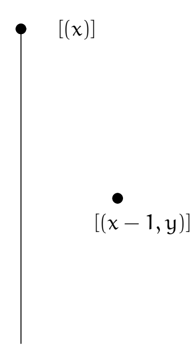


FIGURE 4.9. The set S of Exercise/example 4.7.A, pictured as a subset of $\mathbb{A}^2$

4.7.B. TRICKY EXERCISE. Suppose $S \subset \mathbb{A}^3_{\mathbb{C}}$ is the union of the three axes. Give generators for the ideal $I(S)$. Be sure to prove it! We will see in Exercise 13.1.F that this ideal is not generated by less than three elements.

4.7.C. EXERCISE. Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S. (Compare this to Exercise 4.7.D.)

Note that $I(S)$ is always a radical ideal — if $f \in \sqrt{I(S)}$, then f^n vanishes on S for some $n > 0$, so then f vanishes on S, so $f \in I(S)$.

4.7.D. EXERCISE. Prove that if $J \subset A$ is an ideal, then $I(V(J)) = \sqrt{J}$. (Huge hint: Exercise 4.4.I.)

This exercise and Exercise 4.7.C suggest that V and I are "almost" inverse. More precisely:

4.7.1. Theorem. — $V(\cdot)$ *and* $I(\cdot)$ *give a bijection between closed subsets of* $\operatorname{Spec} A$ *and radical ideals of* A *(where a closed subset gives a radical ideal by* $I(\cdot)$*, and a radical ideal gives a closed subset by* $V(\cdot)$*).*

Theorem 4.7.1 is sometimes called Hilbert's Nullstellensatz, but we reserve that name for Theorem 4.2.3.

4.7.E. IMPORTANT EXERCISE (CF. EXERCISE 4.6.N). Show that $V(\cdot)$ and $I(\cdot)$ give a bijection between *irreducible closed subsets* of $\operatorname{Spec} A$ and *prime* ideals of A. From this conclude that in $\operatorname{Spec} A$ there is a bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$ (where a point determines an irreducible closed subset by taking the closure). Hence *each irreducible closed subset of* $\operatorname{Spec} A$ *has precisely one generic point* — any irreducible closed subset Z can be written uniquely as $\overline{\{z\}}$.

CHAPTER 5

The structure sheaf, and the definition of schemes in general

5.1 The structure sheaf of an affine scheme

The final ingredient in the definition of an affine scheme is the *structure sheaf* $\mathscr{O}_{\operatorname{Spec} A}$, which we think of as the "sheaf of algebraic functions". You should keep in your mind the example of "algebraic functions" on $\mathbb{C}^n$, which you understand well. For example, in $\mathbb{A}^2$, we expect that on the open set $D(xy)$ (away from the two axes), $(3x^4 + y + 4)/x^7y^3$ should be an algebraic function.

These functions will have values at points, but won't be determined by their values at points. But like all sections of sheaves, they will be determined by their germs (see §5.3.3).

It suffices to describe the structure sheaf as a sheaf (of rings) on the base of distinguished open sets (Theorem 3.7.1 and Exercise 4.5.A).

5.1.1. *Definition.* Define $\mathscr{O}_{\operatorname{Spec} A}(D(f))$ to be the localization of A at the multiplicative set of all functions that do not vanish outside of $V(f)$ (i.e. those $g \in A$ such that $V(g) \subset V(f)$, or equivalently $D(f) \subset D(g)$, cf. Exercise 4.5.E). This depends only on $D(f)$, and not on f itself.

5.1.A. GREAT EXERCISE. Show that the natural map $A_f \to \mathscr{O}_{\operatorname{Spec} A}(D(f))$ is an isomorphism. (Possible hint: Exercise 4.5.E.)

If $D(f') \subset D(f)$, define the restriction map $\operatorname{res}_{D(f),D(f')} : \mathscr{O}_{\operatorname{Spec} A}(D(f)) \to \mathscr{O}_{\operatorname{Spec} A}(D(f'))$ in the obvious way: the latter ring is a further localization of the former ring. The restriction maps obviously commute: this is a "presheaf on the distinguished base".

5.1.2. Theorem. — *The data just described give a sheaf on the distinguished base, and hence determine a sheaf on the topological space* Spec A.

This sheaf is called the **structure sheaf**, and will be denoted $\mathscr{O}_{\operatorname{Spec} A}$, or sometimes $\mathscr{O}$ if the subscript is clear from the context. Such a topological space, with sheaf, will be called an **affine scheme**. The notation Spec A will hereafter denote the data of a topological space with a structure sheaf.

Proof. We must show the base identity and base gluability axioms hold (§3.7). We show that they both hold for the open set that is the entire space Spec A, and leave to you the trick which extends them to arbitrary distinguished open sets (Exercises 5.1.B and 5.1.C). Suppose $\operatorname{Spec} A = \cup_{i \in I} D(f_i)$, or equivalently (Exercise 4.5.B) the ideal generated by the f_i is the entire ring A.

We check identity on the base. Suppose that $\operatorname{Spec} A = \cup_{i \in I} D(f_i)$ where i runs over some index set I. Then there is some finite subset of I, which we name $\{1, \dots, n\}$, such that $\operatorname{Spec} A = \cup_{i=1}^n D(f_i)$, i.e. $(f_1, \dots, f_n) = A$ (quasicompactness of $\operatorname{Spec} A$, Exercise 4.5.C). Suppose we are given $s \in A$ such that $\operatorname{res}_{\operatorname{Spec} A, D(f_i)} s = 0$ in A_{f_i} for all i. We wish to show that $s = 0$. The fact that $\operatorname{res}_{\operatorname{Spec} A, D(f_i)} s = 0$ in A_{f_i} implies that there is some m such that for each $i \in \{1, \dots, n\}$, $f_i^m s = 0$. Now $(f_1^m, \dots, f_n^m) = A$ (for example, from $\operatorname{Spec} A = \cup D(f_i) = \cup D(f_i^m)$), so there are $r_i \in A$ with $\sum_{i=1}^n r_i f_i^m = 1$ in A, from which

$$s = \left(\sum r_i f_i^m\right) s = \sum r_i (f_i^m s) = 0.$$

Thus we have checked the "base identity" axiom for $\operatorname{Spec} A$. (Serre has described this as a "partition of unity" argument, and if you look at it in the right way, his insight is very enlightening.)

5.1.B. EXERCISE. Make the tiny changes to the above argument to show base identity for any distinguished open $D(f)$. (Hint: judiciously replace A by A_f in the above argument.)

We next show base gluability. Suppose again $\cup_{i \in I} D(f_i) = \operatorname{Spec} A$, where I is a index set (possibly horribly infinite). Suppose we are given elements in each A_{f_i} that agree on the overlaps $A_{f_i f_j}$. Note that intersections of distinguished open sets are also distinguished open sets.

(Aside: experts will realize that we are trying to show exactness of

$$0 \to A \to \prod_i A_{f_i} \to \prod_{i \neq j} A_{f_i f_j}. \tag{5.1.2.1}$$

Be careful interpreting the right-hand map — signs are involved! The map $A_{f_i} \to A_{f_i f_j}$ should be taken to be the "obvious one" if $i < j$, and negative of the "obvious one" if $i > j$. Base identity corresponds to injectivity at A. The composition of the right two morphisms is trivially zero, and gluability is exactness at $\prod_i A_{f_i}$.)

Assume first that I is finite, say $I = \{1, \dots, n\}$. We have elements $a_i / f_i^{l_i} \in A_{f_i}$ agreeing on overlaps $A_{f_i f_j}$. Letting $g_i = f_i^{l_i}$, using $D(f_i) = D(g_i)$, we can simplify notation by considering our elements as of the form $a_i / g_i \in A_{g_i}$.

The fact that a_i/g_i and a_j/g_j "agree on the overlap" (i.e. in $A_{g_i g_j}$) means that for some m_{ij},

$$(g_i g_j)^{m_{ij}} (g_j a_i - g_i a_j) = 0$$

in A. By taking $m = \max m_{ij}$ (here we use the finiteness of I), we can simplify notation:

$$(g_i g_j)^m (g_j a_i - g_i a_j) = 0$$

for all i, j. Let $b_i = a_i g_i^m$ for all i, and $h_i = g_i^{m+1}$ (so $D(h_i) = D(g_i)$). Then we can simplify notation even more: on each $D(h_i)$, we have a function b_i / h_i, and the overlap condition is

$$h_j b_i = h_i b_j. \tag{5.1.2.2}$$

Now $\cup_i D(h_i) = \operatorname{Spec} A$, implying that $1 = \sum_{i=1}^n r_i h_i$ for some $r_i \in A$. Define

$$r = \sum r_i b_i. \tag{5.1.2.3}$$

This will be the element of A that restricts to each b_j/h_j. Indeed, from the overlap condition (5.1.2.2),

$$rh_j = \sum_i r_i b_i h_j = \sum_i r_i h_i b_j = b_j.$$

We next deal with the case where I is infinite. Choose a finite subset $\{1, \ldots, n\} \subset I$ with $(f_1, \ldots, f_n) = A$ (or equivalently, use quasicompactness of $\operatorname{Spec} A$ to choose a finite subcover by $D(f_i)$). Construct r as above, using (5.1.2.3). We will show that for any $\alpha \in I - \{1, \ldots, n\}$, r restricts to the desired element a_α of A_{f_α}. Repeat the entire process above with $\{1, \ldots, n, \alpha\}$ in place of $\{1, \ldots, n\}$, to obtain $r' \in A$ which restricts to α_α for $i \in \{1, \ldots, n, \alpha\}$. Then by base identity, $r' = r$. (Note that we use base identity to *prove* base gluability. This is an example of how the identity axiom is "prior" to the gluability axiom.) Hence r restricts to $a_\alpha/f_\alpha^{l_\alpha}$ as desired.

5.1.C. EXERCISE. Alter this argument appropriately to show base gluability for any distinguished open $D(f)$.

We have now completed the proof of Theorem 5.1.2. □

The following generalization of Theorem 5.1.2 will be essential in the definition of a *quasicoherent sheaf* in Chapter 14.

5.1.D. IMPORTANT EXERCISE/DEFINITION. Suppose M is an A-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. Define $\tilde{M}(D(f))$ to be the localization of M at the multiplicative set of all functions that do not vanish outside of $V(f)$. Define restriction maps $\operatorname{res}_{D(f),D(g)}$ in the analogous way to $\mathcal{O}_{\operatorname{Spec} A}$. Show that this defines a sheaf on the distinguished base, and hence a sheaf on $\operatorname{Spec} A$. Then show that this is an $\mathcal{O}_{\operatorname{Spec} A}$-module.

5.1.3. *Remark (cf.* (5.1.2.1)*).* In the course of answering the previous exercise, you will show that if $(f_1, \ldots, f_r) = A$, M can be identified with a specific submodule of $M_{f_1} \times \cdots \times M_{f_r}$. Even though $M \to M_{f_i}$ may not be an inclusion for any f_i, $M \to M_{f_1} \times \cdots \times M_{f_r}$ *is* an inclusion. This will be useful later: we will want to show that if M has some nice property, then M_f does too, which will be easy. We will also want to show that if $(f_1, \ldots, f_n) = A$, then if M_{f_i} have this property, then M does too, and we will invoke this.

5.2 Visualizing schemes II: nilpotents

In §4.3, we discussed how to visualize the underlying set of schemes, adding in generic points to our previous intuition of "classical" (or closed) points. Our later discussion of the Zariski topology fit well with that picture. In our definition of the "affine scheme" $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, we have the additional information of nilpotents, which are invisible on the level of points (§4.2.9), so now we figure out to picture them. We will then readily be able to glue them together to picture schemes in general, once we have made the appropriate definitions. As we are building intuition, we will not be rigorous or precise.

To begin, we picture $\operatorname{Spec}\mathbb{C}[x]/(x)$ as a closed subset (a point) of $\operatorname{Spec}\mathbb{C}[x]$: to the quotient $\mathbb{C}[x] \to \mathbb{C}[x]/(x)$, we associate the picture of a closed inclusion. The ring map can be interpreted as restriction of functions: to a polynomial in $\mathbb{C}[x]$, we associate its value at 0 (its residue class modulo (x), by the Remainder Theorem). The quotient $\mathbb{C}[x]/(x^2)$ should fit in between these rings,

$$\begin{tikzcd} \mathbb{C}[x] \arrow[r, two heads] & \mathbb{C}[x]/(x^2) \arrow[r, two heads] & \mathbb{C}[x]/(x) \\ f(x) \arrow[rr, mapsto] & & f(0), \end{tikzcd}$$

and we should picture it in terms of the information the quotient remembers. The image of a polynomial $f(x)$ is the information of its value at 0, and its derivative (cf. Exercise 4.2.R). We thus picture this as being the point, plus a little bit more — a little bit of "fuzz" on the point (see Figure 5.1). (These will later be examples of *closed subschemes*, the schematic version of closed subsets, §9.1.)

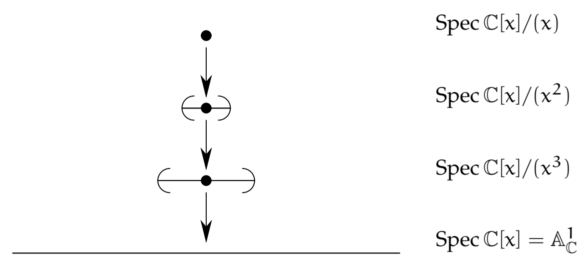


FIGURE 5.1. Picturing quotients of $\mathbb{C}[x]$

Similarly, $\mathbb{C}[x]/(x^3)$ remembers even more information — the second derivative as well. Thus we picture this as the point 0 plus even more fuzz.

More subtleties arise in two dimensions (see Figure 5.2). Consider $\operatorname{Spec}\mathbb{C}[x,y]/(x,y)^2$, which is sandwiched between two rings we know well:

$$\begin{tikzcd} \mathbb{C}[x,y] \arrow[r, two heads] & \mathbb{C}[x,y]/(x,y)^2 \arrow[r, two heads] & \mathbb{C}[x,y]/(x,y) \\ f(x,y) \arrow[rr, mapsto] & & f(0). \end{tikzcd}$$

Again, taking the quotient by $(x,y)^2$ remembers the first derivative, "in both directions". We picture this as fuzz around the point. Similarly, $(x,y)^3$ remembers the second derivative "in all directions".

Consider instead the ideal (x^2,y). What it remembers is the derivative only in the x direction — given a polynomial, we remember its value at 0, and the coefficient of x. We remember this by picturing the fuzz only in the x direction.

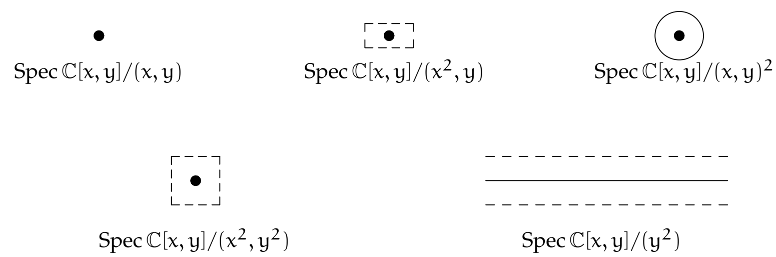


FIGURE 5.2. Picturing quotients of $\mathbb{C}[x, y]$

This gives us some handle on picturing more things of this sort, but now it becomes more an art than a science. For example, $\operatorname{Spec} \mathbb{C}[x, y]/(x^2, y^2)$ we might picture as a fuzzy square around the origin. One feature of this example is that given two ideals I and J of a ring A (such as $\mathbb{C}[x, y]$), your fuzzy picture of $\operatorname{Spec} A/(I, J)$ should be the "intersection" of your picture of $\operatorname{Spec} A/I$ and $\operatorname{Spec} A/J$ in $\operatorname{Spec} A$. (You will make this precise in Exercise 9.1.G(a).) For example, $\operatorname{Spec} \mathbb{C}[x, y]/(x^2, y^2)$ should be the intersection of two thickened lines. (How would you picture $\operatorname{Spec} \mathbb{C}[x, y]/(x^5, y^3)$? $\operatorname{Spec} \mathbb{C}[x, y, z]/(x^3, y^4, z^5, (x + y + z)^2)$? $\operatorname{Spec} \mathbb{C}[x, y]/((x, y)^5, y^3)$?)

This idea captures useful information that you already have some intuition for. For example, consider the intersection of the parabola $y = x^2$ and the x-axis (in the xy-plane). See Figure 5.3. You already have a sense that the intersection has multiplicity two. In terms of this visualization, we interpret this as intersecting (in $\operatorname{Spec} \mathbb{C}[x, y]$):

$$\begin{aligned} \operatorname{Spec} \mathbb{C}[x, y]/(y - x^2) \cap \operatorname{Spec} \mathbb{C}[x, y]/(y) &= \operatorname{Spec} \mathbb{C}[x, y]/(y - x^2, y) \\ &= \operatorname{Spec} \mathbb{C}[x, y]/(y, x^2) \end{aligned}$$

which we interpret as the fact that the parabola and line not just meet with multiplicity two, but that the "multiplicity 2" part is in the direction of the x-axis. You will make this example precise in Exercise 9.1.G(b).

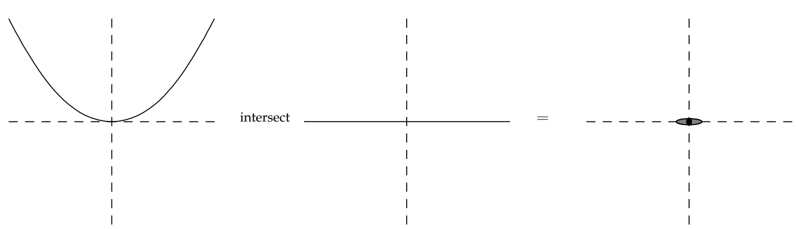


FIGURE 5.3. The scheme-theoretic intersection of the parabola $y = x^2$ and the x-axis is a nonreduced scheme (with fuzz in the x-direction)

5.2.1. We will later make the location of the fuzz somewhat more precise when we discuss associated points (§6.5). We will see that in reasonable circumstances, the fuzz is concentrated on closed subsets (Remark 14.7.2).

5.3 Definition of schemes

We can now define *scheme* in general. First, define an **isomorphism of ringed spaces** $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ as (i) a homeomorphism $f : X \to Y$, and (ii) an isomorphism of sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$, considered to be on the same space via f. (Part (ii), more precisely, is an isomorphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$ of sheaves on Y, or equivalently by adjointness $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X.) In other words, we have a "correspondence" of sets, topologies, and structure sheaves. An **affine scheme** is a ringed space that is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some A. A **scheme** $(X, \mathcal{O}_X)$ is a ringed space such that any point $x \in X$ has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. The scheme can be denoted $(X, \mathcal{O}_X)$, although it is often denoted X, with the structure sheaf implicit.

An **isomorphism of two schemes** $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is an isomorphism as ringed spaces. If $U \subset X$ is an open subset, then $\Gamma(\mathcal{O}_X, U)$ are said to be the **functions on** U; this generalizes in an obvious way the definition of functions on an affine scheme, §4.2.1.

5.3.1. *Remark.* From the definition of the structure sheaf on an affine scheme, several things are clear. First of all, if we are told that $(X, \mathcal{O}_X)$ is an affine scheme, we may recover its ring (i.e. find the ring A such that $\operatorname{Spec} A = X$) by taking the ring of global sections, as $X = D(1)$, so:

$$\begin{aligned} \Gamma(X, \mathcal{O}_X) &= \Gamma(D(1), \mathcal{O}_{\operatorname{Spec} A}) \quad \text{as } D(1) = \operatorname{Spec} A \\ &= A. \end{aligned}$$

(You can verify that we get more, and can "recognize X as the scheme $\operatorname{Spec} A$": we get an isomorphism $f : (\operatorname{Spec} \Gamma(X, \mathcal{O}_X), \mathcal{O}_{\operatorname{Spec} \Gamma(X, \mathcal{O}_X)}) \to (X, \mathcal{O}_X)$. For example, if $\mathfrak{m}$ is a maximal ideal of $\Gamma(X, \mathcal{O}_X)$, $f([\mathfrak{m}]) = V(\mathfrak{m})$.) The following exercise will give you some practice with these notions.

5.3.A. Exercise (which can be strangely confusing). Describe a bijection between the isomorphisms $\operatorname{Spec} A \to \operatorname{Spec} A'$ and the ring isomorphisms $A' \to A$.

More generally, given $f \in A$, $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) \cong A_f$. Thus under the natural inclusion of sets $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, the Zariski topology on $\operatorname{Spec} A$ restricts to give the Zariski topology on $\operatorname{Spec} A_f$ (Exercise 4.4.H), and the structure sheaf of $\operatorname{Spec} A$ restricts to the structure sheaf of $\operatorname{Spec} A_f$, as the next exercise shows.

5.3.B. Important but easy exercise. Suppose $f \in A$. Show that under the identification of $D(f)$ in $\operatorname{Spec} A$ with $\operatorname{Spec} A_f$ (§4.5), there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f})$. Hint: notice that distinguished open sets of $\operatorname{Spec} R_f$ are already distinguished open sets in $\operatorname{Spec} R$.

5.3.C. Easy Exercise. If X is a scheme, and U is *any* open subset, prove that $(U, \mathcal{O}_X|_U)$ is also a scheme.

5.3.2. *Definitions.* We say $(U, \mathcal{O}_X|_U)$ is an **open subscheme of** X. If U is also an affine scheme, we often say U is an **affine open subset**, or an **affine open subscheme**, or sometimes informally just an **affine open**. For example, $D(f)$ is an affine open subscheme of $\operatorname{Spec} A$.

5.3.D. EASY EXERCISE. Show that if X is a scheme, then the affine open sets form a base for the Zariski topology.

5.3.E. EASY EXERCISE. The disjoint union of schemes is defined as you would expect: it is the disjoint union of sets, with the expected topology (thus it is the disjoint union of topological spaces), with the expected sheaf. Once we know what morphisms are, it will be immediate (Exercise 10.1.A) that (just as for sets and topological spaces) disjoint union is the coproduct in the category of schemes.
(a) Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: Exercise 4.6.T.)
(b) *(a first example of a non-affine scheme)* Show that an infinite disjoint union of (non-empty) affine schemes is not an affine scheme. (Hint: affine schemes are quasicompact, Exercise 4.6.D(a).)

5.3.3. Stalks of the structure sheaf: germs, values at a point, and the residue field of a point. Like every sheaf, the structure sheaf has stalks, and we shouldn't be surprised if they are interesting from an algebraic point of view. In fact, we have seen them before.

5.3.F. IMPORTANT EXERCISE. Show that the stalk of $\mathcal{O}_{\operatorname{Spec} A}$ at the point $[\mathfrak{p}]$ is the local ring $A_{\mathfrak{p}}$.

Essentially the same argument will show that the stalk of the sheaf $\tilde{M}$ (defined in Exercise 5.1.D) at $[\mathfrak{p}]$ is $M_{\mathfrak{p}}$. Here is an interesting consequence, or if you prefer, a geometric interpretation of an algebraic fact. A section is determined by its germs (Exercise 3.4.A), meaning that $M \to \prod_{\mathfrak{p}} M_{\mathfrak{p}}$ is an inclusion. So for example an A-module is zero if and only if all its localizations at primes are zero.

5.3.4. *Definition.* We say a ringed space is a **locally ringed space** if its stalks are local rings. (The motivation for the terminology comes from thinking of sheaves in terms of stalks. A *ringed space* is a sheaf whose stalks are rings. A *locally ringed space* is a sheaf whose stalks are local rings.) Thus schemes are locally ringed spaces. Manifolds are another example of locally ringed spaces, see §3.1.1. In both cases, taking quotient by the maximal ideal may be interpreted as evaluating at the point. The maximal ideal of the local ring $\mathcal{O}_{X,p}$ is denoted $\mathfrak{m}_{X,p}$ or $\mathfrak{m}_p$, and the **residue field** $\mathcal{O}_{X,p}/\mathfrak{m}_p$ is denoted $\kappa(p)$. Functions on an open subset U of a locally ringed space have **values** at each point of U. The value at p of such a function lies in $\kappa(p)$. As usual, we say that a function **vanishes** at a point p if its value at p is 0.

As an example, consider a point $[\mathfrak{p}]$ of an affine scheme $\operatorname{Spec} A$. (Of course, this example is "universal", as all points may be interpreted in this way, by choosing an affine neighborhood.) The residue field at $[\mathfrak{p}]$ is $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, which is isomorphic to $K(A/\mathfrak{p})$, the fraction field of the quotient. It is useful to note that localization at

$\mathfrak{p}$ and taking quotient by $\mathfrak{p}$ "commute", i.e. the following diagram commutes.

(5.3.4.1)
$$\begin{tikzcd} & A_{\mathfrak{p}} \arrow[dr, "\text{quotient}"] & \\ A \arrow[ur, "\text{localize}"] \arrow[dr, "\text{quotient}"'] & & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = K(A/\mathfrak{p}) \\ & A/\mathfrak{p} \arrow[ur, "\text{localize, i.e. } K(\cdot)"'] & \end{tikzcd}$$

For example, consider the scheme $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, where k is a field of characteristic not 2. Then $(x^2+y^2)/x(y^2-x^5)$ is a function away from the y-axis and the curve y^2-x^5. Its value at $(2,4)$ (by which we mean $[(x-2,y-4)]$) is $(2^2+4^2)/(2(4^2-2^5))$, as

$$\frac{x^2+y^2}{x(y^2-x^5)} \equiv \frac{2^2+4^2}{2(4^2-2^5)}$$

in the residue field — check this if it seems mysterious. And its value at $[(y)]$, the generic point of the x-axis, is $\frac{x^2}{-x^6} = -1/x^4$, which we see by setting y to 0. This is indeed an element of the fraction field of $k[x,y]/(y)$, i.e. $k(x)$. (If you think you care only about algebraically closed fields, let this example be a first warning: $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ won't be algebraically closed in general, even if A is a finitely generated $\mathbb{C}$-algebra!)

If anything makes you nervous, you should make up an example to make you feel better. Here is one: 27/4 is a function on $\operatorname{Spec}\mathbb{Z} - \{[(2)],[(7)]\}$ or indeed on an even bigger open set. What is its value at $[(5)]$? Answer: $2/(-1) \equiv -2 \pmod 5$. What is its value at the generic point $[(0)]$? Answer: 27/4. Where does it vanish? At $[(3)]$.

5.3.5. *Stray definition: the fiber of an $\mathcal{O}$-module at a point.* If $\mathcal{F}$ is an $\mathcal{O}$-module on a scheme X (or more generally, a locally ringed space), define the **fiber of $\mathcal{F}$ at a point** $p \in X$ by

$$\mathcal{F}|_p := \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} \kappa(p).$$

As a reality check, $\mathcal{O}|_p$ is $\kappa(p)$ by definition.

5.4 Three examples

We now give three extended examples. Our short-term goal is to see that we can really work with the structure sheaf, and can compute the ring of sections of interesting open sets that aren't just distinguished open sets of affine schemes. Our long-term goal is to meet interesting examples that will come up repeatedly in the future.

5.4.1. Example: The plane minus the origin. This example will show you that the distinguished base is something that you can work with. Let $A = k[x,y]$, so $\operatorname{Spec} A = \mathbb{A}^2_k$. Let's work out the space of functions on the open set $U = \mathbb{A}^2 - \{(0,0)\} = \mathbb{A}^2 - \{[(x,y)]\}$.

You can't cut out this set with a single equation (can you see why?), so this isn't a distinguished open set. But in any case, even if we are not sure if this is a distinguished open set, we can describe it as the union of two things which *are* distinguished open sets: $U = D(x) \cup D(y)$. We will find the functions on U by gluing together functions on $D(x)$ and $D(y)$.

The functions on $D(x)$ are, by Definition 5.1.1, $A_x = k[x, y, 1/x]$. The functions on $D(y)$ are $A_y = k[x, y, 1/y]$. Note that $A \hookrightarrow A_x, A_y$. This is because x and y are not zerodivisors. (The ring A is an integral domain — it has no zerodivisors, besides 0 — so localization is always an inclusion, Exercise 2.3.C.) So we are looking for functions on $D(x)$ and $D(y)$ that agree on $D(x) \cap D(y) = D(xy)$, i.e. they are just the same Laurent polynomial. Which things of this first form are also of the second form? Just traditional polynomials —

$$\Gamma(U, \mathcal{O}_{\mathbb{A}^2}) \equiv k[x, y]. \tag{5.4.1.1}$$

In other words, we get no extra functions by throwing out this point. Notice how easy that was to calculate!

5.4.2. *Aside.* Notice that any function on $\mathbb{A}^2 - \{(0,0)\}$ extends over all of $\mathbb{A}^2$. This is an analogue of *Hartogs' Lemma* in complex geometry: you can extend a holomorphic function defined on the complement of a set of codimension at least two on a complex manifold over the missing set. This will work more generally in the algebraic setting: you can extend over points in codimension at least 2 not only if they are "smooth", but also if they are mildly singular — what we will call *normal*. We will make this precise in §12.3.10. This fact will be very useful for us.

5.4.3. We now show an interesting fact: $(U, \mathcal{O}_{\mathbb{A}^2}|_U)$ is a scheme, but it is not an affine scheme. (This is confusing, so you will have to pay attention.) Here's why: otherwise, if $(U, \mathcal{O}_{\mathbb{A}^2}|_U) = (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, then we can recover A by taking global sections:

$$A = \Gamma(U, \mathcal{O}_{\mathbb{A}^2}|_U),$$

which we have already identified in (5.4.1.1) as $k[x, y]$. So if U is affine, then $U \cong \mathbb{A}^2_k$. But this bijection between primes in a ring and points of the spectrum is more constructive than that: *given the prime ideal* I*, you can recover the point as the generic point of the closed subset cut out by* I*, i.e.* $V(I)$*, and given the point* p*, you can recover the ideal as those functions vanishing at* p*, i.e.* $I(p)$. In particular, the prime ideal (x, y) of A should cut out a point of $\operatorname{Spec} A$. But on U, $V(x) \cap V(y) = \varnothing$. Conclusion: U is *not* an affine scheme. (If you are ever looking for a counterexample to something, and you are expecting one involving a non-affine scheme, keep this example in mind!)

5.4.4. Gluing two copies of $\mathbb{A}^1$ together in two different ways. We have now seen two examples of non-affine schemes: an infinite disjoint union of non-empty schemes: Exercise 5.3.E and $\mathbb{A}^2 - \{(0,0)\}$. I want to give you two more examples. They are important because they are the first examples of fundamental behavior, the first pathological, and the second central.

First, I need to tell you how to glue two schemes together. Before that, you should review how to glue topological spaces together along isomorphic open sets. Given two topological spaces X and Y, and open subsets $U \subset X$ and $V \subset Y$ along with a homeomorphism $U \cong V$, we can create a new topological space W, that we think of as gluing X and Y together along $U \cong V$. It is the quotient of the disjoint union $X \coprod Y$ by the equivalence relation $U \cong V$, where the quotient is given the quotient topology. Then X and Y are naturally (identified with) open subsets of W, and indeed cover W. Can you restate this cleanly with an arbitrary (not necessarily finite) number of topological spaces?

Now that we have discussed gluing topological spaces, let's glue schemes together. Suppose you have two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, and open subsets $U \subset X$ and $V \subset Y$, along with a homeomorphism $f : U \xrightarrow{\sim} V$, and an isomorphism of structure sheaves $\mathcal{O}_X \cong f^*\mathcal{O}_Y$ (i.e. an isomorphism *of schemes* $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_Y|_V)$). Then we can glue these together to get a single scheme. Reason: let W be X and Y glued together using the isomorphism $U \cong V$. Then Exercise 3.7.D shows that the structure sheaves can be glued together to get a sheaf of rings. Note that this is indeed a scheme: any point has a neighborhood that is an affine scheme. (Do you see why?)

5.4.A. ESSENTIAL EXERCISE (CF. EXERCISE 3.7.D). For later reference, show that you can glue an arbitrary collection of schemes together. Suppose we are given:

- schemes X_i (as i runs over some index set I, not necessarily finite),
- open subschemes $X_{ij} \subset X_i$,
- isomorphisms $f_{ij} : X_{ij} \to X_{ji}$ with f_{ii} the identity

such that

- (the cocycle condition) the isomorphisms "agree on triple intersections", i.e. $f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{X_{ji} \cap X_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}$.

(The cocycle condition ensures that f_{ij} and f_{ji} are inverses. In fact, the hypothesis that f_{ii} is the identity also follows from the cocycle condition.) Show that there is a unique scheme X (up to unique isomorphism) along with open subset isomorphic to X_i respecting this gluing data in the obvious sense. (Hint: what is X as a set? What is the topology on this set? In terms of your description of the open sets of X, what are the sections of this sheaf over each open set?)

I will now give you two non-affine schemes. In both cases, I will glue together two copies of the affine line $\mathbb{A}^1_k$. Let $X = \operatorname{Spec} k[t]$, and $Y = \operatorname{Spec} k[u]$. Let $U = D(t) = \operatorname{Spec} k[t, 1/t] \subset X$ and $V = D(u) = \operatorname{Spec} k[u, 1/u] \subset Y$. We will get both examples by gluing X and Y together along U and V. The difference will be in how we glue.

5.4.5. Extended example: the affine line with the doubled origin. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow u$ (cf. Exercise 5.3.A). The resulting scheme is called the **affine line with doubled origin**. Figure 5.4 is a picture of it.

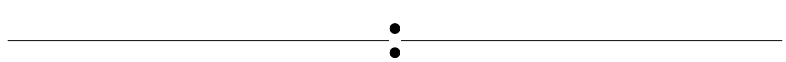


FIGURE 5.4. The affine line with doubled origin

As the picture suggests, intuitively this is an analogue of a failure of Hausdorffness. Now $\mathbb{A}^1$ itself is not Hausdorff, so we can't say that it is a failure of Hausdorffness. We see this as weird and bad, so we will want to make a definition that will prevent this from happening. This will be the notion of *separatedness* (to be discussed in Chapter 11). This will answer other of our prayers as well. For example, on a separated scheme, the "affine base of the Zariski topology" is nice — the intersection of two affine open sets will be affine (Proposition 11.1.8).

5.4.B. EXERCISE. Show that the affine line with doubled origin is not affine. Hint: calculate the ring of global sections, and look back at the argument for $\mathbb{A}^2 - \{(0,0)\}$.

5.4.C. EASY EXERCISE. Do the same construction with $\mathbb{A}^1$ replaced by $\mathbb{A}^2$. You will have defined the **affine plane with doubled origin**. Describe two affine open subsets of this scheme whose intersection is not an affine open subset.

5.4.6. Example 2: the projective line. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow 1/u$. Figure 5.5 is a suggestive picture of this gluing. The resulting scheme is called the **projective line over the field** k, and is denoted $\mathbb{P}^1_k$.

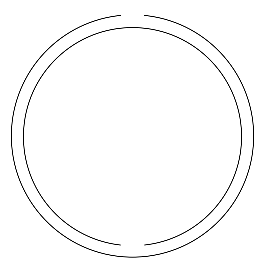


FIGURE 5.5. Gluing two affine lines together to get $\mathbb{P}^1$

Notice how the points glue. Let me assume that k is algebraically closed for convenience. (You can think about how this changes otherwise.) On the first affine line, we have the closed ("traditional") points $[(t-a)]$, which we think of as "a on the t-line", and we have the generic point $[(0)]$. On the second affine line, we have closed points that are "b on the u-line", and the generic point. Then a on the t-line is glued to $1/a$ on the u-line (if $a \neq 0$ of course), and the generic point is glued to the generic point (the ideal (0) of $k[t]$ becomes the ideal (0) of $k[t, 1/t]$ upon localization, and the ideal (0) of $k[u]$ becomes the ideal (0) of $k[u, 1/u]$. And (0) in $k[t, 1/t]$ is (0) in $k[u, 1/u]$ under the isomorphism $t \leftrightarrow 1/u$).

5.4.7. If k is algebraically closed, we can interpret the closed points of $\mathbb{P}^1_k$ in the following way, which may make this sound closer to the way you have seen projective space defined earlier. The points are of the form $[a; b]$, where a and b are not both zero, and $[a; b]$ is identified with $[ac; bc]$ where $c \in k^\times$. Then if $b \neq 0$, this is identified with a/b on the t-line, and if $a \neq 0$, this is identified with b/a on the u-line.

5.4.8. Proposition. — *$\mathbb{P}^1_k$ is not affine.*

Proof. We do this by calculating the ring of global sections. The global sections correspond to sections over X and sections over Y that agree on the overlap. A section on X is a polynomial $f(t)$. A section on Y is a polynomial $g(u)$. If we restrict $f(t)$ to the overlap, we get something we can still call $f(t)$; and similarly for $g(u)$. Now we want them to be equal: $f(t) = g(1/t)$. But the only polynomials in t that are at the same time polynomials in $1/t$ are the constants k. Thus $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$. If $\mathbb{P}^1$ were affine, then it would be $\operatorname{Spec}\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Spec} k$, i.e. one point. But it isn't — it has lots of points. $\square$

We have proved an analogue of a theorem: the only holomorphic functions on $\mathbb{CP}^1$ are the constants!

5.4.9. Important example: Projective space. We now make a preliminary definition of **projective n-space over a field k**, denoted $\mathbb{P}^n_k$, by gluing together $n+1$ open sets each isomorphic to $\mathbb{A}^n_k$. Judicious choice of notation for these open sets will make our life easier. Our motivation is as follows. In the construction of $\mathbb{P}^1$ above, we thought of points of projective space as $[x_0; x_1]$, where (x_0, x_1) are only determined up to scalars, i.e. (x_0, x_1) is considered the same as $(\lambda x_0, \lambda x_1)$. Then the first patch can be interpreted by taking the locus where $x_0 \neq 0$, and then we consider the points $[1; t]$, and we think of t as x_1/x_0; even though x_0 and x_1 are not well-defined, x_1/x_0 *is*. The second corresponds to where $x_1 \neq 0$, and we consider the points $[u; 1]$, and we think of u as x_0/x_1. It will be useful to instead use the notation $x_{1/0}$ for t and $x_{0/1}$ for u.

For $\mathbb{P}^n$, we glue together $n+1$ open sets, one for each of $i = 0, \dots, n+1$. The ith open set U_i will have coordinates $x_{0/i}, \dots, x_{(i-1)/i}, x_{(i+1)/i}, \dots, x_{n/i}$. It will be convenient to write this as

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, \dots, x_{n/i}]/(x_{i/i} - 1)$$

(so we have introduced a "dummy variable" $x_{i/i}$ which we set to 1). We glue the distinguished open set $D(x_{j/i})$ of U_i to the distinguished open set $D(x_{i/j})$ of U_j, by identifying these two schemes by describing the identification of rings

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, \dots, x_{n/i}, 1/x_{j/i}]/(x_{i/i} - 1) \cong$$

$$\operatorname{Spec} k[x_{0/j}, x_{1/j}, \dots, x_{n/j}, 1/x_{i/j}]/(x_{j/j} - 1)$$

via $x_{k/i} = x_{k/j}/x_{i/j}$ and $x_{k/j} = x_{k/i}/x_{j/i}$ (which implies $x_{i/j}x_{j/i} = 1$). We need to check that this gluing information agrees over triple overlaps.

5.4.D. EXERCISE. Check this, as painlessly as possible. (Possible hint: the triple intersection is affine; describe the corresponding ring.)

5.4.10. *Definition.* Note that our definition does not use the fact that k is a field. Hence we may as well define $\mathbb{P}^n_A$ for any *ring* A. This will be useful later.

5.4.E. EXERCISE. Show that the only global sections of the structure sheaf are constants, and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine open sets, but you will only need two of your open sets to see this. There is even some geometric intuition behind this: the complement of the union of two open sets has codimension 2. But "Algebraic Hartogs' Lemma" (discussed informally in §5.4.2, to be stated rigorously in Theorem 12.3.10) says that any function defined on this union extends to be a function on all of projective space. Because we are expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

5.4.F. EXERCISE (GENERALIZING §5.4.7). Show that if k is algebraically closed, the closed points of $\mathbb{P}^n_k$ may be interpreted in the traditional way: the points are of the form $[a_0; \dots ; a_n]$, where the a_i are not all zero, and $[a_0; \dots ; a_n]$ is identified with $[\lambda a_0; \dots ; \lambda a_n]$ where $\lambda \in k^\times$.

We will later give other definitions of projective space (Definition 5.5.4, §17.4.2). Our first definition here will often be handy for computing things. But there is something unnatural about it — projective space is highly symmetric, and that isn't clear from our current definition.

5.4.11. Fun aside: The Chinese Remainder Theorem is a *geometric* fact. The Chinese Remainder theorem is embedded in what we have done, which shouldn't be obvious. I will show this by example, but you should then figure out the general statement. The Chinese Remainder Theorem says that knowing an integer modulo 60 is the same as knowing an integer modulo 3, 4, and 5. Here's how to see this in the language of schemes. What is $\operatorname{Spec} \mathbb{Z}/(60)$? What are the primes of this ring? Answer: those prime ideals containing (60), i.e. those primes dividing 60, i.e. (2), (3), and (5). Figure 5.6 is a sketch of $\operatorname{Spec} \mathbb{Z}/(60)$. They are all closed points, as these are all maximal ideals, so the topology is the discrete topology. What are the stalks? You can check that they are $\mathbb{Z}/4$, $\mathbb{Z}/3$, and $\mathbb{Z}/5$. The nilpotents "at (2)" are indicated by the "fuzz" on that point. (We discussed visualizing nilpotents with "infinitesimal fuzz" in §5.2.) So what are global sections on this scheme? They are sections on this open set (2), this other open set (3), and this third open set (5). In other words, we have a natural isomorphism of rings

$$\mathbb{Z}/60 \to \mathbb{Z}/4 \times \mathbb{Z}/3 \times \mathbb{Z}/5.$$

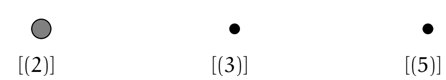


FIGURE 5.6. A picture of the scheme $\operatorname{Spec} \mathbb{Z}/(60)$

5.4.12. $\star$ *Example.* Here is an example of a function on an open subset of a scheme that is a bit surprising. On $X = \operatorname{Spec} k[w, x, y, z]/(wx - yz)$, consider the open subset $D(y) \cup D(w)$. Show that the function x/y on $D(y)$ agrees with z/w on $D(w)$ on their overlap $D(y) \cap D(w)$. Hence they glue together to give a section. You may have seen this before when thinking about analytic continuation in complex geometry — we have a "holomorphic" function which has the description x/y on an open set, and this description breaks down elsewhere, but you can still "analytically continue" it by giving the function a different definition on different parts of the space.

Follow-up for curious experts: This function has no "single description" as a well-defined expression in terms of w, x, y, z! There is a lot of interesting geometry here. This scheme will be a constant source of interesting examples for us. We will later recognize it as the cone over the quadric surface. Here is a glimpse, in terms of words we have not yet defined. Now $\operatorname{Spec} k[w, x, y, z]$ is $\mathbb{A}^4$, and is, not surprisingly, 4-dimensional. We are looking at the set X, which is a hypersurface, and is 3-dimensional. It is a cone over a "smooth" quadric surface in $\mathbb{P}^3$ (flip to Figure 9.2). $D(y)$ is X minus some hypersurface, so we are throwing away a codimension 1 locus. $D(z)$ involves throwing away another codimension 1 locus. You might think that their intersection is then codimension 2, and that maybe failure of extending this weird function to a global polynomial comes because of a failure of our Hartogs' Lemma-type theorem, which will be a failure of normality. But that's not true — $V(y) \cap V(z)$ is in fact codimension 1 — so no Hartogs-type theorem holds. Here is what is actually going on. $V(y)$ involves throwing away the (cone over the) union of two lines ℓ and m_1, one in each "ruling" of the surface, and $V(z)$ also involves throwing away the (cone over the) union of two lines ℓ and m_2. The intersection is the (cone over the) line ℓ, which is a codimension 1 set. Neat fact: despite being "pure codimension 1", it is not cut out even set-theoretically by a single equation. (It is hard to get an example of this behavior. This construction is the simplest example I know.) This means that any expression $f(w, x, y, z)/g(w, x, y, z)$ for our function cannot correctly describe our function on $D(y) \cup D(z)$ — at some point of $D(y) \cup D(z)$ it must be $0/0$. Here's why. Our function can't be defined on $V(y) \cap V(z)$, so g must vanish here. But g can't vanish just on the cone over ℓ — it must vanish elsewhere too. (For the experts among the experts, familiar with closed subschemes: here is why the cone over l is not cut out set-theoretically by a single equation. If $\ell = V(f)$, then $D(f)$ is affine. Let ℓ' be another line in the same ruling as ℓ, and let $C(\ell)$ (resp. ℓ') be the cone over ℓ (resp. ℓ'). Then $C(\ell')$ can be given the structure of a closed subscheme of $\operatorname{Spec} k[w, x, y, z]$ (a notion we'll properly define in §9.1), and can be given the structure of $\mathbb{A}^2$. Then $C(\ell') \cap V(f)$ is a closed subscheme of $D(f)$. Any closed subscheme of an affine scheme is affine. But $\ell \cap \ell' = \varnothing$, so the cone over ℓ intersects the cone over ℓ' in a point, so $C(\ell') \cap V(f)$ is $\mathbb{A}^2$ minus a point, which we have seen is not affine, so we have a contradiction.)

5.5 Projective schemes

Projective schemes are important for a number of reasons. Here are a few. Schemes that were of "classical interest" in geometry — and those that you would have cared about before knowing about schemes — are all projective or quasiprojective. Moreover, schemes of "current interest" tend to be projective or quasiprojective. In fact, it is very hard to even give an example of a scheme satisfying basic properties — for example, finite type and "Hausdorff" ("separated") over a field — that is provably not quasiprojective. For complex geometers: it is hard to find a compact complex variety that is provably not projective (see Remark 11.3.6), and it is quite hard to come up with a complex variety that is provably not an open subset of a projective variety. So projective schemes are really ubiquitous. Also a projective k-scheme is a good approximation of the algebro-geometric version of compactness ("properness", see §11.3).

Finally, although projective schemes may be obtained by gluing together affines, and we know that keeping track of gluing can be annoying, there is a simple means of dealing with them without worrying about gluing. Just as there is a rough dictionary between rings and affine schemes, we will have an analogous dictionary between graded rings and projective schemes. Just as one can work with affine schemes by instead working with rings, one can work with projective schemes by instead working with graded rings. To get an initial sense of how this works, consider Example 9.2.1 (which secretly gives the notion of projective A-schemes in full generality). Recall that any collection of homogeneous elements of $A[x_0, \ldots, x_n]$ describes a closed subscheme of $\mathbb{P}^n_A$. (The $x_0, \ldots, x_n$ are called **projective coordinates** on the scheme. Warning: they are not functions on the scheme. Any closed subscheme of $\mathbb{P}^n_A$ cut out by a set of homogeneous polynomials will soon be called a *projective A-scheme*.) Thus if I is a **homogeneous ideal** in $A[x_0, \ldots, x_n]$ (i.e. generated by homogeneous polynomials), we have defined a closed subscheme of $\mathbb{P}^n_A$ deserving the name $V(I)$. Conversely, given a closed subset S of $\mathbb{P}^n_A$, we can consider those homogeneous polynomials in the projective coordinates, vanishing on S. This homogeneous ideal deserves the name $I(S)$.

5.5.1. *A motivating picture from classical geometry.* For geometric intuition, we recall how one thinks of projective space "classically" (in the classical topology, over the real numbers). $\mathbb{P}^n$ can be interpreted as the lines through the origin in $\mathbb{R}^{n+1}$. Thus subsets of $\mathbb{P}^n$ correspond to unions of lines through the origin of $\mathbb{R}^{n+1}$, and closed subsets correspond to such unions which are closed. (The same is not true with "closed" replaced by "open"!)

One often pictures $\mathbb{P}^n$ as being the "points at infinite distance" in $\mathbb{R}^{n+1}$, where the points infinitely far in one direction are associated with the points infinitely far in the opposite direction. We can make this more precise using the decomposition

$$\mathbb{P}^{n+1} = \mathbb{R}^{n+1} \coprod \mathbb{P}^n$$

by which we mean that there is an open subset in $\mathbb{P}^{n+1}$ identified with $\mathbb{R}^{n+1}$ (the points with last projective coordinate non-zero), and the complementary closed subset identified with $\mathbb{P}^n$ (the points with last projective coordinate zero).

Then for example any equation cutting out some set V of points in $\mathbb{P}^n$ will also cut out some set of points in $\mathbb{R}^{n+1}$ that will be a closed union of lines. We call this the *affine cone* of V. These equations will cut out some union of $\mathbb{P}^1$'s in $\mathbb{P}^{n+1}$, and we call this the *projective cone* of V. The projective cone is the disjoint union of the affine cone and V. For example, the affine cone over $x^2 + y^2 = z^2$ in $\mathbb{P}^2$ is just the "classical" picture of a cone in $\mathbb{R}^3$, see Figure 5.7. We will make this analogy precise in our algebraic setting in §9.2.11. To make a connection with the previous discussion on homogeneous ideals: the homogeneous ideal given by the cone is $(x^2 + y^2 - z^2)$.

5.5.2. The Proj construction.

We will now produce a scheme out of a graded ring. A **graded ring** for us is a ring $S_\bullet = \oplus_{n \in \mathbb{Z}} S_n$ (the subscript is called the **grading**), where multiplication respects the grading, i.e. sends $S_m \times S_n$ to S_{m+n}. Note that S_0 is a subring, and

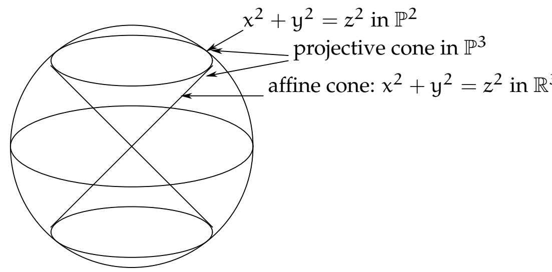


FIGURE 5.7. The affine and projective cone of $x^2 + y^2 = z^2$ in classical geometry

$S_\bullet$ is a S_0-algebra. In our examples so far, we have a graded ring $A[x_0, \dots, x_n]/I$ where I is a homogeneous ideal. We are taking the usual grading on $A[x_0, \dots, x_n]$, where each x_i has weight 1. In most of the examples below, $S_0 = A$, and $S_\bullet$ is generated as an S_0-algebra by S_1.

5.5.3. *Graded rings over* A, *and finitely generated graded rings.* Fix a ring A (the **base ring**). Our motivating example is $S_\bullet = A[x_0, x_1, x_2]$, with the usual grading. If $S_\bullet$ is graded by $\mathbb{Z}$, with $S_0 = A$, we say that $S_\bullet$ is a **graded ring over** A. Hence each S_n is an A-module. The subset $S_+ := \oplus_{i>0} S_i \subset S_\bullet$ is an ideal, called the **irrelevant ideal**. The reason for the name "irrelevant" will be clearer in a few paragraphs. If the irrelevant ideal S_+ is a finitely-generated ideal, we say that $S_\bullet$ is a **finitely generated graded ring over** A. If $S_\bullet$ is generated by S_1 as an A-algebra, we say that $S_\bullet$ is **generated in degree** 1. (We will later find it useful to interpret "$S_\bullet$ is generated in degree 1" as "the natural map $\operatorname{Sym}^\bullet S_1 \to S_\bullet$ is a surjection". The *symmetric algebra* construction will be briefly discussed in §14.5.3.)

5.5.A. EXERCISE. (a) Show that $S_\bullet$ is a finitely-generated graded ring if and only if $S_\bullet$ is a finitely-generated graded A-algebra, i.e. generated over $A = S_0$ by a finite number of homogeneous elements of positive degree. (Hint for the forward implication: show that the generators of S_+ as an ideal are also generators of $S_\bullet$ as an algebra.)

Motivated by our example of $\mathbb{P}^n_A$ and its closed subschemes, we now define a scheme $\operatorname{Proj} S_\bullet$. As we did with Spec of a ring, we will build it first as a set, then as a topological space, and finally as a ringed space. In our preliminary definition of $\mathbb{P}^n_A$, we glued together $n+1$ well-chosen affine pieces, but we don't want to make any choices, so we do this by simultaneously consider "all possible" affines. Our affine building blocks will be as follows. For each homogeneous $f \in S_+$, note that the localization $(S_\bullet)_f$ is naturally a graded ring as well, where $\deg(1/f) = -\deg f$. Consider

$$\operatorname{Spec}((S_\bullet)_f)_0. \tag{5.5.3.1}$$

where $((S_\bullet)_f)_0$ means the 0-graded piece of the graded ring $(S_\bullet)_f$. The notation $((S_\bullet)_f)_0$ is admittedly horrible — the first and third subscripts refer to the grading, and the second refers to localization.

(Before we begin: another possible way of defining $\operatorname{Proj} S_\bullet$ is by gluing together affines, by jumping straight to Exercises 5.5.G, 5.5.H, and 5.5.I. If you prefer that, by all means do so.)

The points of $\operatorname{Proj} S_\bullet$ are the set of homogeneous prime ideals of $S_\bullet$ not containing the irrelevant ideal S_+ (the "relevant prime ideals").

5.5.B. IMPORTANT AND TRICKY EXERCISE. Suppose $f \in S_+$ is homogeneous. Give a bijection between the primes of $((S_\bullet)_f)_0$ and the homogeneous prime ideals of $(S_\bullet)_f$. Describe the latter as a subset of $\operatorname{Proj} S_\bullet$. Hint: From the ring map $((S_\bullet)_f)_0 \to (S_\bullet)_f$, from each homogeneous prime of $(S_\bullet)_f$ we find a homogeneous prime of $((S_\bullet)_f)_0$. The reverse direction is the harder one. Given a prime ideal $P_0 \subset ((S_\bullet)_f)_0$, define $P \subset (S_\bullet)_f$ as $\oplus Q_i$, where $Q_i \subset ((S_\bullet)_f)_i$, and $a \in Q_i$ if and only if $a^{\deg f}/f^i \in P_0$. Note that $Q_0 = P_0$. Show that a is in Q_i if and only if $a^2 \in Q_{2i}$; show that if $a_1, a_2 \in Q_i$ then $(a_1 + a_2)^2 \in Q_{2i}$ and hence $a_1 + a_2 \in Q_i$; then show that P is an ideal; then show that P is prime.)

The interpretation of the points of $\operatorname{Proj} S_\bullet$ with homogeneous prime ideals helps us picture $\operatorname{Proj} S_\bullet$. For example, if $S_\bullet = k[x, y, z]$ with the usual grading, then we picture the homogeneous prime ideal $(z^2 - x^2 - y^2)$ as a subset of $\operatorname{Spec} S_\bullet$; it is a cone (see Figure 5.7). As in §5.5.1, we picture $\mathbb{P}^2_k$ as the "plane at infinity". Thus we picture this equation as cutting out a conic "at infinity". We will make this intuition somewhat more precise in §9.2.11.

5.5.C. EXERCISE (THE ZARISKI TOPOLOGY ON $\operatorname{Proj} S_\bullet$). If I is a homogeneous ideal of $S_\bullet$ contained in S_+, define the **vanishing set of** I, $V(I) \subset \operatorname{Proj} S_\bullet$, to be those homogeneous prime ideals containing I. As in the affine case, let $V(f)$ be $V((f))$, and let $D(f) = \operatorname{Proj} S_\bullet \setminus V(f)$ (the **projective distinguished open set**) be the complement of $V(f)$ (i.e. the open subscheme corresponding to that open set). Show that $D(f)$ is precisely the subset $((S_\bullet)_f)_0$ you described in the previous exercise.

(Caution: the definitions made in the previous exercise can certainly be extended to any ideal in S_+ and $D(f)$ can be defined even if f has degree 0. In what follows, we deliberately make these narrower definitions. For example, we will want the $D(f)$ to form an affine cover, and if f has degree 0, then $D(f)$ needn't be affine.)

As in the affine case, the $V(I)$'s satisfy the axioms of the closed set of a topology, and we call this the **Zariski topology** on $\operatorname{Proj} S_\bullet$. Many statements about the Zariski topology on Spec of a ring carry over to this situation with little extra work. Clearly $D(f) \cap D(g) = D(fg)$, by the same immediate argument as in the affine case (Exercise 4.5.D). As in the affine case (Exercise 4.5.E), if $D(f) \subset D(g)$, then $f^n \in (g)$ for some n, and vice versa.

5.5.D. EASY EXERCISE. Verify that the projective distinguished open sets $D(f)$ (as f runs through the homogeneous elements of S_+) form a base of the Zariski topology.

5.5.E. EXERCISE. Fix a graded ring $S_\bullet$.

(a) Suppose I is any homogeneous ideal of $S_\bullet$ contained in S_+, and f is a homogeneous element. Show that f vanishes on $V(I)$ if and only if $f^n \in I$ for some n. (Hint: Mimic the affine case; see Exercise 4.4.I.)

(b) If $Z \subset \operatorname{Proj} S_\bullet$, define $I(\cdot)$. Show that it is a homogeneous ideal. For any two subsets, show that $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
(c) For any subset $Z \subset \operatorname{Proj} S_\bullet$, show that $V(I(Z)) = \overline{Z}$.

5.5.F. EXERCISE (CF. EXERCISE 4.5.B). Fix a graded ring $S_\bullet$. Show that the following are equivalent.

(a) $V(I) = \varnothing$.
(b) for any f_i (as i runs through some index set) generating I, $\cup D(f_i) = \operatorname{Proj} S_\bullet$.
(c) $\sqrt{I} \supset S_+$.

This is more motivation for the S_+ being "irrelevant": any ideal whose radical contains it is "geometrically irrelevant".

Let's get back to constructing $\operatorname{Proj} S_\bullet$ as a *scheme*.

5.5.G. EXERCISE. Suppose some homogeneous $f \in S_+$ is given. Via the inclusion

$$D(f) = \operatorname{Spec}((S_\bullet)_f)_0 \hookrightarrow \operatorname{Proj} S_\bullet,$$

show that the Zariski topology on $\operatorname{Proj} S_\bullet$ restricts to the Zariski topology on $\operatorname{Spec}((S_\bullet)_f)_0$.

Now that we have defined $\operatorname{Proj} S_\bullet$ as a topological space, we are ready to define the structure sheaf. On $D(f)$, we wish it to be the structure sheaf of $\operatorname{Spec}((S_\bullet)_f)_0$. We will glue these sheaves together using Exercise 3.7.D on gluing sheaves.

5.5.H. EXERCISE. If $f, g \in S_+$ are homogeneous, describe an isomorphism between $\operatorname{Spec}((S_\bullet)_{fg})_0$ and the distinguished open subset $D(g^{\deg f}/f^{\deg g})$ of $\operatorname{Spec}((S_\bullet)_f)_0$.

Similarly, $\operatorname{Spec}((S_\bullet)_{fg})_0$ is identified with a distinguished open subset of $\operatorname{Spec}((S_\bullet)_g)_0$. We then glue the various $\operatorname{Spec}((S_\bullet)_f)_0$ (as f varies) altogether, using these pairwise gluings.

5.5.I. EXERCISE. By checking that these gluings behave well on triple overlaps (see Exercise 3.7.D), finish the definition of the scheme $\operatorname{Proj} S_\bullet$.

5.5.J. EXERCISE (SOME WILL FIND THIS ESSENTIAL, OTHERS WILL PREFER TO IGNORE IT). (Re)interpret the structure sheaf of $\operatorname{Proj} S_\bullet$ in terms of compatible stalks.

5.5.4. *Definition.* We (re)define **projective space** (over a ring A) by $\mathbb{P}^n_A := \operatorname{Proj} A[x_0, \dots, x_n]$. This definition involves no messy gluing, or special choice of patches.

5.5.K. EXERCISE. Check that this agrees with our earlier construction of $\mathbb{P}^n_A$ (Definition 5.4.9). (How do you know that the $D(x_i)$ cover $\operatorname{Proj} A[x_0, \dots, x_n]$?)

Notice that with our old definition of projective space, it would have been a nontrivial exercise to show that $D(x^2 + y^2 - z^2) \subset \mathbb{P}^2_k$ (the complement of a plane conic) is affine; with our new perspective, it is immediate — it is $\operatorname{Spec}(k[x, y, z]_{(x^2+y^2-z^2)})_0$.

5.5.L. EXERCISE. Both parts of this problem ask you to figure out the "right definition" of the vanishing scheme, in analogy with $V(\cdot)$ defined earlier. In both cases, you will be defining a *closed subscheme*, a notion we will introduce in §9.1. A closed subscheme of X is (informally) a particular kind of scheme structure on a closed subset of X.

(a) *(the most important part)* If $S_\bullet$ is generated in degree 1, and $f \in S_+$ is homogeneous, explain how to define $V(f)$ "in" $\operatorname{Proj} S_\bullet$, the **vanishing scheme of** f. (Warning: f in general isn't a function on $\operatorname{Proj} S_\bullet$. We will later interpret it as something close: a section of a line bundle.) Hence define $V(I)$ for any homogeneous ideal I of S_+.

(b) *(harder)* If $S_\bullet$ is a graded ring over A, but not necessarily generated in degree 1, explain how to define the vanishing scheme $V(f)$ "in" $\operatorname{Proj} S_\bullet$. (Hint: On $D(g)$, let $V(f)$ be cut out by all degree 0 equations of the form fh/g^n, where $n \in \mathbb{Z}^+$, and h is homogeneous. Show that this gives a well defined scheme structure on the set $V(f)$. (Once we know what a closed subscheme is, in §9.1, this will be clearly a closed subscheme.) Your calculations will mirror those of Exercise 5.5.H.)

5.5.5. Projective and quasiprojective schemes.

We call a scheme of the form $\operatorname{Proj} S_\bullet$, where $S_\bullet$ is a *finitely generated* graded ring over A, a **projective scheme over** A, or a **projective** A**-scheme**. A **quasiprojective** A**-scheme** is a quasicompact open subscheme of a projective A-scheme. The "A" is omitted if it is clear from the context; often A is a field.

5.5.6. *Unimportant remarks.* (i) Note that $\operatorname{Proj} S_\bullet$ makes sense even when $S_\bullet$ is not finitely generated. This can be useful. But having this more general construction can make things easier. For example, you will later be able to do Exercise 7.4.D without worrying about Exercise 7.4.H.)

(ii) The quasicompact requirement in the definition quasiprojectivity is of course redundant in the Noetherian case (cf. Exercise 4.6.M), which is all that matters to most.

5.5.7. *Silly example.* Note that $\mathbb{P}^0_A = \operatorname{Proj} A[T] \cong \operatorname{Spec} A$. Thus "$\operatorname{Spec} A$ is a projective A-scheme".

5.5.8. Example: $\mathbb{P}V$**.** We can make this definition of projective space even more choice-free as follows. Let V be an $(n+1)$-dimensional vector space over k. (Here k can be replaced by any ring A as usual.) Define

$$\operatorname{Sym}^\bullet V^\vee = k \oplus V^\vee \oplus \operatorname{Sym}^2 V^\vee \oplus \cdots .$$

(The reason for the dual is explained by the next exercise.) If for example V is the dual of the vector space with basis associated to $x_0, \dots, x_n$, we would have $\operatorname{Sym}^\bullet V^\vee = k[x_0, \dots, x_n]$. Then we can define $\mathbb{P}V := \operatorname{Proj} \operatorname{Sym}^\bullet V^\vee$. In this language, we have an interpretation for $x_0, \dots, x_n$: they are the linear functionals on the underlying vector space V.

5.5.M. UNIMPORTANT EXERCISE. Suppose k is algebraically closed. Describe a natural bijection between one-dimensional subspaces of V and the points of $\mathbb{P}V$. Thus this construction canonically (in a basis-free manner) describes the one-dimensional subspaces of the vector space $\operatorname{Spec} V$.

Unimportant remark: you may be surprised at the appearance of the dual in the definition of $\mathbb{P}V$. This is explained by the previous exercise. Most normal (traditional) people define the projectivization of a vector space V to be the space of one-dimensional subspaces of V. Grothendieck considered the projectivization to be the space of one-dimensional *quotients*. One motivation for this is that it gets rid of the annoying dual in the definition above. There are better reasons, that we won't go into here. In a nutshell, quotients tend to be better-behaved than subobjects for coherent sheaves, which generalize the notion of vector bundle. (We will discuss them in Chapter 14.)

On another note related to Exercise 5.5.M: you can also describe a natural bijection between points of V and the points of $\operatorname{Spec}\operatorname{Sym}^{\bullet} V^{\vee}$. This construction respects the affine/projective cone picture of §9.2.11.

5.5.9. *The Grassmannian.* At this point, we could describe the fundamental geometric object known as the *Grassmannian*, and give the "wrong" definition of it. We will instead wait until §7.7 to give the wrong definition, when we will know enough to sense that something is amiss. The right definition will be given in §17.6.

CHAPTER 6

Some properties of schemes

6.1 Topological properties

We will now define some useful properties of schemes. The definitions of *irreducible*, *irreducible component*, *closed point*, *specialization*, *generization*, *generic point*, *connected*, *connected component*, and *quasicompact* were given in §4.5–4.6. You should have pictures in your mind of each of these notions.

Exercise 4.6.O shows that $\mathbb{A}^n$ is irreducible (it was easy). This argument "behaves well under gluing", yielding:

6.1.A. EASY EXERCISE. Show that $\mathbb{P}_k^n$ is irreducible.

6.1.B. EXERCISE. Exercise 4.7.E showed that there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

6.1.C. EASY EXERCISE. Prove that if X is a scheme that has a finite cover $X = \cup_{i=1}^n \operatorname{Spec} A_i$ where A_i is Noetherian, then X is a Noetherian topological space (§4.6.3). (We will soon call such a scheme a *Noetherian scheme*, §6.3.4.)

Thus $\mathbb{P}_k^n$ and $\mathbb{P}_{\mathbb{Z}}^n$ are Noetherian topological spaces: we built them by gluing together a finite number of spectra of Noetherian rings.

6.1.D. EASY EXERCISE. Show that a scheme X is quasicompact if and only if it can be written as a finite union of affine schemes. (Hence $\mathbb{P}_k^n$ is quasicompact.)

6.1.E. IMPORTANT EXERCISE: QUASICOMPACT SCHEMES HAVE CLOSED POINTS. Show that if X is a quasicompact scheme, then every point has a closed point in its closure. Show that every nonempty closed subset of X contains a closed point of X. In particular, every nonempty quasicompact scheme has a closed point. (Warning: there exist non-empty schemes with no closed points, so your argument had better use the quasicompactness hypothesis!)

This exercise will often be used in the following way. If there is some property P of points of a scheme that is "open" (if a point p has P, then there is some neighborhood U of p such that all the points in U have P), then to check if *all* points of a quasicompact scheme have P, it suffices to check only the closed points. This provides a connection between schemes and the classical theory of varieties — the points of traditional varieties are the *closed* points of the corresponding schemes. A first example of this is Exercise 6.2.C. In many good situations, the closed points are dense (such as for varieties), Exercise 6.3.E, but this is not true in some important cases, such as spectra of local rings (e.g. Spec $k[x]_{(x)}$, see Exercise 4.4.J).

6.1.1. Quasiseparated schemes. Quasiseparatedness is a weird notion that comes in handy for certain people. (Warning: we will later realize that this is really a property of *morphisms*, not of schemes §8.3.1.) Most people, however, can ignore this notion, as the schemes they will encounter in real life will all have this property. A topological space is **quasiseparated** if the intersection of any two quasicompact open sets is quasicompact. Thus a scheme is quasiseparated if the intersection of any two affine open subsets is a finite union of affine open subsets.

6.1.F. SHORT EXERCISE. Prove this equivalence.

We will see later that this will be a useful hypothesis in theorems (in conjunction with quasicompactness), and that various interesting kinds of schemes (affine, locally Noetherian, separated, see Exercises 6.1.G, 6.3.B, and 11.1.H resp.) are quasiseparated, and this will allow us to state theorems more succinctly (e.g. "if X is quasicompact and quasiseparated" rather than "if X is quasicompact, and either this or that or the other thing hold").

6.1.G. EXERCISE. Show that affine schemes are quasiseparated.

"Quasicompact and quasiseparated" means something concrete:

6.1.H. EXERCISE. Show that a scheme X is quasicompact and quasiseparated if and only if X can be covered by a finite number of affine open subsets, any two of which have intersection also covered by a finite number of affine open subsets.

So when you see "quasicompact and quasiseparated" as hypotheses in a theorem, you should take this as a clue that you will use this interpretation, and that finiteness will be used in an essential way.

6.1.I. EASY EXERCISE. Show that all projective A-schemes are quasicompact and quasiseparated. (Hint: use the fact that the graded ring in the definition is finitely generated — those finite number of generators will lead you to a covering set.)

6.1.2. Dimension. One very important topological notion is *dimension*. (It is amazing that this is a *topological* idea.) But despite being intuitively fundamental, it is more difficult, so we postpone it until Chapter 12.

6.2 Reducedness and integrality

Recall that one of the alarming things about schemes is that functions are not determined by their values at points, and that was because of the presence of nilpotents (§4.2.9).

6.2.1. Definition. A ring is said to be *reduced* if it has no nonzero nilpotents (§4.2.11). A scheme X is **reduced** if $\mathcal{O}_X(U)$ is reduced for every open set U of X.

An example of a nonreduced affine scheme is Spec $k[x,y]/(y^2, xy)$. A useful representation of this scheme is given in Figure 6.1, although we will only explain in §6.5 why this is a good picture. The fuzz indicates that there is some nonreducedness going on at the origin. Here are two different functions: x and $x+y$. Their values agree at all points (all closed points $[(x-a, y)] = (a, 0)$ and at the generic point $[(y)]$). They are actually the same function on the open set $D(x)$, which is not surprising, as $D(x)$ is reduced, as the next exercise shows. (This explains why the fuzz is only at the origin, where $y = 0$.)

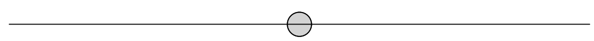


FIGURE 6.1. A picture of the scheme $\operatorname{Spec} k[x,y]/(y^2, xy)$. The fuzz indicates where "the nonreducedness lives".

6.2.A. EXERCISE. Show that $\left(k[x,y]/(y^2, xy)\right)_x$ has no nilpotents. (Possible hint: show that it is isomorphic to another ring, by considering the geometric picture. Exercise 4.2.I may give another hint.)

6.2.B. EXERCISE (REDUCEDNESS IS A **stalk-local** PROPERTY, I.E. CAN BE CHECKED AT STALKS). Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if f and g are two functions on a reduced scheme that agree at all points, then $f = g$. (Two hints: $\mathcal{O}_X(U) \hookrightarrow \prod_{x \in U} \mathcal{O}_{X,x}$ from Exercise 3.4.A, and the nilradical is intersection of all prime ideals from Theorem 4.2.10.)

We remark that the fuzz in Figure 6.1 indicates the points where there is nonreducedness.

6.2.C. EXERCISE. If X is a quasicompact scheme, show that it suffices to check reducedness at closed points. (Hint: Exercise 6.1.E.)

Warning for experts: if a scheme X is reduced, then it is immediate from the definition that its ring of global sections is reduced. However, the converse is not true; the example of the scheme X cut out by $x^2 = 0$ in $\mathbb{P}^2_k$ will come up in §20.1.5, and you already know enough to verify that $\Gamma(X, \mathcal{O}_X) \cong k$.

6.2.D. EXERCISE. Suppose X is quasicompact, and f is a function (a global section of $\mathcal{O}_X$) that vanishes at all points of x. Show that there is some n such that $f^n = 0$. Show that this may fail if X is not quasicompact. (This exercise is less important, but shows why we like quasicompactness, and gives a standard pathology when quasicompactness doesn't hold.) Hint: take an infinite disjoint union of $\operatorname{Spec} A_n$ with $A_n := k[\epsilon]/\epsilon^n$.

Definition. A scheme X is **integral** if it is nonempty, and $\mathcal{O}_X(U)$ is an integral domain for every nonempty open set U of X.

6.2.E. IMPORTANT EXERCISE. Show that a scheme X is integral if and only if it is irreducible and reduced.

6.2.F. EXERCISE. Show that an affine scheme $\operatorname{Spec} A$ is integral if and only if A is an integral domain.

6.2.G. EXERCISE. Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of X. Show that the stalk at η, $\mathcal{O}_{X,\eta}$, is naturally $K(A)$, the fraction field of A. This is called the **function field** $K(X)$ of X. It can be computed on any non-empty open set of X, as any such open set contains the generic point.

6.2.H. EXERCISE. Suppose X is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \varnothing$. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of X (so A is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = K(A)$ (where U is any non-empty open set) is an inclusion. Thus irreducible varieties (an important example of integral schemes defined later) have the convenient property that sections over different open sets can be considered subsets of the same ring. Thus restriction maps (except to the empty set) are always inclusions, and gluing is easy: functions f_i on a cover U_i of U (as i runs over an index set) glue if and only if they are the same element of $K(X)$. This is one reason why (irreducible) varieties are usually introduced before schemes.

Integrality is not stalk-local (the disjoint union of two integral schemes is not integral, as $\operatorname{Spec} A \coprod \operatorname{Spec} B = \operatorname{Spec} A \times B$, cf. Exercise 4.6.T), but it almost is, as is shown in the following believable exercise.

6.2.I. UNIMPORTANT EXERCISE. Show that a locally Noetherian scheme X is integral if and only if X is connected and all stalks $\mathcal{O}_{X,p}$ are integral domains. Thus in "good situations" (when the scheme is Noetherian), integrality is the union of local (stalks are integral domains) and global (connected) conditions.

6.3 Properties of schemes that can be checked "affine-locally"

This section is intended to address something tricky in the definition of schemes. We have defined a scheme as a topological space with a sheaf of rings, that can be covered by affine schemes. Hence we have all of the affine open sets in the cover, but we don't know how to communicate between any two of them. Somewhat more explicitly, if I have an affine cover, and you have an affine cover, and we want to compare them, and I calculate something on my cover, there should be some way of us getting together, and figuring out how to translate my calculation over to your cover. The Affine Communication Lemma 6.3.2 will provide a convenient machine for doing this.

Thanks to this lemma, we can define a host of important properties of schemes. All of these are "affine-local" in that they can be checked on any affine cover, i.e. a covering by open affine sets. We like such properties because we can check them using any affine cover we like. If the scheme in question is quasicompact, then we need only check a finite number of affine open sets.

6.3.1. Proposition. — *Suppose* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are affine open subschemes of a scheme* X*. Then* $\operatorname{Spec} A \cap \operatorname{Spec} B$ *is the union of open sets that are simultaneously distinguished open subschemes of* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$*.*

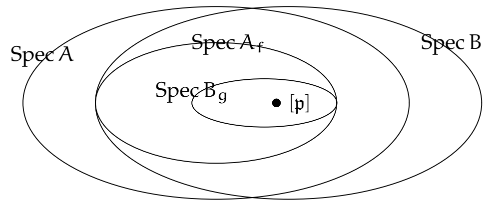


FIGURE 6.2. A trick to show that the intersection of two affine open sets may be covered by open sets that are simultaneously distinguished in both affine open sets

Proof. (See Figure 6.2 for a sketch.) Given any point $p \in \operatorname{Spec} A \cap \operatorname{Spec} B$, we produce an open neighborhood of p in $\operatorname{Spec} A \cap \operatorname{Spec} B$ that is simultaneously distinguished in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Let $\operatorname{Spec} A_f$ be a distinguished open subset of $\operatorname{Spec} A$ contained in $\operatorname{Spec} A \cap \operatorname{Spec} B$ and containing p. Let $\operatorname{Spec} B_g$ be a distinguished open subset of $\operatorname{Spec} B$ contained in $\operatorname{Spec} A_f$ and containing p. Then $g \in \Gamma(\operatorname{Spec} B, \mathscr{O}_X)$ restricts to an element $g' \in \Gamma(\operatorname{Spec} A_f, \mathscr{O}_X) = A_f$. The points of $\operatorname{Spec} A_f$ where g vanishes are precisely the points of $\operatorname{Spec} A_f$ where g' vanishes, so

$$\begin{aligned} \operatorname{Spec} B_g &= \operatorname{Spec} A_f \setminus \{[\mathfrak{p}] : g' \in \mathfrak{p}\} \\ &= \operatorname{Spec} (A_f)_{g'}. \end{aligned}$$

If $g' = g''/f^n$ ($g'' \in A$) then $\operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fg''}$, and we are done. □

The following easy result will be crucial for us.

6.3.2. Affine Communication Lemma. — *Let P be some property enjoyed by some affine open sets of a scheme X, such that*

(i) *if an affine open set* $\operatorname{Spec} A \hookrightarrow X$ *has property* P *then for any* $f \in A$, $\operatorname{Spec} A_f \hookrightarrow X$ *does too.*
(ii) *if* $(f_1, \dots, f_n) = A$, *and* $\operatorname{Spec} A_{f_i} \hookrightarrow X$ *has* P *for all* i, *then so does* $\operatorname{Spec} A \hookrightarrow X$.

Suppose that $X = \cup_{i \in I} \operatorname{Spec} A_i$ *where* $\operatorname{Spec} A_i$ *has property* P. *Then every open affine subset of* X *has* P *too.*

We say such a property is **affine-local**. Note that any property that is stalk-local (a scheme has property P if and only if all its stalks have property Q) is necessarily affine-local (a scheme has property P if and only if all of its affines have property R, where an affine scheme has property R if and only if and only if all its stalks have property Q), but it is sometimes not so obvious what the right definition of Q is; see for example the discussion of normality in the next section.

Proof. Let $\operatorname{Spec} A$ be an affine subscheme of X. Cover $\operatorname{Spec} A$ with a finite number of distinguished open sets $\operatorname{Spec} A_{g_j}$, each of which is distinguished in some $\operatorname{Spec} A_i$. This is possible by Proposition 6.3.1 and the quasicompactness of $\operatorname{Spec} A$ (Exercise 4.6.D(a)). By (i), each $\operatorname{Spec} A_{g_j}$ has P. By (ii), $\operatorname{Spec} A$ has P. □

By choosing property P appropriately, we define some important properties of schemes.

6.3.3. Proposition. — *Suppose A is a ring, and $(f_1, \dots, f_n) = A$.*

(a) *If A is a Noetherian ring, then so is A_{f_i}. If each A_{f_i} is Noetherian, then so is A.*
(b) *If A is reduced, then A_{f_i} is also reduced. If each A_{f_i} is reduced, then so is A.*
(c) *Suppose B is a ring, and A is a B-algebra. (Hence A_g is a B-algebra for all $g \in A$.) If A is a finitely generated B-algebra, then so is A_{f_i}. If each A_{f_i} is a finitely-generated B-algebra, then so is A.*

We will prove these shortly (§6.3.9). But let's first motivate you to read the proof by giving some interesting definitions *assuming* Proposition 6.3.3 is true.

6.3.4. Important Definition. Suppose X is a scheme. If X can be covered by affine open sets $\operatorname{Spec} A$ where A is Noetherian, we say that X is a **locally Noetherian scheme**. If in addition X is quasicompact, or equivalently can be covered by finitely many such affine open sets, we say that X is a **Noetherian scheme**. (We will see a number of definitions of the form "if X has this property, we say that it is locally Q; if further X is quasicompact, we say that it is Q.") By Exercise 6.1.C, the underlying topological space of a Noetherian scheme is Noetherian.

6.3.A. EXERCISE. Show that all open subsets of a Noetherian topological space (hence a Noetherian scheme) are quasicompact.

6.3.B. EXERCISE. Show that locally Noetherian schemes are quasiseparated.

6.3.C. EXERCISE. Show that a Noetherian scheme has a finite number of irreducible components. Show that a Noetherian scheme has a finite number of connected components, each a finite union of irreducible components.

6.3.D. EXERCISE. Show that X is reduced if and only if X can be covered by affine open sets $\operatorname{Spec} A$ where A is reduced.

Our earlier definition of reducedness required us to check that the ring of functions over *any* open set is nilpotent-free. Our new definition lets us check a single affine cover. Hence for example $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are reduced.

6.3.5. *Schemes over a given field, or more generally over a given ring (A-schemes).* You may be particularly interested in working over a particular field, such as $\mathbb{C}$ or $\mathbb{Q}$, or over a ring such as $\mathbb{Z}$. Motivated by this, we define the notion of **A-scheme**, or **scheme over** A, where A is a ring, as a scheme where all the rings of sections of the structure sheaf (over all open sets) are A-algebras, and all restriction maps are maps of A-algebras. (Like some earlier notions such as quasiseparatedness, this will later in Exercise 7.3.G be properly understood as a "relative notion"; it is the data of a morphism $X \to \operatorname{Spec} A$.) Suppose now X is an A-scheme. If X can be covered by affine open sets $\operatorname{Spec} B_i$ where each B_i is a *finitely generated* A-algebra, we say that X is **locally of finite type over** A, or that it is a **locally of finite type A-scheme**. (This is admittedly cumbersome terminology; it will make more sense later, once we know about morphisms in §8.3.9.) If furthermore X is quasicompact, X is (of) **finite type over** A, or a **finite type A-scheme**. Note that a scheme locally of finite type over k or $\mathbb{Z}$ (or indeed any Noetherian ring) is locally Noetherian, and similarly a scheme of finite type over any Noetherian ring is Noetherian. As our key "geometric" examples: (i) $\operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]/I$ is a finite-type $\mathbb{C}$-scheme; and (ii) $\mathbb{P}^n_{\mathbb{C}}$ is a finite type $\mathbb{C}$-scheme. (The field $\mathbb{C}$ may be replaced by an arbitrary ring A.)

6.3.6. *Varieties.* We now make a connection to the classical language of varieties. An affine scheme that is a reduced and of finite type k-scheme is said to be an **affine variety (over** k**)**, or an **affine** k**-variety**. A reduced (quasi-)projective k-scheme is a **(quasi-)projective variety (over** k**)**, or an **(quasi-)projective** k**-variety**. (Warning: in the literature, it is sometimes also assumed in the definition of variety that the scheme is irreducible, or that k is algebraically closed.) We will not define varieties in general until §11.1.7; we will need the notion of separatedness first, to exclude abominations like the line with the doubled origin (Example 5.4.5). But many of the statements we will make in this section about affine k-varieties will automatically apply more generally to k-varieties.

6.3.E. EXERCISE. Show that a point of a locally finite type k-scheme is a closed point if and only if the residue field of the stalk of the structure sheaf at that point is a finite extension of k. (Hint: the Nullstellensatz 4.2.3.) Show that the closed points are dense on such a scheme (even though it needn't be quasicompact, cf. Exercise 6.1.E). (For another exercise on closed points, see 6.1.E. Warning: closed points need not be dense even on quite reasonable schemes, such as that of Exercise 4.4.J.)

6.3.7. $\star\star$ *Exercise (analytification of complex varieties).* (Warning: Any discussion of analytification will be only for readers who are familiar with the notion of a complex analytic varieties, or willing to develop it on their own in parallel with our development of schemes.) Suppose X is a reduced, finite type $\mathbb{C}$-scheme. Define the corresponding complex analytic prevariety X_{an}. (The definition of an analytic prevariety is the same as the definition of a variety without the Hausdorff condition.) Caution: your definition should not depend on a choice of an affine cover of X. (Hint: First explain how to analytify reduced finite type affine $\mathbb{C}$-schemes. Then glue.) Give a bijection between the closed points of X and the points of X_{an}, using the weak Nullstellensatz 4.2.2. (In fact one may construct a continuous map of sets $X \to X_{an}$ generalizing Exercise 4.2.F, but this is more fun than useful.) In Exercise 7.3.J, we will see that analytification can be made into a functor.

6.3.8. *Definition.* The **degree** of a closed point p of a locally finite type k-scheme is the degree of the field extension $\kappa(p)/k$. For example, in $\mathbb{A}^1_k = \operatorname{Spec} k[t]$, the point $[k[t]/p(t)]$ (p irreducible) is $\deg p$. If k is algebraically closed, the degree of every closed point is 1.

6.3.9. *Proof of Proposition 6.3.3.* We divide each part into (i) and (ii) following the statement of the Affine Communication Lemma 6.3.2. (a) (i) If $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of A_f, then we can verify that $J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of A, where

$$J_j = \{r \in A \ : \ r \in I_j\}$$

where $r \in I_j$ means "the image in A_f lies in I_j". (We think of this as $I_j \cap A$, except in general A needn't inject into A_{f_i}.) Clearly J_j is an ideal of A. If $x/f^n \in I_{j+1} \setminus I_j$ where $x \in A$, then $x \in J_{j+1}$, and $x \notin J_j$ (or else $x(1/f)^n \in J_j$ as well). (ii) Suppose

$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of A. Then for each $1 \leq i \leq n$,

$$I_{i,1} \subset I_{i,2} \subset I_{i,3} \subset \cdots$$

is an increasing chain of ideals in A_{f_i}, where $I_{i,j} = I_j \otimes_A A_{f_i}$. It remains to show that for each j, $I_{i,j} \subsetneq I_{i,j+1}$ for some i; the result will then follow.

6.3.F. EXERCISE. Finish this argument.

6.3.G. EXERCISE. Prove (b).

(c) (i) is clear: if A is generated over B by $r_1, \ldots, r_n$, then A_f is generated over B by $r_1, \ldots, r_n, 1/f$.

(ii) Here is the idea. As the f_i generate A, we can write $1 = \sum c_i f_i$ for $c_i \in A$. We have generators of A_i: r_{ij}/f_i^j, where $r_{ij} \in A$. I claim that $\{f_i\}_i \cup \{c_i\} \cup \{r_{ij}\}_{ij}$ generate A as a B-algebra. Here's why. Suppose you have any $r \in A$. Then in A_{f_i}, we can write r as some polynomial in the r_{ij}'s and f_i, divided by some huge power of f_i. So "in each A_{f_i}, we have described r in the desired way", except for this annoying denominator. Now use a partition of unity type argument as in the proof of Theorem 5.1.2 to combine all of these into a single expression, killing the denominator. Show that the resulting expression you build still agrees with r in each of the A_{f_i}. Thus it is indeed r.

6.3.H. EXERCISE. Make this argument precise.

This concludes the proof of Proposition 6.3.3 □

6.3.I. EASY EXERCISE. Suppose $S_\bullet$ is a finitely generated graded ring over A. Show that $\operatorname{Proj} S_\bullet$ is of finite type over $A = S_0$. If S_0 is a Noetherian ring, show that $\operatorname{Proj} S_\bullet$ is a Noetherian scheme, and hence that $\operatorname{Proj} S_\bullet$ has a finite number of irreducible components. Show that any quasiprojective scheme is locally of finite type over A. If A is Noetherian, show that any quasiprojective A-scheme is quasicompact, and hence of finite type over A. Show this need not be true if A is not Noetherian. Better: give an example of a quasiprojective A-scheme that is not quasicompact, necessarily for some non-Noetherian A. (Hint: Silly example 5.5.7.)

6.4 Normality and factoriality

6.4.1. Normality.

We can now define a property of schemes that says that they are "not too far from smooth", called *normality*, which will come in very handy. We will see later that "locally Noetherian normal schemes satisfy Hartogs' Lemma" (Algebraic Hartogs' Lemma 12.3.10 for Noetherian normal schemes): functions defined away from a set of codimension 2 extend over that set. (We saw a first glimpse of this in §5.4.2.) As a consequence, rational functions that have no poles (certain sets of codimension one where the function isn't defined) are defined everywhere. We need definitions of dimension and poles to make this precise.

A scheme X is **normal** if all of its stalks $\mathcal{O}_{X,p}$ are normal, i.e. are integral domains, and integrally closed in their fraction fields. (An integral domain A is **integrally closed** if the only zeros in $K(A)$ to any monic polynomial in $A[x]$ must lie in A itself. The basic example is $\mathbb{Z}$.) As reducedness is a stalk-local property (Exercise 6.2.B), normal schemes are reduced.

6.4.A. EXERCISE. Show that integrally closed domains behave well under localization: if A is an integrally closed domain, and S is a multiplicative subset, show that $S^{-1}A$ is an integrally closed domain. (Hint: assume that $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in S^{-1}A$ has a root in the fraction field. Turn this into another equation in $A[x]$ that also has a root in the fraction field.)

It is no fun checking normality at every single point of a scheme. Thanks to this exercise, we know that if A is an integrally closed domain, then $\operatorname{Spec} A$ is normal. Also, for quasicompact schemes, normality can be checked at closed points, thanks to this exercise, and the fact that for such schemes, any point is a generization of a closed point (see Exercise 6.1.E).

It is not true that normal schemes are integral. For example, the disjoint union of two normal schemes is normal. Thus $\operatorname{Spec} k \coprod \operatorname{Spec} k \cong \operatorname{Spec}(k \times k) \cong \operatorname{Spec} k[x]/(x(x-1))$ is normal, but its ring of global sections is not an integral domain.

6.4.B. UNIMPORTANT EXERCISE. Show that a Noetherian scheme is normal if and only if it is the finite disjoint union of integral Noetherian normal schemes. (Hint: Exercise 6.2.I.)

We are close to proving a useful result in commutative algebra, so we may as well go all the way.

6.4.2. Proposition. — *If A is an integral domain, then the following are equivalent.*

(i) *A is integrally closed.*
(ii) *$A_{\mathfrak{p}}$ is integrally closed for all prime ideals $\mathfrak{p} \subset A$.*
(iii) *$A_{\mathfrak{m}}$ is integrally closed for all maximal ideals $\mathfrak{m} \subset A$.*

Proof. Exercise 6.4.A shows that integral closure is preserved by localization, so (i) implies (ii). Clearly (ii) implies (iii).

It remains to show that (iii) implies (i). This argument involves a pretty construction that we will use again. Suppose A is not integrally closed. We show that there is some $\mathfrak{m}$ such that $A_{\mathfrak{m}}$ is also not integrally closed. Suppose

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{6.4.2.1}$$

(with $a_i \in A$) has a solution s in $K(A) \setminus A$. Let I be the **ideal of denominators of** s:

$$I := \{r \in A \ : \ rs \in A\}.$$

(Note that I is clearly an ideal of A.) Now $I \neq A$, as $1 \notin I$. Thus there is some maximal ideal $\mathfrak{m}$ containing I. Then $s \notin A_{\mathfrak{m}}$, so equation (6.4.2.1) in $A_{\mathfrak{m}}[x]$ shows that $A_{\mathfrak{m}}$ is not integrally closed as well, as desired. $\square$

6.4.C. UNIMPORTANT EXERCISE. If A is an integral domain, show that $A = \cap A_{\mathfrak{m}}$, where the intersection runs over all maximal ideals of A. (We won't use this exercise, but it gives good practice with the ideal of denominators.)

6.4.D. UNIMPORTANT EXERCISE RELATING TO THE IDEAL OF DENOMINATORS. One might naively hope from experience with unique factorization domains that the ideal of denominators is principal. This is not true. As a counterexample, consider our new friend $A = k[w, x, y, z]/(wz - xy)$ (which we last saw in Example 5.4.12, and which we will later recognize as the cone over the quadric surface), and $w/y = x/z \in K(A)$. Show that $I = (y, z)$. We will soon see that it is not principal (Exercise 13.1.C).

6.4.3. Factoriality.
We define a notion which implies normality.

6.4.4. Definition. If all the stalks of a scheme X are unique factorization domains, we say that X is **factorial**. (Unimportant remark: This is sometimes called *locally factorial*, which may falsely suggest that this notion if affine local, which it isn't, see Exercise 6.4.M. But the alternative terminology avoids another confusion: unique factorial domains are sometimes called *factorial rings*, and while we will see that if A is a unique factorial domain then $\operatorname{Spec} A$ is factorial, we will also see in Exercise 6.4.M that the converse does not hold.)

6.4.E. EXERCISE. Show that any localization of a unique factorization domain is a unique factorization domain.

Thus if A is a unique factorization domain, then $\operatorname{Spec} A$ is factorial. (The converse need not hold. This property is *not* affine-local, see Exercise 6.4.M. In fact, we will see that elliptic curves are factorial, yet *no* affine open set is the Spec of a unique factorization domain, §21.9.1.) Hence it suffices to check factoriality by finding an appropriate affine cover.

6.4.5. *Remark: How to check if a ring is a unique factorization domain.* There are very few means of checking that a Noetherian integral domain is a unique factorization domain. Some useful ones are: (0) elementary means (rings with a euclidean algorithm such as $\mathbb{Z}$, $k[t]$, and $\mathbb{Z}[i]$; polynomial rings over a unique factorization domain, by Gauss' Lemma). (1) Exercise 6.4.E, that the localization of a unique factorization domain is also a unique factorization domain. (2) height 1 primes are principal (Proposition 12.3.5). (3) Nagata's Lemma (Exercise 15.2.R). (4) normal and $\mathrm{Cl} = 0$ (Exercise 15.2.Q).

6.4.6. *Factoriality implies normality.* One of the reasons we like factoriality is that it implies normality.

6.4.F. IMPORTANT EXERCISE. Show that unique factorization domains are integrally closed. Hence factorial schemes are normal, and if A is a unique factorization domain, then $\operatorname{Spec} A$ is normal. (However, rings can be integrally closed without being unique factorization domains, as we will see in Exercise 6.4.K. Another example, without proof is given in Exercise 6.4.M; in this example, Spec of the ring is factorial. A variation on Exercise 6.4.K will show that schemes can be normal without being factorial, see Exercise 13.1.D.)

6.4.G. EASY EXERCISE. Show that the following schemes are normal: $\mathbb{A}^n_k$, $\mathbb{P}^n_k$, $\operatorname{Spec} \mathbb{Z}$. (As usual, k is a field. Although it is true that if A is integrally closed then $A[x]$ is as well [**B**, Ch. 5, §1, no. 3, Cor. 2], this is not an easy fact, so do not use it here.)

6.4.H. HANDY EXERCISE (YIELDING MANY OF ENLIGHTENING EXAMPLES LATER). Suppose A is a unique factorization domain with 2 invertible, $f \in A$ has no repeated prime factors, and $z^2 - f$ is irreducible in $A[z]$. Show that $\operatorname{Spec} A[z]/(z^2 - f)$ is normal. Show that if f is *not* square-free, then $\operatorname{Spec} A[z]/(z^2 - f)$ is *not* normal. (Hint: $B := A[z]/(z^2 - f)$ is an integral domain, as $(z^2 - f)$ is prime in $A[z]$. Suppose we have monic $F(T) \in B[T]$ so that $F(T) = 0$ has a root α in $K(B)$. Then by replacing $F(T)$ by $\overline{F}(T)F(T)$, we can assume $F(T) \in A[T]$. Also, $\alpha = g + hz$ where $g, h \in K(A)$. Now α is the root of $Q(T) = 0$ for monic $Q(T) = T^2 - 2gT + (g^2 - h^2 f) \in K(A)[T]$, so we can factor $F(T) = P(T)Q(T)$ in $K(A)[T]$. By Gauss' lemma, $2g$, $g^2 - h^2 f \in A$. Say $g = r/2$, $h = s/t$ (s and t have no common factors, $r, s, t \in A$). Then $g^2 - h^2 f = (r^2t^2 - 4s^2 f)/4t^2$. Then t is a unit, and r is even.)

6.4.I. EXERCISE. Show that the following schemes are normal:

(a) $\operatorname{Spec} \mathbb{Z}[x]/(x^2 - n)$ where n is a square-free integer congruent to $3 \pmod 4$;
(b) $\operatorname{Spec} k[x_1, \dots, x_n]/(x_1^2 + x_2^2 + \cdots + x_m^2)$ where $\operatorname{char} k \neq 2$, $m \geq 3$;
(c) $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ where $\operatorname{char} k \neq 2$ and k is algebraically closed. This is our cone over a quadric surface example from Exercises 5.4.12 and 6.4.D. (Hint: Exercise 6.4.J may help.)

6.4.J. EXERCISE (DIAGONALIZING QUADRICS). Suppose k is an algebraically closed field of characteristic not 2.
(a) Show that any quadratic form in n variables can be "diagonalized" by changing coordinates to be a sum of at most n squares (e.g. $uw - v^2 = ((u + w)/2)^2 + (i(u - w)/2)^2 + (iv)^2$), *where the linear forms appearing in the squares are linearly independent*. (Hint: use induction on the number of variables, by "completing the square" at each step.)
(b) Show that the number of squares appearing depends only on the quadric. For example, $x^2 + y^2 + z^2$ cannot be written as a sum of two squares. (Possible approach: given a basis $x_1, \dots, x_n$ of the linear forms, write the quadratic form as

$$\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where M is a symmetric matrix. Determine how M transforms under a change of basis, and show that the rank of M is independent of the choice of basis.)

The **rank** of the quadratic form is the number of ("linearly independent") squares needed.

6.4.K. EXERCISE (RINGS CAN BE INTEGRALLY CLOSED BUT NOT FACTORIAL). Suppose k is an algebraically closed field of characteristic not 2. Let $A = k[w, x, y, z]/(wz-$

xy), so Spec A is the cone over the quadric surface (cf. Exercises 5.4.12 and 6.4.D).
(a) Show that A is integrally closed. (Hint: Exercises 6.4.I(c) and 6.4.J.)
(b) Show that A is not a unique factorization domain. (Clearly $wz = xy$. But why are w, x, y, and z irreducible? Hint: Since A is a graded integral domain, if a homogeneous element factor, show that the factors must be homogeneous.)

6.4.L. EXERCISE. Suppose A is a k-algebra where $\operatorname{char} k = 0$, and l/k is a finite field extension. Show that if $A \otimes_k l$ is normal (and in particular an integral domain) then A is normal. (This is a case of a more general fact, and stated correctly, the converse is true.) Show that $\operatorname{Spec} k[w,x,y,z]/(wz - xy)$ is normal if k has characteristic 0. Possible hint: reduce to the case where l/k is Galois.

6.4.M. EXERCISE (FACTORIALITY IS NOT AFFINE-LOCAL). Let $A = (\mathbb{Q}[x,y]_{x^2+y^2})_0$ denote the homogeneous degree 0 part of the ring $\mathbb{Q}[x,y]_{x^2+y^2}$. In other words, it consists of quotients $f(x,y)/(x^2+y^2)^n$, where f has pure degree $2n$. Show that the distinguished open sets $D(\frac{x^2}{x^2+y^2})$ and $D(\frac{y^2}{x^2+y^2})$ cover Spec A. (Hint: the sum of those two fractions is 1.) Show that $A_{\frac{x^2}{x^2+y^2}}$ and $A_{\frac{y^2}{x^2+y^2}}$ are unique factorization domains. (Hint for the first: show that each ring is isomorphic to $\mathbb{Q}[t]_{t^2+1}$, where $t = y/x$; this is a localization of the unique factorization domain $\mathbb{Q}[t]$.) Finally, show that A is not a unique factorization domain. Possible hint:

$$\left(\frac{xy}{x^2+y^2}\right)^2 = \left(\frac{x^2}{x^2+y^2}\right)\left(\frac{y^2}{x^2+y^2}\right).$$

Number theorists may prefer a different example: $\mathbb{Z}[\sqrt{-5}]$ is not factorial (as witnessed by $2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$), but is factorial because it is a Dedekind domain. (We will be able to make this precise. You can use the usual norm $|a + b\sqrt{-5}| = a^2 + 5b^2$ to show that 2, 3, $1+\sqrt{-5}$, and $1-\sqrt{-5}$ are all irreducible. See §13.4.14 for why $\mathbb{Z}[\sqrt{-5}]$ it is factorial. (Here again the Picard group is $\mathbb{Z}/2$, but this is a coincidence, or so I think.)

6.5 Associated points of schemes, and drawing fuzzy pictures

The price of metaphor is eternal vigilance. — Norbert Wiener

(This important topic won't be used in an essential way for some time, certainly until we talk about dimension in Chapter 12, so it may be best skipped on a first reading. Better: read this section considering only the case where A is an integral domain, or possibly a reduced Noetherian ring, thereby bypassing some of the annoyances. Then you will at least be comfortable with the notion of a rational function in these situations.)

Recall from just after Definition 6.2.1 (of *reduced*) our "fuzzy" picture of the nonreduced scheme $\operatorname{Spec} k[x,y]/(y^2, xy)$ (see Figure 6.1). When this picture was introduced, we mentioned that the "fuzz" at the origin indicated that the nonreduced behavior was concentrated there. This was verified in Exercise 6.2.A, and indeed the origin is the only point where the stalk of the structure sheaf is nonreduced.

You might imagine that in a bigger scheme, we might have different closed subsets with different amount of "nonreducedness". This intuition will be made precise in this section. We will define *associated points* of a scheme, which will be the most important points of a scheme, encapsulating much of the interesting behavior of the structure sheaf. For example, in Figure 6.1, the associated points are the generic point of the x-axis, and the origin (where "the nonreducedness lives").

The primes corresponding to the associated points of an affine scheme $\operatorname{Spec} A$ will be called *associated primes of* A. In fact this is backwards; we will define associated primes first, and then define associated points.

6.5.1. Properties of associated points. The properties of associated points that it will be most important to remember are as follows. Frankly, it is much more important to remember these facts than it is to remember their proofs. But we will, of course, prove these statements.

(0) They will exist for any locally Noetherian scheme, and for integral schemes. There are a finite number in any affine open set (and hence in any quasicompact open set). This will come for free.

(1) *The generic points of the irreducible components of a locally Noetherian scheme are associated points.* The other associated points are called **embedded points**. Thus in Figure 6.1, the origin is the only embedded point. (By the way, there are easier analogues of these properties where Noetherian hypotheses are replaced by integral conditions; see Exercise 6.5.C.)

(2) *If a locally Noetherian scheme X is reduced, then X has no embedded points.* (This jibes with the intuition of the picture of associated points described earlier.) It follows from **(1)** and **(2)** that if X is integral (i.e. irreducible and reduced, Exercise 6.2.E), then the generic point is the only associated point.

(3) Recall that one nice property of integral schemes X (such as irreducible affine varieties) not shared by all schemes is that for any non-empty open $U \subset X$, the natural map $\Gamma(U, \mathcal{O}_X) \to K(X)$ is an inclusion (Exercise 6.2.H). Thus all sections over any non-empty open set, and stalks, can be thought of as lying in a single field $K(X)$, which is the stalk at the generic point.

More generally, *if X is a locally Noetherian scheme, then for any $U \subset X$, the natural map*

$$\Gamma(U, \mathcal{O}_X) \to \prod_{\text{associated } \mathfrak{p} \text{ in } U} \mathcal{O}_{X,\mathfrak{p}} \tag{6.5.1.1}$$

is an injection.

6.5.2. *Definitions.* We define a **rational function** on a scheme with associated points to be an element of the image of $\Gamma(U, \mathcal{O}_U)$ in (6.5.1.1) for some U containing all the associated points. Equivalently, the set of rational functions is the colimit of $\mathcal{O}_X(U)$ over all open sets containing the associated points. Thus if X is integral, the rational functions are the elements of the stalk at the generic point, and even if there is more than one associated point, it is helpful to think of them in this stalk-like manner. For example, in Figure 6.1, we think of $\frac{x-2}{(x-1)(x-3)}$ as a rational function, but not $\frac{x-2}{x(x-1)}$. The rational functions form a ring, called the **total**

fraction ring of X, denoted $Q(X)$. If $X = \operatorname{Spec} A$ is affine, then this ring is called the **total fraction ring** of A, and is denoted $Q(A)$. (But we will never use this notation.) If X is integral, this is the function field $K(X)$, so this extends our earlier Definition 6.2.G of $K(\cdot)$. It can be more conveniently interpreted as follows, using the injectivity of (6.5.1.1). A rational function is a function defined on an open set containing all associated points, i.e. an ordered pair (U, f), where U is an open set containing all associated points, and $f \in \Gamma(U, \mathcal{O}_X)$. Two such data (U, f) and (U', f') define the same open rational function if and only if the restrictions of f and f' to $U \cap U'$ are the same. If X is reduced, this is the same as requiring that they are defined on an open set of each of the irreducible components. A rational function has a maximal **domain of definition**, because any two actual functions on an open set (i.e. sections of the structure sheaf over that open set) that agree as "rational functions" (i.e. on small enough open sets containing associated points) must be the same function, by the injectivity of (6.5.1.1). We say that a rational function f is **regular** at a point p if p is contained in this maximal domain of definition (or equivalently, if there is some open set containing p where f is defined). For example, in Figure 6.1, the rational function $\frac{x-2}{(x-1)(x-3)}$ has domain of definition consisting of everything but 1 and 3 (i.e. $[(x-1)]$ and $[(x-3)]$), and is regular away from those two points.

The previous facts are intimately related to the following one.

(4) *A function on an affine Noetherian scheme X is a zerodivisor if and only if it vanishes at an associated point of X.*

Motivated by the above four properties, when sketching (locally Noetherian) schemes, we will draw the irreducible components (the closed subsets corresponding to maximal associated points), and then draw "additional fuzz" precisely at the closed subsets corresponding to embedded points. All of our earlier sketches were of this form. (See Figure 6.3.) The fact that these sketches "make sense" implicitly uses the fact that the notion of associated points behaves well with respect to open sets (and localization, cf. Theorem 6.5.4(d)).

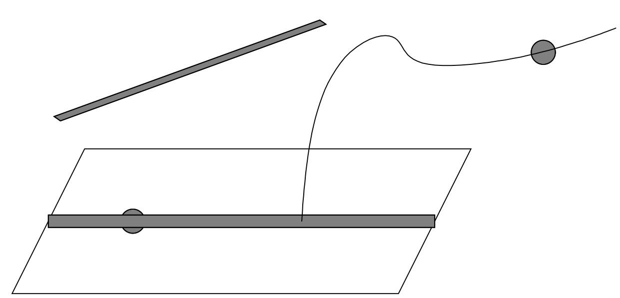


FIGURE 6.3. This scheme has 6 associated points, of which 3 are embedded points. A function is a zerodivisor if it vanishes at one of these six points. It is nilpotent if it vanishes at all six of these points. (In fact, it suffices to vanish at the non-embedded associated points.)

6.5.A. EXERCISE (FIRST PRACTICE WITH MAKING FUZZY PICTURES). Assume the properties **(1)–(4)** of associated points (§6.5.1). Suppose X is a closed subscheme of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$ with associated points at $[(y-x^2)]$, $[(x-1, y-1)]$, and $[(x-2, y-2)]$. (a) Sketch X, including fuzz. (b) Do you have enough information to know if X is reduced? (c) Do you have enough information to know if $x+y-2$ is a zerodivisor? How about $x+y-3$? How about $y-x^2$? (Exercise 6.5.K will verify that such an X actually exists!)

We now finally define associated points, and show that they have the desired properties **(1)** through **(4)**.

6.5.3. *Definition.* We work more generally with modules M over a ring A. A prime $\mathfrak{p} \subset A$ is **associated** to M if $\mathfrak{p}$ is the annihilator of an element m of M ($\mathfrak{p} = \{a \in A : am = 0\}$). The set of primes associated to M is denoted $\operatorname{Ass} M$ (or $\operatorname{Ass}_A M$). Awkwardly, if I is an ideal of A, the associated primes of the module A/I are said to be the associated primes of I. This is not my fault.

6.5.B. EASY EXERCISE. Show that $\mathfrak{p}$ is associated to M if and only if M has a submodule isomorphic to $A/\mathfrak{p}$.

6.5.4. Theorem (properties of associated primes). — *Suppose A is a Noetherian ring, and $M \neq 0$ is finitely generated.*

(a) *The set $\operatorname{Ass} M$ is finite and nonempty.*
(b) *The natural map $M \to \prod_{\mathfrak{p} \in \operatorname{Ass} M} M_{\mathfrak{p}}$ is an injection.*
(c) *The set of zerodivisors of M is $\cup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$.*
(d) *(association commutes with localization) If S is a multiplicative set, then*

$$\operatorname{Ass}_{S^{-1}A} S^{-1}M = \operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A$$

($= \{\mathfrak{p} \in \operatorname{Ass}_A M : \mathfrak{p} \cap S = \varnothing\}$).
(e) *The set $\operatorname{Ass} M$ contains the primes minimal among those containing* $\operatorname{ann} M := \{a \in A : aM = 0\}$.

6.5.5. *Definition.* We define the **associated points** of a locally Noetherian scheme X to be those points $p \in X$ such that, on any affine open set $\operatorname{Spec} A$ containing p, p corresponds to an associated prime of A. This notion is independent of choice of affine neighborhood $\operatorname{Spec} A$: if p has two affine open neighborhoods $\operatorname{Spec} A$ and $\operatorname{Spec} B$ (say corresponding to primes $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$ respectively), then p corresponds to an associated prime of A if and only if it corresponds to an associated prime of $A_{\mathfrak{p}} = \mathcal{O}_{X,p} = B_{\mathfrak{q}}$ if and only if it corresponds to an associated prime of B, by Theorem 6.5.4(d).

6.5.C. STRAIGHTFORWARD EXERCISE. State and prove the analogues of **(1)–(4)** for schemes that are integral rather than locally Noetherian. State and prove the analogues of Theorem 6.5.4 where the hypothesis that A is Noetherian is replaced by the hypothesis that A is an integral domain.

6.5.D. IMPORTANT EXERCISE. Show how Theorem 6.5.4 implies properties **(0)–(4)**. (By **(3)**, I mean the injectivity of (6.5.1.1). The trickiest is probably **(2)**.)

We now prove Theorem 6.5.4.

6.5.E. EXERCISE. Suppose $M \neq 0$ is an A-module. Show that if $I \subset A$ is maximal among all ideals that are annihilators of elements of M, then I is prime, and hence $I \in \operatorname{Ass} M$. Thus if A is Noetherian, then $\operatorname{Ass} M$ is nonempty (part of Theorem 6.5.4(a)).

6.5.F. EXERCISE. Suppose that M is a module over a Noetherian ring A. Show that $m = 0$ if and only if m is 0 in $M_{\mathfrak{p}}$ for each of the maximal associated primes of M. (Hint: use the previous exercise.)

This immediately implies Theorem 6.5.4(b). It also implies Theorem 6.5.4(c): Any nonzero element of $\cup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$ is clearly a zerodivisor. Conversely, if a annihilates a nonzero element of M, then r is contained in a maximal annihilator ideal.

6.5.G. EXERCISE. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of A-modules, show that

$$\operatorname{Ass} M' \subset \operatorname{Ass} M \subset \operatorname{Ass} M' \cup \operatorname{Ass} M''.$$

(Possible hint for the second containment: if $m \in M$ has annihilator $\mathfrak{p}$, then $Am = A/\mathfrak{p}$, cf. Exercise 6.5.B.)

6.5.H. EXERCISE. If M is a finitely generated module over Noetherian A, show that M has a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

where $M_{i+1}/M_i \cong R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. Show that the associated primes are among the $\mathfrak{p}_i$, and thus prove Theorem 6.5.4(a).

6.5.I. EXERCISE. Prove Theorem 6.5.4(d) as follows.
(a) Show that

$$\operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A \subset \operatorname{Ass}_{S^{-1}A} S^{-1}M.$$

(Hint: suppose $\mathfrak{p} \in \operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A$, with $\mathfrak{p} = \operatorname{ann} m$ for $m \in M$.)
(b) Suppose $\mathfrak{q} \in \operatorname{Ass}_{S^{-1}A} S^{-1}M$, which corresponds to $\mathfrak{p} \in A$ (i.e. $\mathfrak{q} = \mathfrak{p}(S^{-1}A)$). Then $\mathfrak{q} = \operatorname{ann}_{S^{-1}A} m$ ($m \in S^{-1}M$), which yields a nonzero element of

$$\operatorname{Hom}_{S^{-1}A}(S^{-1}A/\mathfrak{q}, S^{-1}M).$$

Argue that this group is isomorphic to $S^{-1} \operatorname{Hom}_A(A/\mathfrak{p}, M)$ (see Exercise 2.6.G), and hence $\operatorname{Hom}_A(A/\mathfrak{p}, M) \neq 0$.

6.5.J. EXERCISE. Prove Theorem 6.5.4(e) as follows. If $\mathfrak{p}$ is minimal over $\operatorname{ann} M$, localize at $\mathfrak{p}$, so that $\mathfrak{p}$ is the *only* prime containing $\operatorname{ann} M$. Use Theorem 6.5.4(d).

6.5.K. EXERCISE. Let $I = (y - x^2)^3 \cap (x-1, y-1)^{15} \cap (x-2, y-2)$. Show that $X = \operatorname{Spec} \mathbb{C}[x,y]/I$ satisfies the hypotheses of Exercise 6.5.A. (Side question: Is there a "smaller" example? Is there a "smallest"?)

6.5.6. *Aside: Primary ideals.* The notion of primary ideals is important, although we won't use it. (An ideal $I \subset A$ in a ring is **primary** if $I \neq A$ and if $xy \in I$ implies either $x \in I$ or $y^n \in I$ for some $n > 0$.) The associated primes of an ideal turn out to be precisely those primes appearing in its primary decomposition. See [**E**, §3.3], for example, for more on this topic.

Part III

Morphisms of schemes

CHAPTER 7

Morphisms of schemes

7.1 Introduction

We now describe the morphisms between schemes. We will define some easy-to-state properties of morphisms, but leave more subtle properties for later.

Recall that a scheme is (i) a set, (ii) with a topology, (iii) and a (structure) sheaf of rings, and that it is sometimes helpful to think of the definition as having three steps. In the same way, the notion of morphism of schemes $X \to Y$ may be defined (i) as a map of sets, (ii) that is continuous, and (iii) with some further information involving the sheaves of functions. In the case of affine schemes, we have already seen the map as sets (§4.2.7) and later saw that this map is continuous (Exercise 4.4.G).

Here are two motivations for how morphisms should behave. The first is algebraic, and the second is geometric.

7.1.1. *Algebraic motivation.* We will want morphisms of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ to be precisely the ring maps $A \to B$. We have already seen that ring maps $A \to B$ induce maps of topological spaces in the opposite direction (Exercise 4.4.G); the main new ingredient will be to see how to add the structure sheaf of functions into the mix. Then a morphism of schemes should be something that "on the level of affines, looks like this".

7.1.2. *Geometric motivation.* Motivated by the theory of differentiable manifolds (§4.1.1), which like schemes are ringed spaces, we want morphisms of schemes at the very least to be morphisms of ringed spaces; we now describe what these are. Notice that if $\pi : X \to Y$ is a map of differentiable manifolds, then a differentiable function on Y pulls back to a differentiable function on X. More precisely, given an open subset $U \subset Y$, there is a natural map $\Gamma(U, \mathcal{O}_Y) \to \Gamma(\pi^{-1}(U), \mathcal{O}_X)$. This behaves well with respect to restriction (restricting a function to a smaller open set and pulling back yields the same result as pulling back and then restricting), so in fact we have a map of sheaves on Y: $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. Similarly a morphism of schemes $X \to Y$ should induce a map $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. But in fact in the category of differentiable manifolds a continuous map $X \to Y$ is a map of differentiable manifolds precisely when differentiable functions on Y pull back to differentiable functions on X (i.e. the pullback map from differentiable functions on Y to *functions* on X in fact lies in the subset of *differentiable functions*, i.e. the continuous map $X \to Y$ induces a pullback of differential functions $\mathcal{O}_Y \to \mathcal{O}_X$), so this map of sheaves *characterizes* morphisms in the differentiable category. So we could use this as the *definition* of morphism in the differentiable category.

But how do we apply this to the category of schemes? In the category of differentiable manifolds, a continuous map $X \to Y$ *induces* a pullback of (the sheaf of) functions, and we can ask when this induces a pullback of *differentiable* functions. However, functions are odder on schemes, and we can't recover the pullback map just from the map of topological spaces. A reasonable patch is to hardwire this into the definition of morphism, i.e. to have a continuous map $f : X \to Y$, along with a pullback map $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$. This leads to the definition of the *category* of ringed spaces.

One might hope to define morphisms of schemes as morphisms of ringed spaces. This isn't quite right, as then Motivation 7.1.1 isn't satisfied: as desired, to each morphism $A \to B$ there is a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, but there can be additional morphisms of ringed spaces $\operatorname{Spec} B \to \operatorname{Spec} A$ not arising in this way (see Exercise 7.2.E). A revised definition as morphisms of ringed spaces that locally looks of this form will work, but this is awkward to work with, and we take a different approach. However, we will check that our eventual definition actually is equivalent to this (Exercise 7.3.C).

We begin by formally defining morphisms of ringed spaces.

7.2 Morphisms of ringed spaces

7.2.1. Definition. A **morphism** $\pi : X \to Y$ **of ringed spaces** is a continuous map of topological spaces (which we unfortunately also call π) along with a map $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$, which we think of as a "pullback map". By adjointness (§3.6.1), this is the same as a map $\pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. There is an obvious notion of composition of morphisms, so ringed spaces form a category. Hence we have notion of automorphisms and isomorphisms. You can easily verify that an isomorphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a homeomorphism $f : X \to Y$ along with an isomorphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$ (or equivalently $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$).

If $U \subset Y$ is an open subset, then there is a natural morphism of ringed spaces $(U, \mathcal{O}_Y|_U) \to (Y, \mathcal{O}_Y)$ (which implicitly appeared earlier in Exercise 3.6.G). More precisely, if $U \to Y$ is an isomorphism of U with an open subset V of Y, and we are given an isomorphism $(U, \mathcal{O}_U) \cong (V, \mathcal{O}_Y|_V)$ (via the isomorphism $U \cong V$), then the resulting map of ringed spaces is called an **open immersion** of ringed spaces.

7.2.A. EXERCISE (MORPHISMS OF RINGED SPACES GLUE). Suppose $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (Exercise 3.2.F essentially showed this for topological spaces.)

7.2.B. EASY IMPORTANT EXERCISE: $\mathcal{O}$-MODULES PUSH FORWARD. Given a morphism of ringed spaces $f : X \to Y$, show that sheaf pushforward induces a functor $\mathit{Mod}_{\mathcal{O}_X} \to \mathit{Mod}_{\mathcal{O}_Y}$.

7.2.C. EASY IMPORTANT EXERCISE. Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

7.2.D. KEY EXERCISE. Suppose $\pi^\sharp : B \to A$ is a morphism of rings. Define a morphism of ringed spaces $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ as follows. The map of topological spaces was given in Exercise 4.4.G. To describe a morphism of sheaves $\mathcal{O}_B \to \pi_* \mathcal{O}_A$ on $\operatorname{Spec} B$, it suffices to describe a morphism of sheaves on the distinguished base of $\operatorname{Spec} B$. On $D(g) \subset \operatorname{Spec} B$, we define

$$\mathcal{O}_B(D(g)) \to \mathcal{O}_A(\pi^{-1} D(g)) = \mathcal{O}_A(D(\pi^\sharp g))$$

by $B_g \to A_{\pi^\sharp g}$. Verify that this makes sense (e.g. is independent of g), and that this describes a morphism of sheaves on the distinguished base. (This is the third in a series of exercises. We saw that a morphism of rings induces a map of sets in §4.2.7, a map of topological spaces in Exercise 4.4.G, and now a map of ringed spaces here.)

This will soon be an example of morphism of schemes! In fact we could make that definition right now.

7.2.2. Tentative Definition we won't use (cf. Motivation 7.1.1 in §7.1). A morphism of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces that "locally looks like" the maps of affine schemes described in Key Exercise 7.2.D. Precisely, for each choice of affine open sets $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, the induced map of ringed spaces should be of the form shown in Key Exercise 7.2.D.

We would like this definition to be checkable on an affine cover, and we might hope to use the Affine Communication Lemma to develop the theory in this way. This works, but it will be more convenient to use a clever trick: in the next section, we will use the notion of locally ringed spaces, and then once we have used it, we will discard it like yesterday's garbage.

The map of ringed spaces of Key Exercise 7.2.D is really not complicated. Here is an example. Consider the ring map $\mathbb{C}[x] \to \mathbb{C}[y]$ given by $x \mapsto y^2$ (see Figure 4.6). We are mapping the affine line with coordinate y to the affine line with coordinate x. The map is (on closed points) $a \mapsto a^2$. For example, where does $[(y-3)]$ go to? Answer: $[(x-9)]$, i.e. $3 \mapsto 9$. What is the preimage of $[(x-4)]$? Answer: those prime ideals in $\mathbb{C}[y]$ containing $[(y^2-4)]$, i.e. $[(y-2)]$ and $[(y+2)]$, so the preimage of 4 is indeed ± 2. This is just about the map of sets, which is old news (§4.2.7), so let's now think about functions pulling back. What is the pullback of the function $3/(x-4)$ on $D([(x-4)]) = \mathbb{A}^1 - \{4\}$? Of course it is $3/(y^2-4)$ on $\mathbb{A}^1 - \{-2, 2\}$.

We conclude with an example showing that not every morphism of ringed spaces between affine schemes is of the form of Key Exercise 7.2.D. (In the language of the next section, this morphism of ringed spaces is not a morphism of locally ringed spaces.)

7.2.E. UNIMPORTANT EXERCISE. Recall (Exercise 4.4.J) that $\operatorname{Spec} k[x]_{(x)}$ has two points, corresponding to (0) and (x), where the second point is closed, and the first is not. Consider the map *of ringed spaces* $\operatorname{Spec} k(x) \to \operatorname{Spec} k[x]_{(x)}$ sending the point of $\operatorname{Spec} k(x)$ to $[(x)]$, and the pullback map $f^\sharp \mathcal{O}_{\operatorname{Spec} k[x]_{(x)}} \to \mathcal{O}_{\operatorname{Spec} k(x)}$ is induced by $k \hookrightarrow k(x)$. Show that this map of ringed spaces is not of the form described in Key Exercise 7.2.D.

7.3 From locally ringed spaces to morphisms of schemes

In order to prove that morphisms behave in a way we hope, we will use the notion of a *locally ringed space*. It will not be used later, although it is useful elsewhere in geometry. The notion of locally ringed spaces (and maps between them) is inspired by what we know about manifolds (see Exercise 4.1.B). If $\pi : X \to Y$ is a morphism of manifolds, with $\pi(p) = q$, and f is a function on Y vanishing at q, then the pulled back function $\pi^\sharp f$ on X should vanish on p. Put differently: germs of functions (at $q \in Y$) vanishing at q should pull back to germs of functions (at $p \in X$) vanishing at p.

7.3.1. *Definition.* Recall (Definition 5.3.4) that a *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that the stalks $\mathcal{O}_{X,x}$ are all local rings. A **morphism of locally ringed spaces** $f : X \to Y$ is a morphism of ringed spaces such that the induced map of stalks $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ (Exercise 7.2.C) sends the maximal ideal of the former into the maximal ideal of the latter (a **"homomorphism of local rings"**). This means something rather concrete and intuitive: "if $p \mapsto q$, and g is a function vanishing at q, then it will pull back to a function vanishing at p." Note that locally ringed spaces form a category.

To summarize: we use the notion of locally ringed space only to define morphisms of schemes, and to show that morphisms have reasonable properties. The main things you need to remember about locally ringed spaces are (i) that the functions have values at points, and (ii) that given a map of locally ringed spaces, the pullback of where a function vanishes is precisely where the pulled back function vanishes.

7.3.A. EXERCISE. Show that morphisms of locally ringed spaces glue (cf. Exercise 7.2.A). (Hint: your solution to Exercise 7.2.A may work without change.)

7.3.B. EASY IMPORTANT EXERCISE. (a) Show that $\operatorname{Spec} A$ is a locally ringed space. (Hint: Exercise 5.3.F.) (b) Show that the morphism of ringed spaces $f : \operatorname{Spec} A \to \operatorname{Spec} B$ defined by a ring morphism $f^\sharp : B \to A$ (Exercise 4.4.G) is a morphism of locally ringed spaces.

7.3.2. Key Proposition. — *If* $f : \operatorname{Spec} A \to \operatorname{Spec} B$ *is a morphism of locally ringed spaces then it is the morphism of locally ringed spaces induced by the map* $f^\sharp : B = \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$ *as in Exercise 7.3.B(b).*

Proof. Suppose $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of locally ringed spaces. We wish to show that it is determined by its map on global sections $f^\sharp : B \to A$. We first need to check that the map of points is determined by global sections. Now a point p of $\operatorname{Spec} A$ can be identified with the prime ideal of global functions vanishing on it. The image point $f(p)$ in $\operatorname{Spec} B$ can be interpreted as the unique point q of $\operatorname{Spec} B$, where the functions vanishing at q pull back to precisely those functions vanishing at p. (Here we use the fact that f is a map of locally ringed spaces.) This is precisely the way in which the map of sets $\operatorname{Spec} A \to \operatorname{Spec} B$ induced by a ring map $B \to A$ was defined (§4.2.7).

Note in particular that if $b \in B$, $f^{-1}(D(b)) = D(f^\sharp b)$, again using the hypothesis that f is a morphism of locally ringed spaces.

It remains to show that $f^\sharp : \mathcal{O}_{\operatorname{Spec} B} \to f_*\mathcal{O}_{\operatorname{Spec} A}$ is the morphism of sheaves given by Exercise 7.2.D (cf. Exercise 7.3.B(b)). It suffices to check this on the distinguished base (Exercise 3.7.C(a)). We now want to check that for any map of locally ringed spaces inducing the map of sheaves $\mathcal{O}_{\operatorname{Spec} B} \to f_*\mathcal{O}_{\operatorname{Spec} A}$, the map of sections on any distinguished open set $D(b) \subset \operatorname{Spec} B$ is determined by the map of global sections $B \to A$.

Consider the commutative diagram

$$\begin{tikzcd}
B \arrow[r, equal] & \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \arrow[r, "f^\sharp_{\operatorname{Spec} B}"] \arrow[d, "\operatorname{res}_{\operatorname{Spec} B, D(b)}"'] & \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \arrow[r, equal] \arrow[d, "\operatorname{res}_{\operatorname{Spec} A, D(f^\sharp b)}"] & A \\
B_b \arrow[r, equal] & \Gamma(D(b), \mathcal{O}_{\operatorname{Spec} B}) \arrow[r, "f^\sharp_{D(b)}"] & \Gamma(D(f^\sharp b), \mathcal{O}_{\operatorname{Spec} A}) \arrow[r, equal] & A_{f^\sharp b} = A \otimes_B B_b.
\end{tikzcd}$$

The vertical arrows (restrictions to distinguished open sets) are localizations by b, so the lower horizontal map $f^\sharp_{D(b)}$ is determined by the upper map (it is just localization by b). $\square$

We are ready for our definition.

7.3.3. Definition. If X and Y are schemes, then a morphism $\pi : X \to Y$ as locally ringed spaces is called a **morphism of schemes**. We have thus defined the **category of schemes**, which we denote *Sch*. (We then have notions of **isomorphism** — just the same as before, §5.3.4 — and **automorphism**. We note that the *target* Y of π is sometimes called the **base scheme** or the **base**, when we are interpreting π as a family of schemes parametrized by Y — this may become clearer once we have defined the fibers of morphisms in §10.3.2.)

The definition in terms of locally ringed spaces easily implies Tentative Definition 7.2.2:

7.3.C. IMPORTANT EXERCISE. Show that a morphism of schemes $f : X \to Y$ is a morphism of ringed spaces that looks locally like morphisms of affines. Precisely, if $\operatorname{Spec} A$ is an affine open subset of X and $\operatorname{Spec} B$ is an affine open subset of Y, and $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, then the induced morphism of ringed spaces is a morphism of affine schemes. (In case it helps, note: if $W \subset X$ and $Y \subset Z$ are both open immersions of ringed spaces, then any morphism of ringed spaces $X \to Y$ induces a morphism of ringed spaces $W \to Z$, by composition $W \to X \to Y \to Z$.) Show that it suffices to check on a set $(\operatorname{Spec} A_i, \operatorname{Spec} B_i)$ where the $\operatorname{Spec} A_i$ form an open cover of X.

In practice, we will use the affine cover interpretation, and forget completely about locally ringed spaces. In particular, put imprecisely, the category of affine schemes is the category of rings with the arrows reversed. More precisely:

7.3.D. EXERCISE. Show that the category of rings and the opposite category of affine schemes are equivalent (see §2.2.21 to read about equivalence of categories).

In particular, here is something surprising: there can be interesting maps from one point to another. For example, here are two different maps from the point

$\operatorname{Spec}\mathbb{C}$ to the point $\operatorname{Spec}\mathbb{C}$: the identity (corresponding to the identity $\mathbb{C} \to \mathbb{C}$), and complex conjugation. (There are even more such maps!)

It is clear (from the corresponding facts about locally ringed spaces) that morphisms glue (Exercise 7.3.A), and the composition of two morphisms is a morphism. Isomorphisms in this category are precisely what we defined them to be earlier (§5.3.4).

7.3.4. The category of complex schemes (or more generally the category of k-schemes where k is a field, or more generally the category of A-schemes where A is a ring, or more generally the category of S-schemes where S is a scheme). The category of S-schemes Sch_S (where S is a scheme) is defined as follows. The objects are morphisms of the form

$$\begin{tikzcd} X \arrow[d] \\ S \end{tikzcd}$$

(The morphism to S is called the **structure morphism**. A motivation for this terminology is the fact that if $S = \operatorname{Spec} A$, the structure morphism gives the functions on each open set the structure of an A-algebra, cf. §6.3.5.) The morphisms in the category of S-schemes are defined to be commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ S \arrow[r, "="] & S \end{tikzcd}$$

which is more conveniently written as a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S. & \end{tikzcd}$$

When there is no confusion (if the base scheme is clear), simply the top row of the diagram is given. In the case where $S = \operatorname{Spec} A$, where A is a ring, we get the notion of an A-scheme, which is the same as the same definition as in §6.3.5, but in a more satisfactory form. For example, complex geometers may consider the category of $\mathbb{C}$-schemes.

The next two examples are important. The first will show you that you can work with these notions in a straightforward, hands-on way. The second will show that you can work with these notions in a formal way.

7.3.E. IMPORTANT EXERCISE. (This exercise will give you some practice with understanding morphisms of schemes by cutting up into affine open sets.) Make sense of the following sentence: "$\mathbb{A}^{n+1} \setminus \{\vec{0}\} \to \mathbb{P}^n$ given by

$$(x_0, x_1, \dots, x_{n+1}) \mapsto [x_0; x_1; \dots; x_n]$$

is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you will have to divide these up into affines, and describe the maps, and check that they glue.

7.3.F. ESSENTIAL EXERCISE. Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. Hint: Show that this is true when X is affine. Use the fact that morphisms glue, Exercise 7.3.A. (This is even true in the category of locally ringed spaces. You are free to prove it in this generality, but it is easier in the category of schemes.)

In particular, there is a canonical morphism from a scheme to Spec of its ring of global sections. (Warning: Even if X is a finite-type k-scheme, the ring of global sections might be nasty! In particular, it might not be finitely generated, see 21.9.7.)

7.3.G. EASY EXERCISE. Show that this definition of A-scheme agrees with the earlier definition of §6.3.5.

7.3.5. ⋆ *Side fact for experts:* Γ *and* Spec *are adjoints.* We have a contravariant functor Spec from rings to locally ringed spaces, and a contravariant functor Γ from locally ringed spaces to rings. In fact $(\Gamma, \operatorname{Spec})$ is an adjoint pair! Thus we could have defined Spec by requiring it to be right-adjoint to Γ.

7.3.H. EASY EXERCISE. If $S_\bullet$ is a finitely generated graded A-algebra, describe a natural "structure morphism" $\operatorname{Proj} S_\bullet \to \operatorname{Spec} A$.

7.3.I. EASY EXERCISE. Show that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.) If k is a field, show that $\operatorname{Spec} k$ is the final object in the category of k-schemes.

7.3.J. ⋆⋆ EASY EXERCISE FOR THOSE WITH APPROPRIATE BACKGROUND: THE ANALYTIFICATION FUNCTOR. Recall the analytification construction of Exercise 6.3.7. For each morphism of reduced finite-type $\mathbb{C}$-schemes $f : X \to Y$ (over $\mathbb{C}$), define a morphism of complex analytic prevarieties $f_{an} : X_{an} \to Y_{an}$ (the **analytification of** f). Show that analytification gives a functor from the category of reduced finite type $\mathbb{C}$-schemes to the category of complex analytic prevarieties

7.3.6. Definition: The functor of points, and S-valued points of a scheme. If S is a scheme, then S**-valued points** of a scheme X, denoted $X(S)$, are defined to be maps $S \to X$. If A is a ring, then A**-valued points** of a scheme X, denoted $X(A)$, are defined to be the $(\operatorname{Spec} A)$-valued points of the scheme. We denote S-valued points of X by $X(S)$ and A-valued points of X by $X(A)$.

If you are working over a base scheme B — for example, complex algebraic geometers will consider only schemes and morphisms over $B = \operatorname{Spec} C$ — then in the above definition, there is an implicit structure map $S \to B$ (or $\operatorname{Spec} A \to B$ in the case of $X(A)$). For example, for a complex geometer, if X is a scheme over $\mathbb{C}$, the $\mathbb{C}(t)$-valued points of X correspond to commutative diagrams of the form

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C}(t) \arrow[rr] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & \operatorname{Spec} \mathbb{C} & \end{tikzcd}$$

where $g : X \to \operatorname{Spec}\mathbb{C}$ is the structure map for X, and f corresponds to the obvious inclusion of rings $\mathbb{C} \to \mathbb{C}(t)$. (Warning: a k-valued point of a k-scheme X is sometimes called a "rational point" of X, which is dangerous, as for most of the world, "rational" refers to $\mathbb{Q}$. We will use the safer phrase "k-valued point" of X.)

The terminology "S-valued point" is unfortunate, because we earlier defined the notion of points of a scheme, and S-valued points are not (necessarily) points! But this definition is well-established in the literature.

7.3.K. EXERCISE. (a) (easy) Show that a morphism of schemes $X \to Y$ induces a map of S-valued points $X(S) \to Y(S)$. (b) Note that morphisms of schemes $X \to Y$ are not determined by their "underlying" map of points. (What is an example?) Show that they *are* determined by their induced maps of S-valued points, as S varies over all schemes. (Hint: pick $S = X$. In the course of doing this exercise, you will largely prove Yoneda's Lemma in the guise of Exercise 10.1.D.)

Here is another reason S-valued points are a reasonable notion: *the A-valued points of an affine scheme* $\operatorname{Spec}\mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_r)$ *(where $f_i \in \mathbb{Z}[x_1, \dots, x_n]$ are relations) are precisely the solutions to the equations*

$$f_1(x_1, \dots, x_n) = \cdots = f_r(x_1, \dots, x_n) = 0$$

in the ring A. For example, the rational solutions to $x^2 + y^2 = 16$ are precisely the $\mathbb{Q}$-valued points of $\operatorname{Spec}\mathbb{Z}[x, y]/(x^2 + y^2 - 16)$. The integral solutions are precisely the $\mathbb{Z}$-valued points. So A-valued points of an affine scheme (finite type over $\mathbb{Z}$) can be interpreted simply. In the special case where A is local, A-valued points of a general scheme have a good interpretation too:

7.3.L. EXERCISE (MORPHISMS FROM Spec OF A LOCAL RING TO X). Suppose X is a scheme, and $(A, \mathfrak{m})$ is a local ring. Suppose we have a scheme morphism $\pi : \operatorname{Spec} A \to X$ sending $[\mathfrak{m}]$ to x. Show that any open set containing x contains the image of π. Show that there is a bijection between $\operatorname{Hom}(\operatorname{Spec} A, X)$ and $\{x \in X, \text{local homomorphisms } \mathcal{O}_{x,X} \to A\}$.

Another reason this notion is good is that the notation $X(S)$ suggests the interpretation of X as a (contravariant) functor h_X from schemes to sets — the **functor of** (scheme-valued) **points** of the scheme X (cf. Example 2.2.20).

A related reason this notion is good is that "products of S-valued points" behave as you might hope, see §10.1.3.

On the other hand, maps to projective space can be confusing. There are some maps we can write down easily, as shown by applying the next exercise in the case $X = \operatorname{Spec} A$, where A is a B-algebra.

7.3.M. EASY (BUT SURPRISINGLY ENLIGHTENING) EXERCISE. Suppose B is a ring. If X is a B-scheme, and $f_0, \dots, f_n$ are $n+1$ functions on X with no common zeros, then show that $[f_0; \dots; f_n]$ gives a morphism $X \to \mathbb{P}^n_B$.

You might hope that this gives all morphisms. But this isn't the case. Indeed, even the identity morphism $X = \mathbb{P}^1_k \to \mathbb{P}^1_k$ isn't of this form, as the source $\mathbb{P}^1$ has no nonconstant global functions with which to build this map. (There are similar examples with an affine source.) However, there is a correct generalization (characterizing *all* maps from schemes to projective schemes) in Theorem 17.4.1. This result roughly states that this works, so longer as the f_i are not quite functions, but sections of a line bundle. Our desire to understand maps to projective schemes in a clean way will be one important motivation for understanding line bundles.

We will see more ways to describe maps to projective space in the next section. A different description directly generalizing Exercise 7.3.M will be given in Exercise 16.3.F, which will turn out (in Theorem 17.4.1) to be a "universal" description.

Incidentally, before Grothendieck, it was considered a real problem to figure out the right way to interpret points of projective space with "coordinates" in a ring. These difficulties were due to a lack of functorial reasoning. And the clues to the right answer already existed (the same problems arise for maps from a smooth real manifold to $\mathbb{RP}^n$) — if you ask such a geometric question (for projective space is geometric), the answer is necessarily geometric, not purely algebraic!

7.3.7. *Visualizing schemes III: picturing maps of schemes when nilpotents are present.* You now know how to visualize the points of schemes (§4.3), and nilpotents (§5.2 and §6.5). The following imprecise exercise will give you some sense of how to visualize maps of schemes when nilpotents are involved. Suppose $a \in \mathbb{C}$. Consider the map of rings $\mathbb{C}[x] \to \mathbb{C}[\epsilon]/\epsilon^2$ given by $x \mapsto a\epsilon$. Recall that $\operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ may be pictured as a point with a tangent vector (§5.2). How would you picture this map if $a \neq 0$? How does your picture change if $a = 0$? (The tangent vector should be "crushed" in this case.)

Exercise 13.1.G will extend this considerably; you may enjoy reading its statement now.

7.4 Maps of graded rings and maps of projective schemes

As maps of rings correspond to maps of affine schemes in the opposite direction, maps of graded rings (over a base ring A) sometimes give maps of projective schemes in the opposite direction. This is an imperfect generalization: not every map of graded rings gives a map of projective schemes (§7.4.1); not every map of projective schemes comes from a map of graded rings (later); and different maps of graded rings can yield the same map of schemes (Exercise 7.4.C).

7.4.A. ESSENTIAL EXERCISE. Suppose that $f : S_\bullet \longrightarrow R_\bullet$ is a morphism of finitely-generated graded rings over A. By map of finitely generated graded rings, we mean a map of rings that preserves the grading as a map of "graded semigroups". In other words, there is a $d > 0$ such that S_n maps to R_{dn} for all n. Show that this induces a morphism of schemes $\operatorname{Proj} R_\bullet \setminus V(f(S_+)) \to \operatorname{Proj} S_\bullet$. (Hint: Suppose x is a homogeneous element of S_+. Define a map $D(f(x)) \to D(x)$. Show that they glue together (as x runs over all homogeneous elements of S_+). Show that this defines a map from all of $\operatorname{Proj} R_\bullet \setminus V(f(S_+))$.) In particular, if

$$V(f(S_+)) = \varnothing, \tag{7.4.0.1}$$

then we have a morphism $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$.

7.4.B. EXERCISE. Show that if $f : S_\bullet \to R_\bullet$ satisfies $\sqrt{(f(S_+))} = R_+$, then hypothesis (7.4.0.1) is satisfied. (Hint: Exercise 5.5.F.) This algebraic formulation of the more geometric hypothesis can sometimes be easier to verify.

Let's see Exercise 7.4.A in action. We will schematically interpret the map of complex projective manifolds $\mathbb{P}^1$ to $\mathbb{P}^2$ given by

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ & & \\ [s;t] & \longmapsto & [s^{20}; s^9 t^{11}; t^{20}] \end{array}$$

Notice first that this is well-defined: $[\lambda s; \lambda t]$ is sent to the same point of $\mathbb{P}^2$ as $[s;t]$. The reason for it to be well-defined is that the three polynomials s^{20}, $s^9 t^{11}$, and t^{20} are all homogeneous of degree 20.

Algebraically, this corresponds to a map of graded rings in the opposite direction

$$\mathbb{C}[x,y,z] \mapsto \mathbb{C}[s,t]$$

given by $x \mapsto s^{20}$, $y \mapsto s^9 t^{11}$, $z \mapsto t^{20}$. You should interpret this in light of your solution to Exercise 7.4.A, and compare this to the affine example of §4.2.8.

7.4.1. Notice that there is no map of complex manifolds $\mathbb{P}^2 \to \mathbb{P}^1$ given by $[x;y;z] \to [x;y]$, because the map is not defined when $x = y = 0$. This corresponds to the fact that the map of graded rings $\mathbb{C}[s,t] \to \mathbb{C}[x,y,z]$ given by $s \mapsto x$ and $t \mapsto y$, doesn't satisfy hypothesis (7.4.0.1).

7.4.C. UNIMPORTANT EXERCISE. This exercise shows that different maps of graded rings can give the same map of schemes. Let $R_\bullet = k[x,y,z]/(xz, yz, z^2)$ and $S_\bullet = k[a,b,c]/(ac, bc, c^2)$, where every variable has degree 1. Show that $\operatorname{Proj} R_\bullet \cong \operatorname{Proj} S_\bullet \cong \mathbb{P}^1_k$. Show that the maps $S_\bullet \to R_\bullet$ given by $(a,b,c) \mapsto (x,y,z)$ and $(a,b,c) \mapsto (x,y,0)$ give the same (iso)morphism $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$. (The real reason is that all of these constructions are insensitive to what happens in a finite number of degrees. This will be made precise in a number of ways later, most immediately in Exercise 7.4.F.)

7.4.2. Veronese subrings.

Here is a useful construction. Suppose $S_\bullet$ is a finitely generated graded ring. Define the n**th Veronese subring** of $S_\bullet$ by $S_{n\bullet} = \oplus_{j=0}^{\infty} S_{nj}$. (The "old degree" n is "new degree" 1.)

7.4.D. EXERCISE. Show that the map of graded rings $S_{n\bullet} \hookrightarrow S_\bullet$ induces an *isomorphism* $\operatorname{Proj} S_\bullet \to \operatorname{Proj} S_{n\bullet}$. (Hint: if $f \in S_+$ is homogeneous of degree divisible by n, identify $D(f)$ on $\operatorname{Proj} S_\bullet$ with $D(f)$ on $\operatorname{Proj} S_{n\bullet}$. Why do such distinguished open sets cover $\operatorname{Proj} S_\bullet$?)

7.4.E. EXERCISE. If $S_\bullet$ is generated in degree 1, show that $S_{n\bullet}$ is also generated in degree 1. (You may want to consider the case of the polynomial ring first.)

7.4.F. EXERCISE. Use the previous exercise to show that if $R_\bullet$ and $S_\bullet$ are the same finitely generated graded rings except in a finite number of nonzero degrees (make this precise!), then $\operatorname{Proj} R_\bullet \cong \operatorname{Proj} S_\bullet$.

7.4.G. EXERCISE. Suppose $S_\bullet$ is generated over S_0 by $f_1, \ldots, f_n$. Find a d such that $S_{d\bullet}$ is generated in "new" degree 1 (= "old" degree d). (This is surprisingly tricky, so here is a hint. Suppose there are generators $x_1, \ldots, x_n$ of degrees $d_1, \ldots,$

d_n respectively. Show that any monomial $x_1^{a_1} \cdots x_n^{a_n}$ of degree at least $nd_1 \dots d_n$ has $a_i \geq (\prod_j d_j)/d_i$ for some i. Show that the $nd_1 \dots d_n$th Veronese subring is generated by elements in "new" degree 1.) This, in combination with the previous exercise, shows that there is little harm in assuming that finitely generated graded rings are generated in degree 1, as after a regrading, this is indeed the case. This is handy, as it means that, using Exercise 7.4.D, we can assume that any finitely-generated graded ring is generated in degree 1. We will see that as a consequence we can place every Proj in some projective space via the construction of Exercise 9.2.G.

7.4.H. LESS IMPORTANT EXERCISE. Show that $S_{n\bullet}$ is a finitely generated graded ring. (Possible approach: use the previous exercise, or something similar, to show there is some N such that $S_{nN\bullet}$ is generated in degree 1, so the graded ring $S_{nN\bullet}$ is finitely generated. Then show that for each $0 < j < N$, $S_{nN\bullet+nj}$ is a finitely generated module over $S_{nN\bullet}$.)

7.5 Rational maps from integral schemes

Informally speaking, a "rational map" is a "a morphism defined almost everywhere", much as a rational function is a name for a function defined almost everywhere. We will later see that in good situations that where a rational map is defined, it is uniquely defined (the Reduced-to-separated Theorem 11.2.1), and has a largest *"domain of definition"* (§11.2.2). For this section, unless otherwise stated, *we assume* X *and* Y *to be integral*. (The reader interested in more general notions should consider first the case where the schemes in question are reduced but not necessarily irreducible.) A key example will be irreducible varieties, and the language of rational maps is most often used in this case. Many notions can make sense in more generality (without reducedness hypotheses for example), but I'm not sure if there is a widely accepted definition.

7.5.1. *Definition.* A **rational map** from X to Y, denoted $X \dashrightarrow Y$, is a morphism on a dense open set, with the equivalence relation $(f : U \to Y) \sim (g : V \to Y)$ if there is a dense open set $Z \subset U \cap V$ such that $f|_Z = g|_Z$. (In §11.2.2, we will improve this to: if $f|_{U\cap V} = g|_{U\cap V}$ in good circumstances — when Y is separated.) People often use the word "map" for "morphism", which is quite reasonable, except that a rational map need not be a map. So to avoid confusion, when one means "rational map", one should never just say "map".

7.5.2. *Rational maps more generally.* The right generality for the notion of rational map, to a situation where no serious pathologies arise, is where X has associated points — where it is integral or locally Noetherian (§6.5) — and where Y is arbitrary. In this case, the dense open set of X is required to contain the associated points. (We will see in §11.2 that rational maps to separated schemes behave particularly well, and they are usually considered in this situation.)

7.5.3. An obvious example of a rational map is a morphism. Another important example is the projection $\mathbb{P}^n_A \dashrightarrow \mathbb{P}^{n-1}_A$ given by $[x_0; \cdots ; x_n] \to [x_0; \cdots ; x_{n-1}]$.

(How precisely is this a rational map in the sense of Definition 7.5.1? What is its domain of definition?) A third example is the following.

7.5.A. EASY EXERCISE. Interpret rational functions on an integral scheme (§6.5.2) as rational maps to $\mathbb{A}^1_{\mathbb{Z}}$. (This is analogous to functions corresponding to morphisms to $\mathbb{A}^1_{\mathbb{Z}}$, which will be described in §7.6.1.)

7.5.B. EASY EXERCISE. Show that a rational map $X \to Y$ from an integral scheme X is the same as a $K(X)$-valued point (§7.3.6) of Y.

A rational map $f : X \dashrightarrow Y$ is **dominant** (or in some sources, *dominating*) if for some (and hence every) representative $U \to Y$, the image is dense in Y. Equivalently, f is dominant if it sends the generic point of X to the generic point of Y. A little thought will convince you that you can compose (in a well-defined way) a dominant map $f : X \dashrightarrow Y$ with a rational map $g : Y \dashrightarrow Z$. Integral schemes and dominant rational maps between them form a category which is geometrically interesting.

7.5.C. EASY EXERCISE. Show that dominant rational maps of integral schemes give morphisms of function fields in the opposite direction.

It is not true that morphisms of function fields always give dominant rational maps, or even rational maps. For example, $\operatorname{Spec} k[x]$ and $\operatorname{Spec} k(x)$ have the same function field ($k(x)$), but there is no rational map $\operatorname{Spec} k[x] \dashrightarrow \operatorname{Spec} k(x)$. Reason: that would correspond to a morphism from an open subset U of $\operatorname{Spec} k[x]$, say $\operatorname{Spec} k[x, 1/f(x)]$, to $\operatorname{Spec} k(x)$. But there is no map of rings $k(x) \to k[x, 1/f(x)]$ for any one $f(x)$. However, maps of function fields indeed give dominant rational maps of integral finite type k-schemes (and in particular, irreducible varieties, to be defined in §11.1.7), see Proposition 7.5.5 below.

(If you want more evidence that the topologically-defined notion of dominance is simultaneously algebraic, you can show that if $\phi : A \to B$ is a ring morphism, then the corresponding morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is dominant if and only if ϕ has nilpotent kernel.)

A rational map $f : X \to Y$ is said to be **birational** if it is dominant, and there is another rational map (a "rational inverse") that is also dominant, such that $f \circ g$ is (in the same equivalence class as) the identity on Y, and $g \circ f$ is (in the same equivalence class as) the identity on X. This is the notion of isomorphism in the category of integral schemes and dominant rational maps. We say X and Y are **birational** (to each other) if there exists a birational map $X \dashrightarrow Y$. Birational maps induce isomorphisms of function fields. The fact that maps of function fields correspond to rational maps in the opposite direction for integral finite type k-schemes, to be proved in Proposition 7.5.5, shows that a map between integral finite type k-schemes that induces an isomorphism of function fields is birational. An integral finite type k-scheme is said to be **rational** if it is birational to $\mathbb{A}^n_k$ for some k. A *morphism* is **birational** if it is birational as a rational map. We will later see (Proposition 11.2.3) that two integral affine k-varieties X and Y are birational if there are open sets $U \subset X$ and $V \subset Y$ that are isomorphic ($U \cong V$). In particular, an integral affine k-variety is rational if "it has a big open subset that is a big open subset of affine space $\mathbb{A}^n_k$".

7.5.4. Rational maps of irreducible varieties.

7.5.5. Proposition. — *Suppose X, Y are integral finite type k-schemes, and we are given $\phi^\sharp : K(Y) \hookrightarrow K(X)$. Then there exists a dominant rational map $\phi : X \dashrightarrow Y$ inducing $\phi^\sharp$.*

Proof. By replacing Y with an affine open set, we may assume Y is affine, say $Y = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then we have $\phi^\sharp x_1, \dots, \phi^\sharp x_n \in K(X)$. Let U be an open subset of the domains of definition of these rational functions. Then we get a morphism $U \to \mathbb{A}^n_k$. But this morphism factors through $Y \subset \mathbb{A}^n$, as $x_1, \dots, x_n$ satisfy the relations $f_1, \dots, f_r$.

We see that the morphism is dense as follows. If the set-theoretic image is not dense, it is contained in a proper closed subset. Let f be a function vanishing on the closed subset. Then the pullback of f to U is 0 (as U is reduced), implying that $\phi^\sharp(f) = 0$, and f doesn't vanish on all of Y, so f is not the 0-element of $K(Y)$. But this contradicts the fact that $\phi^\sharp$ is an inclusion. $\square$

7.5.D. EXERCISE. Let K be a finitely generated field extension of k. (Informal definition: a field extension K over k is **finitely generated** if there is a finite "generating set" $x_1, \dots, x_n$ in K such that every element of K can be written as a rational function in $x_1, \dots, x_n$ with coefficients in k.) Show that there exists an irreducible affine k-variety with function field K. (Hint: Consider the map $k[t_1, \dots, t_n] \to K$ given by $t_i \mapsto x_i$, and show that the kernel is a prime ideal $\mathfrak{p}$, and that $k[t_1, \dots, t_n]/\mathfrak{p}$ has fraction field K. Interpreted geometrically: consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \dots, t_n]$ given by the ring map $t_i \mapsto x_i$, and take the closure of the one-point image.)

7.5.E. EXERCISE. Describe an equivalence of categories between (a) finitely generated field extensions of k, and inclusions extending the identity on k, and (b) integral affine k-varieties, and dominant rational maps defined over k.

In particular, an integral affine k-variety X is rational if its function field $K(X)$ is a purely transcendent extension of k, i.e. $K(X) \cong k(x_1, \dots, x_n)$ for some n. (This needs to be said more precisely: the map $k \hookrightarrow K(X)$ induced by $X \to \operatorname{Spec} k$ should agree with the "obvious" map $k \hookrightarrow k(x_1, \dots, x_n)$ under this isomorphism.)

7.5.6. *Definition: degree of a rational map of varieties.* If $\pi : X \dashrightarrow Y$ is a dominant rational map of integral affine k-varieties of the same dimension, the degree of the field extension $K(X)/K(Y)$ is called the **degree** of the rational map. We will interpret this degree in terms of counting preimages of points of Y later.

7.5.7. More examples of rational maps.

A recurring theme in these examples is that domains of definition of rational maps to projective schemes extend over nonsingular codimension one points. We will make this precise in the Curve-to-projective Extension Theorem 17.5.1, when we discuss curves.

The first example is the classical formula for Pythagorean triples. Suppose you are looking for rational points on the circle C given by $x^2 + y^2 = 1$ (Figure 7.1). One rational point is $p = (1, 0)$. If q is another rational point, then pq is a line of rational (non-infinite) slope. This gives a rational map from the conic C to $\mathbb{A}^1$, given by

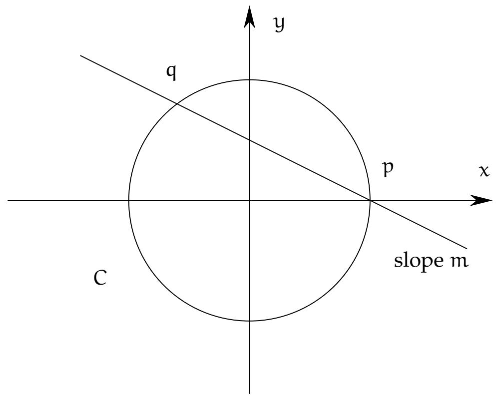


FIGURE 7.1. Finding primitive Pythagorean triples using geometry

$(x, y) \mapsto y/(x-1)$. (Something subtle just happened: we were talking about $\mathbb{Q}$-points on a circle, and ended up with a rational map of schemes.) Conversely, given a line of slope m through p, where m is rational, we can recover q by solving the equations $y = m(x-1)$, $x^2 + y^2 = 1$. We substitute the first equation into the second, to get a quadratic equation in x. We know that we will have a solution $x = 1$ (because the line meets the circle at $(x, y) = (1, 0)$), so we expect to be able to factor this out, and find the other factor. This indeed works:

$$\begin{array}{rrcl} & x^2 + (m(x-1))^2 & = & 1 \\ \Longrightarrow & (m^2+1)x^2 + (-2m^2)x + (m^2-1) & = & 0 \\ \Longrightarrow & (x-1)((m^2+1)x - (m^2-1)) & = & 0 \end{array}$$

The other solution is $x = (m^2-1)/(m^2+1)$, which gives $y = -2m/(m^2+1)$. Thus we get a birational map between the conic C and $\mathbb{A}^1$ with coordinate m, given by $f : (x, y) \mapsto y/(x-1)$ (which is defined for $x \neq 1$), and with inverse rational map given by $m \mapsto ((m^2-1)/(m^2+1), -2m/(m^2+1))$ (which is defined away from $m^2 + 1 = 0$).

We can extend this to a rational map $C \dashrightarrow \mathbb{P}^1$ via the inclusion $\mathbb{A}^1 \to \mathbb{P}^1$. Then f is given by $(x, y) \mapsto [y; x-1]$. We then have an interesting question: what is the domain of definition of f? It appears to be defined everywhere except for where $y = x - 1 = 0$, i.e. everywhere but p. But in fact it can be extended over p! Note that $(x, y) \mapsto [x+1; -y]$ (where $(x, y) \neq (-1, 0)$) agrees with f on their common domains of definition, as $[x+1; -y] = [y; x-1]$. Hence this rational map can be extended farther than we at first thought. This will be a special case of the Curve-to-projective Extension Theorem 17.5.1.

(For the curious: we are working with schemes over $\mathbb{Q}$. But this works for any scheme over a field of characteristic not 2. What goes wrong in characteristic 2?)

7.5.F. EXERCISE. Use the above to find a "formula" yielding all Pythagorean triples.

7.5.G. EXERCISE. Show that the conic $x^2 + y^2 = z^2$ in $\mathbb{P}^2_k$ is isomorphic to $\mathbb{P}^1_k$ for any field k of characteristic not 2. (In the special case where k is algebraically closed, you can also show this using diagonalizability of quadratic forms, §9.2.7.)

7.5.8. In fact, any conic in $\mathbb{P}^2_k$ with a k-valued point (i.e. a point with residue field k) of rank 3 (after base change to $\overline{k}$, so "rank" makes sense, see Exercise 6.4.J) is isomorphic to $\mathbb{P}^1_k$. (The hypothesis of having a k-valued point is certainly necessary: $x^2 + y^2 + z^2 = 0$ over $k = \mathbb{R}$ is a conic that is not isomorphic to $\mathbb{P}^1_k$.)

7.5.H. EXERCISE. Find all rational solutions to $y^2 = x^3 + x^2$, by finding a birational map to $\mathbb{A}^1_k$, mimicking what worked with the conic. (In Exercise 21.8.K, we will see that these points form a group, and that this is a degenerate elliptic curve.)

You will obtain a rational map to $\mathbb{P}^1$ that is not defined over the node $x = y = 0$, and *cannot* be extended over this codimension 1 set. This is an example of the limits of our future result, the Curve-to-projective Extension Theorem 17.5.1, showing how to extend rational maps to projective space over codimension 1 sets: the codimension 1 sets have to be nonsingular.

7.5.I. EXERCISE. Use a similar idea to find a birational map from the quadric $Q = \{x^2 + y^2 = w^2 + z^2\}$ to $\mathbb{P}^2$. Use this to find all rational points on Q. (This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational map is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.)

7.5.J. EXERCISE (THE CREMONA TRANSFORMATION, A USEFUL CLASSICAL CONSTRUCTION). Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $[x;y;z] \to [1/x;1/y;1/z]$. What is the the domain of definition? (It is bigger than the locus where $xyz \neq 0$!) You will observe that you can extend it over codimension 1 sets. This again foreshadows the Curve-to-projective Extension Theorem 17.5.1.

7.5.9. $\star$ **Complex curves that are not rational (fun but inessential).**

We now describe two examples of curves C such that do not admit a nonconstant rational map from $\mathbb{P}^1_{\mathbb{C}}$. Both proofs are by Fermat's method of *infinite descent*. By Exercise 7.5.B, these results can be interpreted as the fact that these curves have no "nontrivial" $\mathbb{C}(t)$-valued points, where by "nontrivial" we mean any such point is secretly a $\mathbb{C}$-valued point. You may notice that if you consider the same examples with $\mathbb{C}(t)$ replaced by $\mathbb{Q}$ (and where C is a curve over $\mathbb{Q}$ rather than $\mathbb{C}$), you get two fundamental questions in number theory and geometry. The analog of Exercise 7.5.L is the question of rational points on elliptic curves, and you may realize that the analog of Exercise 7.5.K is even more famous. Also, the arithmetic analogue of Exercise 7.5.L(a) is the "four squares theorem" (there are not four integer squares in arithmetic progression), first stated by Fermat. These examples will give you a glimpse of how and why facts over number fields are often parallelled by facts over function fields of curves. This parallelism is a recurring deep theme in the subject.

7.5.K. EXERCISE. If $n > 2$, show that $\mathbb{P}^1_{\mathbb{C}}$ has no dominant rational maps to the "Fermat curve" $x^n + y^n = z^n$ in $\mathbb{P}^2_{\mathbb{C}}$. Hint: reduce this to showing that there is no "nonconstant" solution $(f(t), g(t), h(t))$ to $f(t)^n + g(t)^n = h(t)^n$, where $f(t)$, $g(t)$, and $h(t)$ are rational functions in t. By clearing denominators, reduce this to showing that there is no nonconstant solution where $f(t)$, $g(t)$, and $h(t)$ are relatively prime polynomials. For this, assume there is a solution, and consider one of the lowest positive degree. Then use the fact that $\mathbb{C}[t]$ is a unique factorization domain, and $h(t)^n - g(t)^n = \prod_{i=1}^n (h(t) - \zeta^i g(t))$, where ζ is a primitive nth root of unity. Argue that each $h(t) - \zeta^i g(t)$ is an nth power. Then use

$$(h(t) - g(t))\, \alpha\, (h(t) - \zeta g(t)) = \beta \left(h(t) - \zeta^2 g(t)\right)$$

for suitably chosen α and β to get a solution of smaller degree. (How does this argument fail for $n = 2$?)

7.5.L. EXERCISE. Suppose a, b, and c are distinct complex numbers. By the following steps, show that if $x(t)$ and $y(t)$ are two rational functions of t (elements of $\mathbb{C}(t)$) such that

$$y(t)^2 = (x(t) - a)(x(t) - b)(x(t) - c), \tag{7.5.9.1}$$

then $x(t)$ and $y(t)$ are constants ($x(t), y(t) \in \mathbb{C}$). (Here $\mathbb{C}$ may be replaced by any field K; slight extra care is needed if K is not algebraically closed.)

(a) Suppose $P, Q \in \mathbb{C}[t]$ are relatively prime polynomials such that four distinct linear combinations of them are perfect squares. Show that P and Q are constant (i.e. $P, Q \in \mathbb{C}$). Hint: By renaming P and Q, show that you may assume that the perfect squares are P, $P - Q$, $P - \lambda Q$ (for some $\lambda \in \mathbb{C}$). Define u and v to be square roots of P and Q respectively. Show that $u - v$, $u + v$, $u - \sqrt{\lambda}v$, $u + \sqrt{\lambda}v$ are perfect squares, and that u and v are relatively prime. If p and q are not both constant, note that $0 < \max(\deg u, \deg v) < \max(\deg P, \deg Q)$. Assume from the start that P and Q were chosen as a counterexample with minimal $\max(\deg P, \deg Q)$ to obtain a contradiction. (Aside: It is possible to have *three* distinct linear combinations that are perfect squares. Such examples essentially correspond to primitive Pythagorean triples in $\mathbb{C}(t)$ — can you see how?)

(b) Suppose $(x, y) = (p/q, r/s)$ is a solution to (7.5.9.1), where $p, q, r, s \in \mathbb{C}[t]$, and p/q and r/s are in lowest terms. Clear denominators to show that $r^2 q^3 = s^2(p - aq)(p - bq)(p - cq)$. Show that $s^2 | q^3$ and $q^3 | sr$, and hence that $s^2 = \delta q^3$ for some $\delta \in \mathbb{C}$. From $r^2 = \delta(p - aq)(p - bq)(p - cq)$, show that $(p - aq)$, $(p - bq)$, $(p - cq)$ are perfect squares. Show that q is also a perfect square, and then apply part (a).

7.6 ⋆ Representable functors and group schemes

7.6.1. Maps to $\mathbb{A}^1$ correspond to functions. If X is a scheme, there is a bijection between the maps $X \to \mathbb{A}^1$ and global sections of the structure sheaf: by Exercise 7.3.F, maps $f : X \to \mathbb{A}^1_{\mathbb{Z}}$ correspond to maps to ring maps $f^\sharp : \mathbb{Z}[t] \to \Gamma(X, \mathcal{O}_X)$, and $f^\sharp(t)$ is a function on X; this is reversible.

This map is very natural in an informal sense: you can even picture this map to $\mathbb{A}^1$ as being *given* by the function. (By analogy, a function on a smooth manifold is a map to $\mathbb{R}$.) But it is natural in a more precise sense: this bijection is functorial in X. We will ponder this example at length, and see that it leads us to two important advanced notions: representable functors and group schemes.

7.6.A. EASY EXERCISE. Suppose X is a $\mathbb{C}$-scheme. Verify that there is a natural bijection between maps $X \to \mathbb{A}^1_{\mathbb{C}}$ in the category of $\mathbb{C}$-schemes and functions on X.

7.6.2. Representable functors. We restate the bijection of §7.6.1 as follows. We have two different contravariant functors from *Sch* to *Sets*: maps to $\mathbb{A}^1$ (i.e. $H : X \mapsto \operatorname{Mor}(X, \mathbb{A}^1_{\mathbb{Z}})$), and functions on X ($F : X \mapsto \Gamma(X, \mathcal{O}_X)$). The "naturality" of the bijection — the functoriality in X — is precisely the statement that the bijection gives a natural isomorphism of functors (§2.2.21): given any $f : X \to X'$, the diagram

$$\begin{tikzcd} H(X') \arrow[r] \arrow[d] & H(X) \arrow[d] \\ F(X') \arrow[r] & F(X) \end{tikzcd}$$

(where the vertical maps are the bijections given in §7.6.1) commutes.

More generally, if Y is an element of a category $\mathcal{C}$ (we care about the special case $\mathcal{C} = Sch$), recall the contravariant functor $h_Y : \mathcal{C} \to Sets$ defined by $h_Y(X) = \operatorname{Mor}(X, Y)$ (Example 2.2.20). We say a contravariant functor from $\mathcal{C}$ to *Sets* is **representable by** Y if it is naturally isomorphic to the representable functor h_Y. We say it is **representable** if it is representable by *some* Y.

7.6.B. IMPORTANT EASY EXERCISE (REPRESENTING OBJECTS ARE UNIQUE UP TO UNIQUE ISOMORPHISM). Show that if a contravariant functor F is representable by Y and by Z, then we have a unique isomorphism $Y \to Z$ induced by the natural isomorphism of functors $h_Y \to h_Z$. Hint: this is a version of the universal property arguments of §2.3: once again, we are recognizing an object (up to unique isomorphism) by maps to that object. This exercise is essentially Exercise 2.3.Y(b). (This extends readily to Yoneda's Lemma, Exercise 10.1.D. You are welcome to try that now.)

You have implicitly seen this notion before: you can interpret the existence of products and fibered products in a category as examples of representable functors. (You may wish to work out how a natural isomorphism $h_{Y \times Z} \cong h_Y \times h_Z$ induces the projection maps $Y \times Z \to Y$ and $Y \times Z \to Z$.)

7.6.C. EXERCISE. In this exercise, $\mathbb{Z}$ may be replaced by any ring.
(a) *(affine n-space represents the functor of n functions)* Show that the functor $X \mapsto \{(f_1, \dots, f_n) : f_i \in \Gamma(X, \mathcal{O}_X)\}$ is represented by $\mathbb{A}^n_{\mathbb{Z}}$. Show that $\mathbb{A}^1_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \cong \mathbb{A}^2_{\mathbb{Z}}$ (i.e. $\mathbb{A}^2$ satisfies the universal property of $\mathbb{A}^1 \times \mathbb{A}^1$).
(b) *(The functor of invertible functions is representable)* Show that the functor taking X to invertible functions on X is representable by $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$. **Definition:** This scheme is called $\mathbb{G}_m$.

7.6.D. LESS IMPORTANT EXERCISE. Fix a ring A. Consider the functor H from the category of locally ringed spaces to *Sets* given by $H(X) = \{A \to \Gamma(X, \mathcal{O}_X)\}$. Show that this functor is representable (by Spec A). This gives another (admittedly odd) motivation for the definition of Spec A, closely related to that of §7.3.5.

7.6.3. $\star\star$ **Group schemes (or more generally, group objects in a category).**

(The rest of §7.6 is intended to be double-starred, and should be read only for entertainment.) We return again to Example 7.6.1. Functions on X are better than a set: they form a group. (Indeed they even form a ring, but we will worry about this later.) Given a morphism $X \to Y$, pullback of functions $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ is a group homomorphism. So we should expect $\mathbb{A}^1$ to have some group-like structure. This leads us to the notion of *group scheme*, or more generally a *group object* in a category, which we now define.

Suppose $\mathcal{C}$ is a category with a final object and with products. (We know that *Sch* has a final object $Z = \operatorname{Spec} \mathbb{Z}$, by Exercise 7.3.I. We will later see that it has products, §10.1. But you can remove this hypothesis from the definition of group object, so we won't worry about this.)

A **group object** in $\mathcal{C}$ is an element X along with three morphisms:

- *Multiplication:* $m : X \times X \to X$
- *Inverse:* $i : X \to X$
- *Identity element:* $e : Z \to X$ (*not* the identity map)

These morphisms are required to satisfy several conditions.

(i) associativity axiom:

$$\begin{tikzcd} X \times X \times X \arrow[r, "(m,id)"] \arrow[d, "(id,m)"'] & X \times X \arrow[d, "m"] \\ X \times X \arrow[r, "m"] & X \end{tikzcd}$$

commutes. (Here id means the equality $X \to X$.)

(ii) identity axiom: $X \xrightarrow{\sim} Z \times X \xrightarrow{e \times id} X \times X \xrightarrow{m} X$ and $X \xrightarrow{\sim} X \times Z \xrightarrow{id \times e} X \times X \xrightarrow{m} X$ are both the identity map $X \to X$. (This corresponds to group axiom: multiplication by the identity element is the identity map.)

(iii) inverse axiom: $X \xrightarrow{i,id} X \times X \xrightarrow{m} X$ and $X \xrightarrow{id,i} X \times X \xrightarrow{m} X$ are both the map that is the composition $X \longrightarrow Z \xrightarrow{e} X$.

As motivation, you can check that a group object in the category of sets is in fact the same thing as a group. (This is symptomatic of how you take some notion and make it categorical. You write down its axioms in a categorical way, and if all goes well, if you specialize to the category of sets, you get your original notion. You can apply this to the notion of "rings" in an exercise below.)

A **group scheme** is defined to be a group object in the category of schemes. A **group scheme** over a ring A (or a scheme S) is defined to be a group object in the category of A-schemes (or S-schemes).

7.6.E. EXERCISE. Give $\mathbb{A}^1_{\mathbb{Z}}$ the structure of a group scheme, by describing the three structural morphisms, and showing that they satisfy the axioms. (Hint: the morphisms should not be surprising. For example, inverse is given by $t \mapsto -t$. Note that we know that the product $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$ exists, by Exercise 7.6.C(a).)

7.6.F. EXERCISE. Show that if G is a group object in a category $\mathcal{C}$, then for any $X \in \mathcal{C}$, $\mathrm{Mor}(X, G)$ has the structure of a group, and the group structure is preserved by pullback (i.e. $\mathrm{Mor}(\cdot, G)$ is a contravariant functor to *Groups*).

7.6.G. EXERCISE. Show that the group structure described by the previous exercise translates the group scheme structure on $\mathbb{A}^1_{\mathbb{Z}}$ to the group structure on $\Gamma(X, \mathcal{O}_X)$, via the bijection of §7.6.1.

7.6.H. EXERCISE. Define the notion of **ring scheme**, and **abelian group scheme**.

The language of S-valued points (Definition 7.3.6) has the following advantage: notice that the points of a group scheme need not themselves form a group (consider $\mathbb{A}^1_{\mathbb{Z}}$). But Exercise 7.6.F shows that the S-valued points of a group indeed form a group.

7.6.4. *Group schemes, more functorially.* There was something unsatisfactory about our discussion of the group scheme nature of the bijection in §7.6.1: we observed that the right side (functions on X) formed a group, then we developed the axioms of a group scheme, then we cleverly figured out the maps that made $\mathbb{A}^1_{\mathbb{Z}}$ into a group scheme, then we showed that this induced a group structure on the left side of the bijection ($\mathrm{Mor}(X, \mathbb{A}^1)$) that precisely corresponded to the group structure on the right side (functions on X).

The picture is more cleanly explained as follows.

7.6.I. EXERCISE. Suppose we have a contravariant functor F from *Sch* (or indeed any category) to *Groups*. Suppose further that F composed with the forgetful functor *Groups* $\to$ *Sets* is representable by an object Y. Show that the group operations on $F(X)$ (as X varies through *Sch*) uniquely determine $m : Y \times Y \to Y$, $i : Y \to Y$, $e : Z \to Y$ satisfying the axioms defining a group scheme, such that the group operation on $\mathrm{Mor}(X, Y)$ is the same as that on $F(X)$.

In particular, the definition of a group object in a category was forced upon us by the definition of group. More generally, you should expect that any class of objects that can be interpreted as sets with additional structure should fit into this picture.

You should apply this exercise to $\mathbb{A}^1_X$, and see how the explicit formulas you found in Exercise 7.6.E are forced on you.

7.6.J. EXERCISE. Work out the maps m, i, and e in the group schemes of Exercise 7.6.C.

7.6.K. EXERCISE. (a) Define **morphism of group schemes**.
(b) Define the group scheme GL_n, and describe the determinant map $\det : GL_n \to \mathbb{G}_m$.
(c) Make sense of the statement: $\cdot^n : \mathbb{G}_m \to \mathbb{G}_m$ given by $t \mapsto t^n$ is a morphism of group schemes.

7.6.L. EXERCISE (KERNELS OF MAPS OF GROUP SCHEMES). Suppose $F : G_1 \to G_2$ is a morphism of group schemes. Consider the contravariant functor *Sch* $\to$ *Groups* given by $X \mapsto \ker(\mathrm{Mor}(X, G_1) \to \mathrm{Mor}(X, G_2))$. If this is representable, by a group scheme G_0, say, show that $G_0 \to G_1$ is the kernel of F in the category of group schemes.

7.6.M. EXERCISE. Show that the kernel of $\cdot^n$ (Exercise 7.6.K) is representable. Show that over a field k of characteristic p, this group scheme is nonreduced. (Clarification: $\mathbb{G}_m$ over a field k means $\operatorname{Spec} k[t, t^{-1}]$, with the same group operations. Better: it represents the group of invertible functions in the category of k-schemes. We can similarly define $\mathbb{G}_m$ over an arbitrary scheme.)

7.6.N. EXERCISE. Show (as easily as possible) that $\mathbb{A}^1_k$ is a ring scheme.

7.6.5. *Aside: Hopf algebras.* Here is a notion that we won't use, but it is easy enough to define now. Suppose $G = \operatorname{Spec} A$ is an affine group scheme, i.e. a group scheme that is an affine scheme. The categorical definition of group scheme can be restated in terms of the ring A. Then these axioms define a **Hopf algebra**. For example, we have a "comultiplication map" $A \to A \otimes A$.

7.6.O. EXERCISE. As $\mathbb{A}^1_k$ is a group scheme, $k[t]$ has a Hopf algebra structure. Describe the comultiplication map $k[t] \to k[t] \otimes_k k[t]$.

7.7 ⋆⋆ The Grassmannian (initial construction)

The Grassmannian is a useful geometric construction that is "the geometric object underlying linear algebra". In (classical) geometry over a field $K = \mathbb{R}$ or $\mathbb{C}$, just as projective space parametrizes one-dimensional subspaces of a given n-dimensional vector space, the Grassmannian parametrizes k-dimensional subspaces of n-dimensional space. The Grassmannian $G(k, n)$ is a manifold of dimension $k(n-k)$ (over the field). The manifold structure is given as follows. Given a basis $(v_1, \dots, v_n)$ of n-space, "most" k-planes can be described as the span of the k vectors

$$\langle v_1 + \sum_{i=k+1}^{n} a_{1i} v_i, v_2 + \sum_{i=k+1}^{n} a_{2i} v_i, \dots, v_k + \sum_{i=k+1}^{n} a_{ki} v_i \rangle. \tag{7.7.0.1}$$

(Can you describe which k-planes are *not* of this form? Hint: row reduced echelon form. Aside: the stratification of $G(k, n)$ by normal form is the decomposition of the Grassmannian into *Schubert cells*. You may be able to show using the normal form that each Schubert cell is isomorphic to an affine space.) Any k-plane of this form can be described in such a way uniquely. We use this to identify those k-planes of this form with the manifold $K^{k(n-k)}$ (with coordinates a_{ji}). This is a large affine patch on the Grassmannian (called the "open Schubert cell" with respect to this basis). As the v_i vary, these patches cover the Grassmannian (why?), and the manifold structures agree (a harder fact).

We now *define* the Grassmannian in algebraic geometry, over a ring A. Suppose $v = (v_1, \dots, v_n)$ is a basis for A^n. More precisely: $v_i \in A^n$, and the map $A^n \to A^n$ given by $(a_1, \dots, a_n) \mapsto a_1 v_1 + \cdots + a_n v_n$ is an isomorphism.

7.7.A. EXERCISE. Show that any two bases are related by an invertible $n \times n$ matrix over A — a matrix with entries in A whose determinant is an invertible element of A.

For each such v, we consider the scheme $U_v \cong \mathbb{A}_A^{k(n-k)}$, with coordinates a_{ji} ($k+1 \leq i \leq n$, $1 \leq j \leq k$), which we imagine as corresponding to the k-plane spanned by the vectors (7.7.0.1).

7.7.B. EXERCISE. Given two bases v and w, explain how to glue U_v to U_w along appropriate open sets. You may find it convenient to work with coordinates a_{ji} where i runs from 1 to n, not just $k+1$ to n, but imposing $a_{ji} = \delta_{ji}$ (i.e. 1 when $i = j$ and 0 otherwise). This convention is analogous to coordinates $x_{i/j}$ on the patches of projective space (§5.4.9). Hint: the relevant open subset of U_v will be where a certain determinant doesn't vanish.

7.7.C. EXERCISE/DEFINITION. By checking triple intersections, verify that these patches (over all possible bases) glue together to a single scheme (Exercise 5.4.A). This is the **Grassmannian** $G(k, n)$ over the ring A.

Although this definition is pleasantly explicit (it is immediate that the Grassmannian is covered by $\mathbb{A}^{k(n-k)}$'s), and perhaps more "natural" than our original definition of projective space in §5.4.9 (we aren't making a choice of basis; we use *all* bases), there are several things unsatisfactory about this definition of the Grassmannian. In fact the Grassmannian is always projective; this isn't obvious with this definition. Furthermore, the Grassmannian comes with a natural closed immersion into $\mathbb{P}^{\binom{n}{k}-1}$ (the *Plücker embedding*). We will address these issues in §17.6, by giving a better description, as a moduli space.

CHAPTER 8

Useful classes of morphisms of schemes

We now define an unreasonable number of types of morphisms. Some (often finiteness properties) are useful because every "reasonable morphism" has such properties, and they will be used in proofs in obvious ways. Others correspond to geometric behavior, and you should have a picture of what each means.

One of Grothendieck's lessons is that things that we often think of as properties of *objects* are better understood as properties of *morphisms*. One way of turning properties of objects into properties of morphisms is as follows. If P is a property of schemes, we say that a *morphism* $f : X \to Y$ has P if for every affine open $U \subset X$, $f^{-1}(U)$ has P. We will see this for $P =$ quasicompact, quasiseparated, affine, and more. (As you might hope, in good circumstances, P will satisfy the hypotheses of the Affine Communication Lemma 6.3.2.) Informally, you can think of such a morphism as one where all the fibers have P. (You can quickly define the fiber of a morphism as a topological space, but once we define fiber product, we will define the *scheme-theoretic* fiber, and then this discussion will make sense.) But it means more than that: it means that the "P-ness" is really not just fiber-by-fiber, but behaves well as the fiber varies. (For comparison, a smooth morphism of manifolds means more than that the fibers are smooth.)

8.0.1. *What to expect of any "reasonable" type of morphism.* You will notice that essentially all classes of morphisms have three properties.

(i) They are "local on the target". In other words, to check if a morphism $f : X \to Y$ is in the class, then it suffices to check on an open cover on Y. In particular, as schemes are built out of rings (i.e. affine schemes), it should be possible to check on an affine cover, as described in the previous paragraph.
(ii) They are closed under composition: if $f : X \to Y$ and $g : Y \to Z$ are both in this class, then so is $g \circ f$.
(iii) They are closed under "base change" or "fibered product". We will discuss fibered product of schemes in Chapter 10.

When anyone tells you a new class of morphism, you should immediately ask yourself (or them) whether these three properties hold. And it is essentially true that a class of morphism is "reasonable" if and only if it satisfies these three properties.

8.1 Open immersions

An **open immersion of schemes** is defined to be an open immersion as ringed spaces (§7.2.1). In other words, a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of schemes is an open immersion if f factors as

$$(X, \mathcal{O}_X) \xrightarrow[\sim]{g} (U, \mathcal{O}_Y|_U) \xhookrightarrow{h} (Y, \mathcal{O}_Y)$$

where g is an isomorphism, and $U \hookrightarrow Y$ is an inclusion of an open set. It is immediate that isomorphisms are open immersions. We say that $(U, \mathcal{O}_Y|_U)$ is an **open subscheme** of $(Y, \mathcal{O}_Y)$, and often sloppily say that $(X, \mathcal{O}_X)$ is an open subscheme of $(Y, \mathcal{O}_Y)$. This is a bit confusing, and not too important: at the level of sets, open subschemes are subsets, while open immersions are bijections onto subsets.

8.1.A. IMPORTANT BUT EASY EXERCISE. Suppose $i : U \to Z$ is an open immersion, and $f : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists. (Hint: I'll even tell you what it is: $(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)})$.) In particular, if $U \hookrightarrow Z$ and $V \hookrightarrow Z$ are open immersions, $U \times_Z V \cong U \cap V$.

8.1.B. EASY EXERCISE. Suppose $f : X \to Y$ is an open immersion. Show that if Y is locally Noetherian, then X is too. Show that if Y is Noetherian, then X is too. However, show that if Y is quasicompact, X need not be. (Hint: let Y be affine but not Noetherian, see Exercise 4.6.D(b).)

"Open immersions" are scheme-theoretic analogues of open subsets. "Closed immersions" are scheme-theoretic analogues of closed subsets, but they have a surprisingly different flavor, as we will see in §9.1.

8.2 Algebraic interlude: Integral morphisms, the Lying Over Theorem, and Nakayama's lemma

To set up our discussion in the next section on integral morphisms, we develop some algebraic preliminaries. A clever trick we use can also be used to show Nakayama's lemma, so we discuss that as well.

Suppose $\phi : B \to A$ is a ring homomorphism. We say $a \in A$ is **integral** over B if a satisfies some monic polynomial

$$a^n + ?a^{n-1} + \cdots + ? = 0$$

where the coefficients lie in $\phi(B)$. A ring *homomorphism* $\phi : B \to A$ is **integral** if every element of A is integral over $\phi(B)$. An integral ring homomorphism ϕ is an **integral** *extension* if ϕ is an *inclusion* of rings. You should think of integral homomorphisms and integral extensions as ring-theoretic generalizations of the notion of algebraic extensions of fields.

8.2.A. EXERCISE. Show that if $\phi : B \to A$ is a ring homomorphism, $(b_1, \dots, b_n) = 1$ in B, and $B_{b_i} \to A_{\phi(b_i)}$ is integral for all i, then ϕ is integral.

8.2.B. EXERCISE. (a) Show that the property of a *homomorphism* $\phi : B \to A$ being integral is always preserved by localization and quotient of B, and quotient of A, but not localization of A. More precisely: suppose ϕ is integral. Show that the induced maps $T^{-1}B \to \phi(T)^{-1}A$, $B/J \to A/\phi(J)A$, and $B \to A/I$ are integral

(where T is a multiplicative subset of B, J is an ideal of B, and I is an ideal of A), but $B \to S^{-1}A$ need not be integral (where S is a multiplicative subset of A). (Hint for the latter: show that $k[t] \to k[t]$ is an integral homomorphism, but $k[t] \to k[t]_{(t)}$ is not.)
(b) Show that the property of f being an integral *extension* is preserved by localization of B, but not localization or quotient of A. (Hint for the latter: $k[t] \to k[t]$ is an integral extension, but $k[t] \to k[t]/(t)$ is not. In fact the property of f being an integral extension is not preserved by quotient of B either. Pieter Mostert's counterexample: let $B = k[x,y]/(y^2)$, $A = k[x,y,z]/(z^2, xz-y)$, and $J = (x)$.)

8.2.C. EXERCISE. Show that if $C \to B$ and $B \to A$ are both integral homomorphisms, then so is their composition.

The following lemma uses a useful but sneaky trick.

8.2.1. Lemma. — *Suppose $\phi : B \to A$ is a ring homomorphism. Then $a \in A$ is integral over B if and only if it is contained in a subalgebra of A that is a finitely generated B-module.*

Proof. If a satisfies a monic polynomial equation of degree n, then the B-submodule of A generated by $1, a, \dots, a^{n-1}$ is closed under multiplication, and hence a subalgebra of A.

Assume conversely that a is contained in a subalgebra A' of A that is a finitely generated B-module. Choose a finite generating set $m_1, \dots, m_n$ of A' (as a B-module). Then $am_i = \sum b_{ij} m_j$, for some $b_{ij} \in B$. Thus

$$(aI_{n\times n} - [b_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{8.2.1.1}$$

We can't invert the matrix $(aI_{n\times n} - [b_{ij}]_{ij})$, but we almost can. Recall that an $n \times n$ matrix M has an *adjugate matrix* $\operatorname{adj}(M)$ such that $\operatorname{adj}(M)M = \det(M)\mathrm{Id}_n$. (The ijth entry of $\operatorname{adj}(M)$ is the determinant of the matrix obtained from M by deleting the ith column and jth row, times $(-1)^{i+j}$. You have likely seen this in the form of a formula for M^{-1} when there *is* an inverse; see for example [**DF**, p. 440].) The coefficients of $\operatorname{adj}(M)$ are polynomials in the coefficients of M. Multiplying (8.2.1.1) by $\operatorname{adj}(aI_{n\times n} - [b_{ij}]_{ij})$, we get

$$\det(aI_{n\times n} - [b_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So $\det(aI - [b_{ij}])$ annihilates every element of A', i.e. $\det(aI - [b_{ij}]) = 0$. But expanding the determinant yields an integral equation for a with coefficients in B. $\square$

8.2.2. Corollary (finite implies integral). — *If A is a finite B-algebra (a finitely generated B-module), then ϕ is an integral homomorphism.*

The converse is false: integral does not imply finite, as $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$ is an integral homomorphism, but $\overline{\mathbb{Q}}$ is not a finite $\mathbb{Q}$-module. (A field extension is integral if it is algebraic.)

8.2.D. EXERCISE. Suppose $\phi : B \to A$ is a ring homomorphism. Show that the elements of A integral over B form a subalgebra of A.

8.2.3. Remark: transcendence theory. These ideas lead to the main facts about transcendence theory we will need for a discussion of dimension of varieties, see Exercise/Definition 12.2.A.

8.2.4. The Lying Over and Going-Up Theorems. The Lying Over Theorem is a useful property of integral extensions.

8.2.5. The Lying Over Theorem (Cohen-Seidenberg). — *Suppose* $\phi : B \to A$ *is an integral* extension. *Then for any prime ideal* $\mathfrak{q} \subset B$*, there is a prime ideal* $\mathfrak{p} \subset A$ *such that* $\mathfrak{p} \cap B = \mathfrak{q}$.

8.2.6. Geometric translation: $\operatorname{Spec} A \to \operatorname{Spec} B$ is surjective. (A map of schemes is **surjective** if the underlying map of sets is surjective.)

Although this is a theorem in algebra, the name can be interpreted geometrically: the theorem asserts that the corresponding morphism of schemes is surjective, and that "above" every prime $\mathfrak{q}$ "downstairs", there is a prime $\mathfrak{p}$ "upstairs", see Figure 8.1. (For this reason, it is often said that $\mathfrak{p}$ "lies over" $\mathfrak{q}$ if $\mathfrak{p} \cap B = \mathfrak{q}$.) The following exercise sets up the proof.

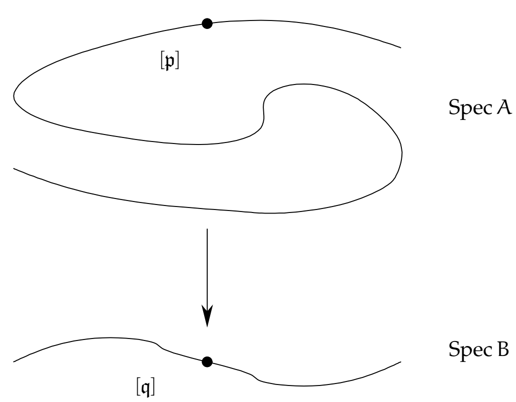


FIGURE 8.1. A picture of the Lying Over Theorem 8.2.5: if $\phi : A \to B$ is an integral extension, then $\operatorname{Spec} A \to \operatorname{Spec} B$ is surjective

8.2.E. $\star$ EXERCISE. Show that the special case where A is a field translates to: if $B \subset A$ is a subring with A integral over B, then B is a field. Prove this. (Hint: you must show that all nonzero elements in B have inverses in B. Here is the start: If $b \in B$, then $1/b \in A$, and this satisfies some integral equation over B.)

⋆ *Proof of the Lying Over Theorem 8.2.5.* We first make a reduction: by localizing at $\mathfrak{q}$ (preserving integrality by Exercise 8.2.B), we can assume that $(B, \mathfrak{q})$ is a local ring. Then let $\mathfrak{p}$ be any *maximal* ideal of A. Consider the following diagram.

$$\begin{tikzcd} A \arrow[r, two heads] & A/\mathfrak{p} & \text{field} \\ B \arrow[u, hook] \arrow[r, two heads] & B/(\mathfrak{p} \cap B) \arrow[u, hook] & \end{tikzcd}$$

(Do you see why the right vertical arrow is an integral extension?) By Exercise 8.2.E, $B/(\mathfrak{p} \cap B)$ is a field too, so $\mathfrak{p} \cap B$ is a maximal ideal, hence it is $\mathfrak{q}$. □

8.2.F. IMPORTANT EXERCISE (THE GOING-UP THEOREM). Suppose $\phi : B \to A$ is an integral *homomorphism* (not necessarily an integral extension). Show that if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain of prime ideals of B, and $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of A such that $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ (and $m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ so that this remains true. (Hint: reduce to the case $m = 1, n = 2$; reduce to the case where $\mathfrak{q}_1 = (0)$ and $\mathfrak{p}_1 = (0)$; use the Lying Over Theorem.)

8.2.7. Nakayama's lemma.

The trick in the proof of Lemma 8.2.1 can be used to quickly prove Nakayama's lemma. This name is used for several different but related results, which we discuss here. (A geometric interpretation will be given in Exercise 14.7.D.) We may as well prove it while the trick is fresh in our minds.

8.2.8. Nakayama's Lemma version 1. — *Suppose A is a ring, I is an ideal of A, and M is a finitely-generated A-module, such that $M = IM$. Then there exists an $a \in A$ with $a \equiv 1 \pmod I$ with $aM = 0$.*

Proof. Say M is generated by $m_1, \ldots, m_n$. Then as $M = IM$, we have $m_i = \sum_j a_{ij} m_j$ for some $a_{ij} \in I$. Thus

$$(\mathrm{Id}_n - Z) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0 \tag{8.2.8.1}$$

where Id_n is the $n \times n$ identity matrix in A, and $Z = (a_{ij})$. Multiplying both sides of (8.2.8.1) on the left by $\mathrm{adj}(\mathrm{Id}_n - Z)$, we obtain

$$\det(\mathrm{Id}_n - Z) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

But when you expand out $\det(\mathrm{Id}_n - Z)$, as Z has entries in I, you get something that is $1 \pmod I$. □

Here is why you care. Suppose I is contained in all maximal ideals of A. (The intersection of all the maximal ideals is called the *Jacobson radical*, but we won't use this phrase. For comparison, recall that the nilradical was the intersection of the *prime ideals* of A.) Then any $a \equiv 1 \pmod{I}$ is invertible. (We are not using Nakayama yet!) Reason: otherwise $(a) \neq A$, so the ideal (a) is contained in some maximal ideal $\mathfrak{m}$ — but $a \equiv 1 \pmod{\mathfrak{m}}$, contradiction. As a is invertible, we have the following.

8.2.9. Nakayama's Lemma version 2. — *Suppose A is a ring, I is an ideal of A contained in all maximal ideals, and M is a finitely-generated A-module. (The most interesting case is when A is a local ring, and I is the maximal ideal.) Suppose $M = IM$. Then $M = 0$.*

8.2.G. EXERCISE (NAKAYAMA'S LEMMA VERSION 3). Suppose A is a ring, and I is an ideal of A contained in all maximal ideals. Suppose M is a *finitely generated* A-module, and $N \subset M$ is a submodule. If $N/IN \to M/IM$ an isomorphism, then $M = N$. (This can be useful, although it won't be relevant for us.)

8.2.H. IMPORTANT EXERCISE (NAKAYAMA'S LEMMA VERSION 4: GENERATORS OF $M/\mathfrak{m}M$ LIFT TO GENERATORS OF M). Suppose $(A, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated A-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

8.2.I. UNIMPORTANT AND EASY EXERCISE (NAKAYAMA'S LEMMA VERSION 5). Prove Nakayama version 1 (Lemma 8.2.8) without the hypothesis that M is finitely generated, but with the hypothesis that $I^n = 0$ for some n. (This argument does *not* use the trick.) This result is quite useful, although we won't use it.

8.2.J. IMPORTANT EXERCISE GENERALIZING LEMMA 8.2.1. Suppose S is a subring of a ring A, and $r \in A$. Suppose there is a faithful $S[r]$-module M that is finitely generated as an S-module. Show that r is integral over S. (Hint: change a few words in the proof of Nakayama's Lemma version 1.)

8.2.K. EXERCISE. Suppose A is an integral domain, and $\tilde{A}$ is the integral closure of A in $K(A)$, i.e. those elements of $K(A)$ integral over A, which form a subalgebra by Exercise 8.2.D. Show that $\tilde{A}$ is integrally closed in $K(\tilde{A}) = K(A)$.

8.3 Finiteness conditions on morphisms

8.3.1. Quasicompact and quasiseparated morphisms.

A morphism $f : X \to Y$ of schemes is **quasicompact** if for every open affine subset U of Y, $f^{-1}(U)$ is quasicompact. (Equivalently, the preimage of any quasicompact open subset is quasicompact.)

We will like this notion because (i) we know how to take the maximum of a finite set of numbers, and (ii) most reasonable schemes will be quasicompact.

Along with quasicompactness comes the weird notion of quasiseparatedness. A morphism $f : X \to Y$ is **quasiseparated** if for every affine open subset U of Y, $f^{-1}(U)$ is a quasiseparated scheme (§6.1.1). This will be a useful hypothesis in theorems (in conjunction with quasicompactness). Various interesting kinds of morphisms (locally Noetherian source, affine, separated, see Exercises 8.3.B(b), 8.3.D, and 11.1.H resp.) are quasiseparated, and this will allow us to state theorems more succinctly.

8.3.A. EASY EXERCISE. Show that the composition of two quasicompact morphisms is quasicompact. (It is also true that the composition of two quasiseparated morphisms is quasiseparated. This is not easy to show directly, but will follow easily once we understand it in a more sophisticated way, see Exercise 11.1.13(b).)

8.3.B. EASY EXERCISE.
(a) Show that any morphism from a Noetherian scheme is quasicompact.
(b) Show that any morphism from a locally Noetherian scheme is quasiseparated. (Hint: Exercise 6.3.B.) Thus those readers working only with locally Noetherian schemes may take quasiseparatedness as a standing hypothesis.

8.3.C. EXERCISE. (Obvious hint for both parts: the Affine Communication Lemma 6.3.2.)
(a) *(quasicompactness is affine-local on the target)* Show that a morphism $f : X \to Y$ is quasicompact if there is a cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasicompact.
(b) *(quasiseparatedness is affine-local on the target)* Show that a morphism $f : X \to Y$ is quasiseparated if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasiseparated.

Following Grothendieck's philosophy of thinking that the important notions are properties of morphisms, not of objects, we can restate the definition of quasicompact (resp. quasiseparated) scheme as a scheme that is quasicompact (resp. quasiseparated) over the final object $\operatorname{Spec} \mathbb{Z}$ in the category of schemes (Exercise 7.3.I).

8.3.2. Affine morphisms.
A morphism $f : X \to Y$ is **affine** if for every affine open set U of Y, $f^{-1}(U)$ (interpreted as an open subscheme of X) is an affine scheme.

8.3.D. FAST EXERCISE. Show that affine morphisms are quasicompact and quasiseparated. (Hint for the second: Exercise 6.1.G.)

8.3.E. EXERCISE (A NONQUASISEPARATED SCHEME). Let $X = \operatorname{Spec} k[x_1, x_2, \dots]$, and let U be $X - [\mathfrak{m}]$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \dots)$. Take two copies of X, glued along U. Show that the result is not quasiseparated. Hint: This open immersion $U \subset X$ came up earlier in Exercise 4.6.D(b) as an example of a nonquasicompact open subset of an affine scheme.

8.3.3. Proposition (the property of "affineness" is affine-local on the target). — *A morphism $f : X \to Y$ is affine if there is a cover of Y by affine open sets U such that $f^{-1}(U)$ is affine.*

This proof is the hardest part of this section. For part of the proof (which will start in §8.3.5), it will be handy to have a lemma.

8.3.4. Qcqs Lemma. — *If X is a quasicompact quasiseparated scheme and $s \in \Gamma(X, \mathcal{O}_X)$, then the natural map $\Gamma(X, \mathcal{O}_X)_s \to \Gamma(X_s, \mathcal{O}_X)$ is an isomorphism.*

Here X_s means the locus on X where s doesn't vanish. We avoid the notation $D(s)$ to avoid any suggestion that X is affine.

To repeat the brief reassuring comment on the "quasicompact quasiseparated" hypothesis: this just means that X can be covered by a finite number of affine open subsets, any two of which have intersection also covered by a finite number of affine open subsets (Exercise 6.1.H). The hypothesis applies in lots of interesting situations, such as if X is affine (Exercise 6.1.G) or Noetherian (Exercise 6.3.B). And conversely, whenever you see quasicompact quasiseparated hypotheses (e.g. Exercises 14.3.E, 14.3.H), they are most likely there because of this lemma. To remind ourselves of this fact, we call it the Qcqs Lemma.

Proof. Cover X with finitely many affine open sets $U_i = \operatorname{Spec} A_i$. Let $U_{ij} = U_i \cap U_j$. Then

$$0 \to \Gamma(X, \mathscr{O}_X) \to \prod_i A_i \to \prod_{i,j} \Gamma(U_{ij}, \mathscr{O}_X)$$

is exact. By the quasiseparated hypotheses, we can cover each U_{ij} with a finite number of affines $U_{ijk} = \operatorname{Spec} A_{ijk}$, so we have that

$$0 \to \Gamma(X, \mathscr{O}_X) \to \prod_i A_i \to \prod_{i,j,k} A_{ijk}$$

is exact. Localizing at s (an exact functor, Exercise 2.6.F(a)) gives

$$0 \to \Gamma(X, \mathscr{O}_X)_s \to \left(\prod_i A_i\right)_s \to \left(\prod_{i,j,k} A_{ijk}\right)_s$$

As localization commutes with *finite* products (Exercise 2.3.L(b)),

$$0 \to \Gamma(X, \mathscr{O}_X)_s \to \prod_i (A_i)_{s_i} \to \prod_{i,j,k} (A_{ijk})_{s_{ijk}} \tag{8.3.4.1}$$

is exact, where the global function s induces functions $s_i \in A_i$ and $s_{ijk} \in A_{ijk}$.

But similarly, the scheme X_s can be covered by affine opens $\operatorname{Spec}(A_i)_{s_i}$, and $\operatorname{Spec}(A_i)_{s_i} \cap \operatorname{Spec}(A_j)_{s_j}$ are covered by a finite number of affine opens $\operatorname{Spec}(A_{ijk})_{s_{ijk}}$, so we have

$$0 \to \Gamma(X, \mathscr{O}_X)_s \to \prod_i (A_i)_{s_i} \to \prod_{i,j,k} (A_{ijk})_{s_{ijk}}. \tag{8.3.4.2}$$

Notice that the maps $\prod_i (A_i)_{s_i} \to \prod_{i,j,k} (A_{ijk})_{s_{ijk}}$ in (8.3.4.1) and (8.3.4.2) are the same, and we have described the kernel of the map in two ways, so $\Gamma(X, \mathscr{O}_X)_s \to \Gamma(X_s, \mathscr{O}_X)$ is indeed an isomorphism. (Notice how the quasicompact and quasiseparated hypotheses were used in an easy way: to obtain finite products, which would commute with localization.) $\square$

8.3.5. Proof of Proposition 8.3.3. As usual, we use the Affine Communication Lemma 6.3.2. We check our two criteria. First, suppose $f : X \to Y$ is affine over $\operatorname{Spec} B$, i.e. $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$. Then $f^{-1}(\operatorname{Spec} B_s) = \operatorname{Spec} A_{f^\sharp s}$.

Second, suppose we are given $f : X \to \operatorname{Spec} B$ and $(s_1, \dots, s_n) = B$ with X_{s_i} affine ($\operatorname{Spec} A_i$, say). We wish to show that X is affine too. Let $A = \Gamma(X, \mathscr{O}_X)$. Then $X \to \operatorname{Spec} B$ factors through the tautological map $g : X \to \operatorname{Spec} A$ (arising from the (iso)morphism $A \to \Gamma(X, \mathcal{O}_X)$, Exercise 7.3.F).

$$\begin{tikzcd} \cup_i X_{f^\sharp s_i} = X \arrow[rr, "g"] \arrow[dr, "f"'] & & \operatorname{Spec} A \arrow[dl, "h"] \\ & \cup_i D(s_i) = \operatorname{Spec} B & \end{tikzcd}$$

Then $h^{-1}D(s_i) = D(h^\sharp s_i) \cong \operatorname{Spec} A_{h^\sharp s_i}$ (the preimage of a distinguished open set is a distinguished open se1t), and $f^{-1}D(s_i) = \operatorname{Spec} A_i$. Now X is quasicompact and quasiseparated by the affine-locality of these notions (Exercise 8.3.C), so the hypotheses of the Qcqs Lemma 8.3.4 are satisfied. Hence we have an induced isomorphism of $A_{h^\sharp s_i} = \Gamma(X, \mathcal{O}_X)_{h^\sharp s_i} \cong \Gamma(X_{s_i}, \mathcal{O}_X) = A_i$. Thus g induces an isomorphism $\operatorname{Spec} A_i \to \operatorname{Spec} A_{h^\sharp s_i}$ (an isomorphism of rings induces an isomorphism of affine schemes, by strangely confusing exercise 5.3.A). Thus g is an isomorphism over each $\operatorname{Spec} A_{h^\sharp s_i}$, which cover $\operatorname{Spec} A$, and thus g is an isomorphism. Hence $X \cong \operatorname{Spec} A$, so is affine as desired. □

The affine-locality of affine morphisms (Proposition 8.3.3) has some nonobvious consequences, as shown in the next exercise.

8.3.F. USEFUL EXERCISE. Suppose Z is a closed subset of an affine scheme X locally cut out by one equation. (In other words, $\operatorname{Spec} A$ can be covered by smaller open sets, and on each such set Z is cut out by one equation.) Show that the complement Y of Z is affine. (This is clear if Y is globally cut out by one equation f; then if $X = \operatorname{Spec} A$ then $Y = \operatorname{Spec} A_f$. However, Y is not always of this form, see Exercise 6.4.M.)

8.3.6. Finite and integral morphisms.

Before defining finite and integral morphisms, we give an example to keep in mind. If L/K is a field extension, then $\operatorname{Spec} L \to \operatorname{Spec} K$ (i) is always affine; (ii) is integral if L/K is algebraic; and (iii) is finite if L/K is finite.

An affine morphism $f : X \to Y$ is **finite** if for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ is the spectrum of a B-algebra that is a finitely-generated B-*module*. Warning about terminology (finite vs. finitely-generated): Recall that if we have a ring homomorphism $A \to B$ such that B is a finitely-generated A-*module* then we say that B is a **finite** A-algebra. This is stronger than being a finitely-generated A-*algebra*.

By definition, finite morphisms are affine.

8.3.G. EXERCISE (THE PROPERTY OF FINITENESS IS AFFINE-LOCAL ON THE TARGET). Show that a morphism $f : X \to Y$ is finite if there is a cover of Y by affine open sets $\operatorname{Spec} A$ such that $f^{-1}(\operatorname{Spec} A)$ is the spectrum of a finite A-algebra.

The following four examples will give you some feeling for finite morphisms. In each example, you will notice two things. In each case, the maps are always finite-to-one (as maps of sets). We will verify this in general in Exercise 8.3.K. You will also notice that the morphisms are **closed** as maps of topological spaces, i.e. the images of closed sets are closed. We will show that finite morphisms are always closed in Exercise 8.3.M (and give a second proof in §9.2.5). Intuitively, you should think of finite as being closed plus finite fibers, although this isn't quite true. We will make this precise later.

Example 1: Branched covers. Consider the morphism $\operatorname{Spec} k[t] \to \operatorname{Spec} k[u]$ given by $u \mapsto p(t)$, where $p(t) \in k[t]$ is a degree n polynomial (see Figure 8.2). This is finite: $k[t]$ is generated as a $k[u]$-module by $1, t, t^2, \dots, t^{n-1}$.

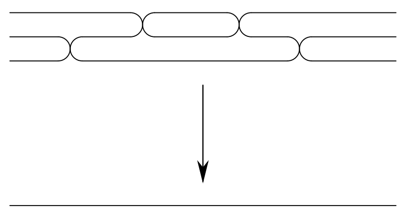


FIGURE 8.2. The "branched cover" $\mathbb{A}^1_k \to \mathbb{A}^1_k$ of the "u-line" by the "t-line" given by $u \mapsto p(t)$ is finite

Example 2: Closed immersions (to be defined soon, in §9.1). If I is an ideal of a ring A, consider the morphism $\operatorname{Spec} A/I \to \operatorname{Spec} A$ given by the obvious map $A \to A/I$ (see Figure 8.3 for an example, with $A = k[t]$, $I = (t)$). This is a finite morphism (A/I is generated as a A-module by the element $1 \in A/I$).

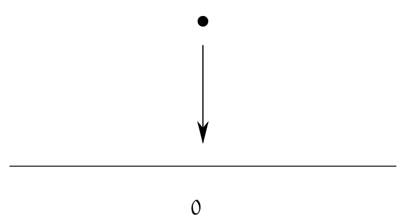


FIGURE 8.3. The "closed immersion" $\operatorname{Spec} k \to \operatorname{Spec} k[t]$ is finite

Example 3: Normalization (to be defined in §10.6). Consider the morphism $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x, y]/(y^2 - x^2 - x^3)$ corresponding to $k[x, y]/(y^2 - x^2 - x^3) \to k[t]$ given by $(x, y) \mapsto (t^2 - 1, t^3 - t)$ (check that this is a well-defined ring map!), see Figure 8.4. This is a finite morphism, as $k[t]$ is generated as a $(k[x, y]/(y^2 - x^2 - x^3))$-module by 1 and t. (The figure suggests that this is an isomorphism away from the "node" of the target. You can verify this, by checking that it induces an isomorphism between $D(t^2 - 1))$ in the source and $D(x)$ in the target. We will meet this example again!)

8.3.H. IMPORTANT EXERCISE (EXAMPLE 4, FINITE MORPHISMS TO $\operatorname{Spec} k$). Show that if $X \to \operatorname{Spec} k$ is a finite morphism, then X is a discrete finite union of points, each with residue field a finite extension of k, see Figure 8.5. (An example is $\operatorname{Spec} \mathbb{F}_8 \times \mathbb{F}_4[x, y]/(x^2, y^4) \times \mathbb{F}_4[t]/(t^9) \times \mathbb{F}_2 \to \operatorname{Spec} \mathbb{F}_2$.) Do *not* just quote some

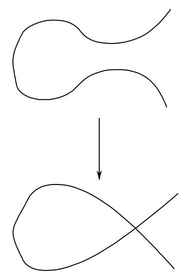


FIGURE 8.4. The "normalization" $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x, y]/(y^2 - x^2 - x^3)$ given by $(x, y) \mapsto (t^2 - 1, t^3 - t)$ is finite

fancy theorem! (Possible approach: Show that any integral domain A which is a finite k-algebra must be a field. Show that every prime $\mathfrak{p}$ of A is maximal. Show that the irreducible components of $\operatorname{Spec} A$ are closed points. Show $\operatorname{Spec} A$ is discrete and hence finite. Show that the residue fields $K(A/\mathfrak{p})$ of A are finite field extensions of k.)

FIGURE 8.5. A picture of a finite morphism to $\operatorname{Spec} k$. Bigger fields are depicted as bigger points.

8.3.I. EASY EXERCISE (CF. EXERCISE 8.2.C). Show that the composition of two finite morphisms is also finite.

8.3.J. EXERCISE: FINITE MORPHISMS TO $\operatorname{Spec} A$ ARE PROJECTIVE. If B is a finite A-algebra, define a graded ring $S_\bullet$ by $S_0 = A$, and $S_n = B$ for $n > 0$. (What is the multiplicative structure? Hint: you know how to multiply elements of B together, and how to multiply elements of A with elements of B.) Describe an isomorphism $\operatorname{Proj} S_\bullet \cong \operatorname{Spec} B$.

8.3.K. IMPORTANT EXERCISE. Show that finite morphisms have finite fibers. (This is a useful exercise, because you will have to figure out how to get at points in a fiber of a morphism: given $f : X \to Y$, and $y \in Y$, what are the points of $f^{-1}(y)$? Hint: if $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are both affine, and $y = [\mathfrak{p}]$, then we can throw out everything in A outside $\overline{y}$ by modding out by $\mathfrak{p}$; you can show that the preimage is $A/\mathfrak{p}$. Then we have reduced to the case where Y is the Spec of an integral domain, and $[\mathfrak{p}] = [0]$ is the generic point. We can throw out the rest of the points by localizing at 0. You can show that the preimage is $(A_{\mathfrak{p}})/\mathfrak{p}A_{\mathfrak{p}}$ (cf. (5.3.4.1)). that finiteness behaves well with respect to the operations you made done, you have reduced the problem to Exercise 8.3.H.)

8.3.7. *Example.* The open immersion $\mathbb{A}^2 - \{(0,0)\} \to \mathbb{A}^2$ has finite fibers, but is not affine (as $\mathbb{A}^2 - \{(0,0)\}$ isn't affine, §5.4.1) and hence not finite.

8.3.L. EASY EXERCISE. Show that the open immersion $\mathbb{A}^1 - \{0\} \to \mathbb{A}^1$ has finite fibers and is affine, but is not finite.

8.3.8. *Definition.* A morphism $\pi : X \to Y$ of schemes is **integral** if π is affine, and for every affine open subset $\operatorname{Spec} B \subset Y$, with $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$, the induced map $B \to A$ is an integral homomorphism of rings. This is an affine-local condition by Exercises 8.2.A and 8.2.B, and the Affine Communication Lemma 6.3.2. It is closed under composition by Exercise 8.2.C. Integral morphisms are mostly useful because finite morphisms are integral by Corollary 8.2.2. Note that the converse implication doesn't hold (witness $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$, as discussed after the statement of Corollary 8.2.2).

8.3.M. EXERCISE. Prove that integral morphisms are closed, i.e. that the image of closed subsets are closed. (Hence finite morphisms are closed. A second proof will be given in §9.2.5.) Hint: Reduced to the affine case. If $f^* : B \to A$ is a ring map, inducing finite $f : \operatorname{Spec} A \to \operatorname{Spec} B$, then suppose $I \subset A$ cuts out a closed set of $\operatorname{Spec} A$, and $J = (f^*)^{-1}(I)$, then note that $B/J \subset A/I$, and apply the Lying Over Theorem 8.2.5 here.

8.3.N. UNIMPORTANT EXERCISE. Suppose $f : B \to A$ is integral. Show that for any ring homomorphism $B \to C$, $C \to A \otimes_B C$ is integral. (Hint: We wish to show that any $\sum_{i=1}^n a_i \otimes c_i \in A \otimes_B C$ is integral over C. Use the fact that each of the finitely many a_i are integral over B, and then Exercise 8.2.D.) Once we know what "base change" is, this will imply that the property of integrality of a morphism is preserved by base change.

8.3.9. Morphisms (locally) of finite type.

A morphism $f : X \to Y$ is **locally of finite type** if for every affine open set $\operatorname{Spec} B$ of Y, and every affine open subset $\operatorname{Spec} A$ of $f^{-1}(\operatorname{Spec} B)$, the induced morphism $B \to A$ expresses A as a finitely generated B-algebra. By the affine-locality of finite-typeness of B-schemes (see Proposition 6.3.3), this is equivalent to: $f^{-1}(\operatorname{Spec} B)$ can be covered by affine open subsets $\operatorname{Spec} A_i$ so that each A_i is a finitely generated B-algebra.

A morphism is **of finite type** if it is locally of finite type and quasicompact. Translation: for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ can be covered with *a finite number of* open sets $\operatorname{Spec} A_i$ so that the induced morphism $B \to A_i$ expresses A_i as a finitely generated B-algebra.

8.3.10. *Side remark.* It is a common practice to name properties as follows: P= locally P plus quasicompact. Two exceptions are "ringed space" (§7.3) and "finite presentation" (§8.3.13).

8.3.O. EXERCISE (THE NOTIONS "LOCALLY OF FINITE TYPE" AND "FINITE TYPE" ARE AFFINE-LOCAL ON THE TARGET). Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by affine open sets $\operatorname{Spec} B_i$ such that $f^{-1}(\operatorname{Spec} B_i)$ is locally of finite type over B_i.

Example: the "structure morphism" $\mathbb{P}^n_A \to \operatorname{Spec} A$ is of finite type, as $\mathbb{P}^n_A$ is covered by $n+1$ open sets of the form $\operatorname{Spec} A[x_1, \dots, x_n]$.

Our earlier definition of schemes of "finite type over k" (or "finite type k-schemes") from §6.3.5 is now a special case of this more general notion: a scheme X is of finite type over k means that we are given a morphism $X \to \operatorname{Spec} k$ (the "structure morphism") that is of finite type.

Here are some properties enjoyed by morphisms of finite type.

8.3.P. EXERCISE (FINITE = INTEGRAL + FINITE TYPE). (a) (easier) Show that finite morphisms are of finite type.
(b) Show that a morphism is finite if and only if it is integral and of finite type.

8.3.Q. EXERCISES (NOT HARD, BUT IMPORTANT).

(a) Show that every open immersion is locally of finite type, and hence that every quasicompact open immersion is of finite type. Show that every open immersion into a locally Noetherian scheme is of finite type.
(b) Show that the composition of two morphisms locally of finite type is locally of finite type. (Hence as the composition of two quasicompact morphisms is quasicompact, the composition of two morphisms of finite type is of finite type.)
(c) Suppose $f : X \to Y$ is locally of finite type, and Y is locally Noetherian. Show that X is also locally Noetherian. If $X \to Y$ is a morphism of finite type, and Y is Noetherian, show that X is Noetherian.

8.3.11. *Definition.* A morphism f is **quasifinite** if it is of finite type, and for all $y \in Y$, $f^{-1}(y)$ is a finite set. The main point of this definition is the "finite fiber" part; the "finite type" hypothesis will ensure that this notion is "preserved by fibered product," Exercise 10.4.C.

Combining Exercise 8.3.K with Exercise 8.3.P(a), we see that finite morphisms are quasifinite. There are quasifinite morphisms which are not finite, such as $\mathbb{A}^2 - \{(0,0)\} \to \mathbb{A}^2$ (Example 8.3.7). A key example of a morphism with finite fibers that is not quasifinite is $\operatorname{Spec} \mathbb{C}(t) \to \operatorname{Spec} \mathbb{C}$. Another is $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$.

8.3.12. *How to picture quasifinite morphisms.* If $X \to Y$ is a finite morphism, then any quasi-compact open subset $U \subset X$ is quasi-finite over Y. In fact *every* reasonable quasifinite morphism arises in this way. (This simple-sounding statement is in fact a deep and important result — Zariski's Main Theorem.) Thus the right way to visualize quasifiniteness is as a finite map with some (closed locus of) points removed.

8.3.13. $\star\star$ **Morphisms (locally) of finite presentation.**

There is a variant often useful to non-Noetherian people. A ring A is a **finitely presented** B-algebra (or $B \to A$ is **finitely presented**) if

$$A = B[x_1, \dots, x_n]/(r_1(x_1, \dots, x_n), \dots, r_j(x_1, \dots, x_n))$$

("A has a finite number of generators and a finite number of relations over B"). If A is Noetherian, then finitely presented is the same as finite type, as the "finite number of relations" comes for free, so most of you will not care. A morphism $f : X \to Y$ is **locally of finite presentation** (or **locally finitely presented**) if for each affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B) = \cup_i \operatorname{Spec} A_i$ with $B \to A_i$ finitely presented. A morphism is of **finite presentation** (or **finitely presented**) if it is locally of finite presentation and quasiseparated and quasicompact. If X is locally Noetherian, then locally of finite presentation is the same as locally of finite type, and finite presentation is the same as finite type. So if you are a Noetherian person, you don't need to worry about this notion.

This definition is a violation of the general principle that erasing "locally" is the same as adding "quasicompact and" (Remark 8.3.10). But it is well motivated: finite presentation means "finite in all possible ways" (the ring corresponding to each affine open set has a finite number of generators, and a finite number of relations, and a finite number of such affine open sets cover, and their intersections are also covered by a finite number affine open sets) — it is all you would hope for in a scheme without it actually being Noetherian. Exercise 10.4.G makes this precise, and explains how this notion often arises in practice.

8.3.R. EXERCISE. Show that the notion of "locally of finite presentation" is affine-local on the target.

8.3.S. EXERCISE. Show that the notion of "locally of finite presentation" is affine-local on the source.

8.3.T. EXERCISE. Show that the composition of two finitely presented morphisms is finitely presented.

8.4 Images of morphisms: Chevalley's theorem and elimination theory

In this section, we will answer a question that you may have wondered about long before hearing the phrase "algebraic geometry". If you have a number of polynomial equations in a number of variables with indeterminate coefficients, you would reasonably ask what conditions there are on the coefficients for a (common) solution to exist. Given the algebraic nature of the problem, you might hope that the answer should be purely algebraic in nature — it shouldn't be "random", or involve bizarre functions like exponentials or cosines. This is indeed the case, and it can be profitably interpreted as a question about images of maps of varieties or schemes, in which guise it is answered by Chevalley's theorem.

In special cases, the image is nicer still. For example, we have seen that finite morphisms are closed (the image of closed subsets under finite morphisms are closed, Exercise 8.3.M). We will prove a classical result, the Fundamental Theorem of Elimination Theory 8.4.5, which essentially generalizes this (as explained in §9.2.5) to maps from projective space. We will use it repeatedly.

8.4.1. Chevalley's theorem.

If $f : X \to Y$ is a morphism of schemes, the notion of the image of f as *sets* is clear: we just take the points in Y that are the image of points in X. We know that the image can be open (open immersions), and we will soon see that it can be closed (closed immersions), and hence locally closed (locally closed immersions). But it can be weirder still: consider the morphism $\mathbb{A}^2_k \to \mathbb{A}^2_k$ given by $(x, y) \mapsto (x, xy)$. The image is the plane, with the x-axis removed, but the origin put back in. This isn't so horrible. We make a definition to capture this phenomenon. A **constructible subset** of a Noetherian topological space is a subset which belongs to the smallest family of subsets such that (i) every open set is in the family, (ii) a finite intersection of family members is in the family, and (iii) the complement of a family member is also in the family. For example the image of $(x, y) \mapsto (x, xy)$ is constructible. (A generalization of the notion of constructibility to more general topological spaces is mentioned in Exercise 8.4.G.)

8.4.A. EXERCISE: CONSTRUCTIBLE SUBSETS ARE FINITE UNIONS OF LOCALLY CLOSED SUBSETS. Recall that a subset of a topological space X is *locally closed* if it is the intersection of an open subset and a closed subset. (Equivalently, it is an open subset of a closed subset, or a closed subset of an open subset. We will later have trouble extending this to open and closed and locally closed subschemes, see Exercise 9.1.L.) Show that a subset of a Noetherian topological space X is constructible if and only if it is the finite disjoint union of locally closed subsets. As a consequence, if $X \to Y$ is a continuous map of Noetherian topological spaces, then the preimage of a constructible set is a constructible set.

8.4.B. EXERCISE (USED IN EXERCISE 25.5.E).
(a) Show that a constructible subset of a Noetherian scheme is closed if and only if it is "stable under specialization". More precisely, if Z is a constructible subset of a Noetherian scheme X, then Z is closed if and only if for every pair of points y_1 and y_2 with $y_1 \in \overline{y_2}$, if $y_2 \in Z$, then $y_1 \in Z$. Hint for the "if" implication: show that Z can be written as $\coprod_{i=1}^n U_i \cap Z_i$ where $U_i \subset X$ is open and $Z_i \subset X$ is closed. Show that Z can be written as $\coprod_{i=1}^n U_i \cap Z_i$ (with possibly different n, U_i, Z_i) where each Z_i is irreducible and meets U_i. Now use "stability under specialization" and the generic point of Z_i to show that $Z_i \subset Z$ for all i, so $Z = \cup Z_i$.)
(b) Show that a constructible subset of a Noetherian scheme is open if and only if it is "stable under generization". (Hint: this follows in one line from (a).)

The image of a morphism of schemes can be stranger than constructible. Indeed if S is *any* subset of a scheme Y, it can be the image of a morphism: let X be the disjoint union of spectra of the residue fields of all the points of S, and let $f : X \to Y$ be the natural map. This is quite pathological, but in any reasonable situation, the image is essentially no worse than arose in the previous example of $(x, y) \mapsto (x, xy)$. This is made precise by Chevalley's theorem.

8.4.2. Chevalley's theorem. — *If $\pi : X \to Y$ is a finite type morphism of Noetherian schemes, the image of any constructible set is constructible. In particular, the image of π is constructible.*

Proof. We begin with a series of reductions.

8.4.C. EXERCISE.
(a) Reduce to the case where Y is affine, say $Y = \operatorname{Spec} B$.
(b) Reduce further to the case where X is affine.
(c) Reduce further to the case where $X = \mathbb{A}^n_B = \operatorname{Spec} B[t_1, \dots, t_n]$.
(d) By induction on n, reduce further to the case where $X = \mathbb{A}^1_B = \operatorname{Spec} B[t]$.
(e) Reduce to showing that for any Noetherian ring B, and any *irreducible locally closed* subset $Z \subset \mathbb{A}^1_B$, the image of Z under the projection $\pi : \mathbb{A}^1_B \to \operatorname{Spec} B$ is constructible.
(f) Reduce to showing that for any Noetherian *integral domain* B (with $\pi : \mathbb{A}^1_B \to B$), and any irreducible locally closed subset $Z \subset \mathbb{A}^1_B$, where $\pi|_Z : Z \to \operatorname{Spec} B$ is dominant, $\pi(Z)$ is constructible. (Hint: replace $\operatorname{Spec} B$ from (e) by the closure of the image of the generic point of Z.)
(g) Use Noetherian induction to show that it suffices to show that for any Noetherian integral domain B (with $\pi : \mathbb{A}^1_B \to B$), and any locally closed subset $Z \subset \mathbb{A}^1_B$ dominant over $\operatorname{Spec} B$, $\pi(Z)$ contains a non-empty open subset of $\operatorname{Spec} B$.

8.4.D. EXERCISE. Reduce to showing the following statement. Given Noetherian integral domains B and C, where C is a B-algebra generated by a single element t (possibly with some relations), and the induced map $\pi : \operatorname{Spec} C \to \operatorname{Spec} B$ is dominant (with π thus inducing an inclusion $B \hookrightarrow C$), then for any nonzero $g \in C$, $\pi(D(g))$ contains a nonempty open subset of $\operatorname{Spec} B$. Hint: choose $\operatorname{Spec} C$ so that its set is the closure of Z in $\mathbb{A}^1_B$ in the statement given in Exercise 8.4.C(g), and choose $g \in C$ such that $D(g) \subset Z$. (Optional: draw a picture.)

We now prove this statement. If $C = B[t]/I$, then we deal first with the case $I = 0$, and second with $I \neq 0$.

8.4.E. EXERCISE. Prove the statement of Exercise 8.4.D in the case $C = B[t]$ as follows. Write $g = \sum_{i=0}^n b_i t^i$, where $b_i \in B$ and $b_n \neq 0$. Show that $D(b_n) \subset \pi(D(g))$.

We now deal with the remaining case $I \neq 0$.

8.4.F. EXERCISE. Suppose $\sum_{i=0}^n b_i t^i \in I$, where $b_n \neq 0$. Show that $\operatorname{Spec} C \to \operatorname{Spec} B$ is finite over $D(b_n)$. More precisely, show that C_{b_n} is generated as a B_{b_n}-module by (the images of) $1, t, \dots, t^{n-1}$.

Thus by replacing B by B_{b_n}, we may assume that $\operatorname{Spec} C \to \operatorname{Spec} B$ is finite. But finite morphisms are closed (Exercise 8.3.M), so the image of $V(g)$ is closed, and doesn't contain the generic point of $\operatorname{Spec} B$ (why?). Thus its complement is dense and open in $\operatorname{Spec} B$, so in particular $\pi(D(g))$ contains a dense open subset of $\operatorname{Spec} B$. $\square$

8.4.G. $\star\star$ EXERCISE (CHEVALLEY'S THEOREM FOR LOCALLY FINITELY PRESENTED MORPHISMS). If you are macho and are embarrassed by Noetherian rings, the following extension of Chevalley's theorem will give you a sense of one of the standard ways of removing Noetherian hypotheses.
(a) Suppose that A is a finitely presented B-algebra (B not necessarily Noetherian), so $A = B[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Show that the image of $\operatorname{Spec} A \to \operatorname{Spec} B$ is a finite union of locally closed subsets of $\operatorname{Spec} B$. Hint: describe $\operatorname{Spec} A \to \operatorname{Spec} B$ as the base change of

$$\operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n, a_1, \dots, a_N]/(g_1, \dots, g_n) \to \operatorname{Spec} \mathbb{Z}[a_1, \dots, a_N],$$

where the images of a_i in $\operatorname{Spec} B$ are the coefficients of the f_j (there is one a_i for each coefficient of each f_j), and $g_i \mapsto f_i$.
(b) Show that if $\pi : X \to Y$ is a quasicompact locally finitely presented morphism, and Y is quasicompact, then $\pi(X)$ is a finite union of locally closed subsets. (For hardened experts only: [**EGA**, 0_{III}.9.1] gives a definition of constructibility, and local constructability, in more generality. The general form of Chevalley's constructibility theorem [**EGA**, IV_1.1.8.4] is that the image of a locally constructible set, under a finitely presented map, is also locally constructible.)

8.4.3. $\star$ **Elimination of quantifiers.** A basic sort of question that arises in any number of contexts is when a system of equations has a solution. Suppose for example you have some polynomials in variables $x_1, \dots, x_n$ over an algebraically closed field $\overline{k}$, some of which you set to be zero, and some of which you set to be nonzero. (This question is of fundamental interest even before you know any scheme theory!) Then there is an algebraic condition on the coefficients which will tell you if this is the case. Define the Zariski topology on $\overline{k}^n$ in the obvious way: closed subsets are cut out by equations.

8.4.H. EXERCISE (ELIMINATION OF QUANTIFIERS, OVER AN ALGEBRAICALLY CLOSED FIELD). Fix an algebraically closed field $\overline{k}$. Suppose

$$f_1, \dots, f_p, g_1, \dots, g_q \in \overline{k}[A_1, \dots, A_m, X_1, \dots X_n]$$

are given. Show that there is a Zariski-constructible subset Y of $\overline{k}^m$ such that

$$f_1(a_1, \dots, a_m, X_1, \dots, X_n) = \cdots = f_p(a_1, \dots, a_m, X_1, \dots, X_n) = 0 \tag{8.4.3.1}$$

and

$$g_1(a_1, \dots, a_m, X_1, \dots, X_n) \neq 0 \quad \cdots \quad g_p(a_1, \dots, a_m, X_1, \dots, X_n) \neq 0 \tag{8.4.3.2}$$

has a solution $(X_1, \dots, X_n) = (x_1, \dots, x_n) \in \overline{k}^n$ if and only if $(a_1, \dots, a_m) \in Y$. Hints: if Z is a finite type scheme over $\overline{k}$, and the closed points are denoted Z^{cl} ("cl" is for either "closed" or "classical"), then under the inclusion of topological spaces $Z^{\mathrm{cl}} \hookrightarrow Z$, the Zariski topology on Z induces the Zariski topology on Z^{cl}. Note that we can identify $(\mathbb{A}^p_{\overline{k}})^{\mathrm{cl}}$ with $\overline{k}^p$ by the Nullstellensatz (Exercise 6.3.E). If X is the locally closed subset of $\mathbb{A}^{m+n}$ cut out by the equalities and inequalities

(8.4.3.1) and (8.4.3.2), we have the diagram

$$\begin{tikzcd} X^{\mathrm{cl}} \arrow[r, hook] \arrow[d, "\pi^{\mathrm{cl}}"'] & X \arrow[r, hook, "\text{loc. cl.}"] \arrow[d, "\pi"] & \mathbb{A}^{m+n} \arrow[dl] \\ \overline{k}^m \arrow[r, hook] & \mathbb{A}^m & \end{tikzcd}$$

where $Y = \operatorname{im} \pi^{\mathrm{cl}}$. By Chevalley's theorem 8.4.2, $\operatorname{im} \pi$ is constructible, and hence so is $(\operatorname{im} \pi) \cap \overline{k}^m$. It remains to show that $(\operatorname{im} \pi) \cap \overline{k}^m = Y$ ($= \operatorname{im} \pi^{\mathrm{cl}}$). You might use the Nullstellensatz.

This is called "elimination of quantifiers" because it gets rid of the quantifier "there exists a solution". The analogous statement for real numbers, where inequalities are also allowed, is a special case of Tarski's celebrated theorem of elimination of quantifiers for real closed fields.

8.4.4. The Fundamental Theorem of Elimination Theory.

8.4.5. Theorem (Fundamental Theorem of Elimination Theory). — *The morphism* $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ *is closed (sends closed sets to closed sets).*

A great deal of classical algebra and geometry is contained in this theorem as special cases. Here are some examples.

First, let $A = k[a, b, c, \dots, i]$, and consider the closed subscheme of $\mathbb{P}^2_A$ (taken with coordinates x, y, z) corresponding to $ax + by + cz = 0$, $dx + ey + fz = 0$, $gx + hy + iz = 0$. Then we are looking for the locus in $\operatorname{Spec} A$ where these equations have a non-trivial solution. This indeed corresponds to a Zariski-closed set — where

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 0.$$

Thus the idea of the determinant is embedded in elimination theory.

As a second example, let $A = k[a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n]$. Now consider the closed subscheme of $\mathbb{P}^1_A$ (taken with coordinates x and y) corresponding to $a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m = 0$ and $b_0 x^n + b_1 x^{m-1} y + \cdots + b_n y^n = 0$. Then there is a polynomial in the coefficients $a_0, \dots, b_n$ (an element of A) which vanishes if and only if these two polynomials have a common non-zero root — this polynomial is called the *resultant*.

More generally, this question boils down to the following question. Given a number of homogeneous equations in $n+1$ variables with indeterminate coefficients, Theorem 8.4.5 implies that one can write down equations in the coefficients that will precisely determine when the equations have a nontrivial solution.

Proof of the Fundamental Theorem of Elimination Theory 8.4.5. Suppose $Z \hookrightarrow \mathbb{P}^n_A$ is a closed *subset*. We wish to show that $\pi(Z)$ is closed. (See Figure 8.6.)

Suppose $y \notin \pi(Z)$ is a *closed* point of $\operatorname{Spec} A$. We will check that there is a distinguished open neighborhood $D(f)$ of y in $\operatorname{Spec} A$ such that $D(f)$ doesn't meet $\pi(Z)$. (If we could show this for *all* points of $\pi(Z)$, we would be done. But I prefer to concentrate on closed points first for simplicity.) Suppose y corresponds to the maximal ideal $\mathfrak{m}$ of A. We seek $f \in A - \mathfrak{m}$ such that $\pi^* f$ vanishes on Z.

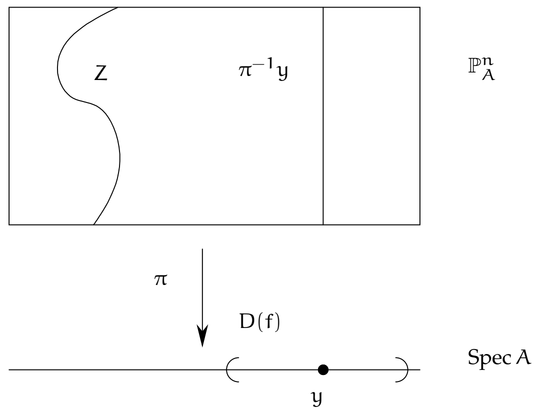


FIGURE 8.6.

Let $U_0, \dots, U_n$ be the usual affine open cover of $\mathbb{P}^n_A$. The closed subsets $\pi^{-1}y$ and Z do not intersect. On the affine open set U_i, we have two closed subsets $Z \cap U_i$ and $\pi^{-1}y \cap U_i$ that do not intersect, which means that the ideals corresponding to the two closed sets generate the unit ideal, so in the ring of functions $A[x_{0/i}, x_{1/i}, \dots, x_{n/i}]/(x_{i/i} - 1)$ on U_i, we can write

$$1 = a_i + \sum m_{ij} g_{ij}$$

where $m_{ij} \in \mathfrak{m}$, and a_i vanishes on Z. Note that $a_i, g_{ij} \in A[x_{0/i}, \dots, x_{n/i}]/(x_{i/i} - 1)$, so by multiplying by a sufficiently high power x_i^n of x_i, we have an equality

$$x_i^N = a_i' + \sum m_{ij} g_{ij}'$$

in $S_\bullet = A[x_0, \dots, x_n]$. We may take N large enough so that it works for all i. Thus for N' sufficiently large, we can write any monomial in $x_1, \dots, x_n$ of degree N' as something vanishing on Z plus a linear combination of elements of $\mathfrak{m}$ times other polynomials. Hence

$$S_{N'} = I(Z)_{N'} + \mathfrak{m} S_{N'}$$

where $I(Z)_\bullet$ is the graded ideal of functions vanishing on Z. By Nakayama's lemma (version 1, Lemma 8.2.8), taking $M = S_{N'}/I(Z)_{N'}$, we see that there exists $f \in A - \mathfrak{m}$ such that

$$fS_{N'} \subset I(Z)_{N'}.$$

Thus we have found our desired f.

We now tackle Theorem 8.4.5 in general, by simply extending the above argument so that y need not be a *closed* point. Suppose $y = [\mathfrak{p}]$ not in the image of Z. Applying the above argument in Spec $A_\mathfrak{p}$, we find $S_{N'} \otimes A_\mathfrak{p} = I(Z)_{N'} \otimes A_\mathfrak{p} + \mathfrak{m} S_{N'} \otimes A_\mathfrak{p}$, from which $g(S_{N'}/I(Z)_{N'}) \otimes A_\mathfrak{p} = 0$ for some $g \in A_\mathfrak{p} - \mathfrak{p}A_\mathfrak{p}$, from which $(S_{N'}/I(Z)_{N'}) \otimes A_\mathfrak{p} = 0$. As $S_{N'}$ is a finitely generated A-module, there is some $f \in A - \mathfrak{p}$ with $fS_N \subset I(Z)$ (if the module-generators of $S_{N'}$ are $h_1, \dots, h_a$, and $f_1, \dots, f_a$ are annihilate the generators $h_1, \dots, h_a$, respectively, then take

$f = \prod f_i$), so once again we have found $D(f)$ containing $\mathfrak{p}$, with (the pullback of) f vanishing on Z. $\square$

Notice that projectivity was crucial to the proof: we used graded rings in an essential way.

CHAPTER 9

Closed immersions and related notions

9.1 Closed immersions and closed subschemes

Just as open immersions (the scheme-theoretic version of open set) are locally modeled on open sets $U \subset Y$, the analogue of closed subsets also has a local model. This was foreshadowed by our understanding of closed subsets of $\operatorname{Spec} B$ as roughly corresponding to ideals. If $I \subset B$ is an ideal, then $\operatorname{Spec} B/I \hookrightarrow \operatorname{Spec} B$ is a morphism of schemes, and we have checked that on the level of topological spaces, this describes $\operatorname{Spec} B/I$ as a closed subset of $\operatorname{Spec} B$, with the subspace topology (Exercise 4.4.H). This morphism is our "local model" of a closed immersion.

9.1.1. *Definition.* A morphism $f : X \to Y$ is a **closed immersion** if it is an affine morphism, and for each open subset $\operatorname{Spec} B \subset Y$, with $f^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$, the map $B \to A$ is surjective (i.e. of the form $B \to B/I$, our desired local model). If X is a *subset* of Y (and f on the level of sets is the inclusion), we say that X is a **closed subscheme** of Y. The difference between a closed immersion and a closed subscheme is confusing and unimportant; the same issue for open immersions/subschemes was discussed in §8.1.

9.1.A. EASY EXERCISE. Show that closed immersions are finite, hence of finite type.

9.1.B. EASY EXERCISE. Show that the composition of two closed immersions is a closed immersion.

9.1.C. EXERCISE. Show that the property of being a closed immersion is affine-local on the target.

A closed immersion $f : X \hookrightarrow Y$ determines an *ideal sheaf* on Y, as the kernel $\mathcal{I}_{X/Y}$ of the map of $\mathcal{O}_Y$-modules

$$\mathcal{O}_Y \to f_*\mathcal{O}_X$$

(An **ideal sheaf** on Y is what it sounds like: it is a sheaf of ideals. It is a sub-$\mathcal{O}_Y$-module $\mathcal{I} \hookrightarrow \mathcal{O}_Y$. On each open subset, it gives an ideal $\mathcal{I}(U) \hookrightarrow \mathcal{O}_Y(U)$.) We thus have an exact sequence (of $\mathcal{O}_Y$-modules) $0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to 0$.

Thus for each affine open subset $\operatorname{Spec} B \hookrightarrow Y$, we have an ideal $I_B \subset B$, and we can recover X from this information: the I_B (as $\operatorname{Spec} B \hookrightarrow Y$ varies over the affine opens) defines an $\mathcal{O}$-module on the base, hence an $\mathcal{O}_Y$-module on Y, and the cokernel of $\mathcal{I} \hookrightarrow \mathcal{O}_Y$ is $\mathcal{O}_X$. It will be useful to understand when the information of the I_B (for all affine opens $\operatorname{Spec} B \hookrightarrow Y$) actually determine a closed subscheme. Our life is complicated by the fact that the answer is "not always", as shown by the following example.

9.1.D. UNIMPORTANT EXERCISE. Let $X = \operatorname{Spec} k[x]_{(x)}$, the germ of the affine line at the origin, which has two points, the closed point and the generic point η. Define $\mathcal{I}(X) = \{0\} \subset \mathcal{O}_X(X) = k[x]_{(x)}$, and $\mathcal{I}(\eta) = k(x) = \mathcal{O}_X(\eta)$. Show that this sheaf of ideals does not correspond to a closed subscheme. (Possible hint: do the next exercise first.)

The next exercise gives a necessary condition.

9.1.E. EXERCISE. Suppose $\mathcal{I}_{X/Y}$ is a sheaf of ideals corresponding to a closed immersion $X \hookrightarrow Y$. Suppose $\operatorname{Spec} B_f$ is a distinguished open of the affine open $\operatorname{Spec} B \hookrightarrow Y$. Show that the natural map $(I_B)_f \to I_{(B_f)}$ is an isomorphism.

It is an important and useful fact that this is sufficient:

9.1.F. ESSENTIAL (HARD) EXERCISE: A USEFUL CRITERION FOR WHEN IDEALS IN AFFINE OPEN SETS DEFINE A CLOSED SUBSCHEME. Suppose Y is a scheme, and for each affine open subset $\operatorname{Spec} B$ of Y, $I_B \subset B$ is an ideal. Suppose further that for each affine open subset $\operatorname{Spec} B \hookrightarrow Y$ and each $f \in B$, restriction of functions from $B \to B_f$ induces an isomorphism $I_{(B_f)} = (I_B)_f$. Show that this data arises from a (unique) closed subscheme $X \hookrightarrow Y$ by the above construction. In other words, the closed immersions $\operatorname{Spec} B/I \hookrightarrow \operatorname{Spec} B$ glue together in a well-defined manner to obtain a closed immersion $X \hookrightarrow Y$.

This is a hard exercise, so as a hint, here are three different ways of proceeding; some combination of them may work for you. *Approach 1.* For each affine open $\operatorname{Spec} B$, we have a closed subscheme $\operatorname{Spec} B/I \hookrightarrow \operatorname{Spec} B$. (i) For any two affine open subschemes $\operatorname{Spec} A$ and $\operatorname{Spec} B$, show that the two closed subschemes $\operatorname{Spec} A/I_A \hookrightarrow \operatorname{Spec} A$ and $\operatorname{Spec} B/I_B \hookrightarrow \operatorname{Spec} B$ restrict to the *same* closed subscheme of their intersection. (Hint: cover their intersection with open sets simultaneously distinguished in both affine open sets, Proposition 6.3.1.) Thus for example we can glue these two closed subschemes together to get a closed subscheme of $\operatorname{Spec} A \cup \operatorname{Spec} B$. (ii) Use Exercise 5.4.A on gluing schemes (or the ideas therein) to glue together the closed immersions in all affine open subschemes simultaneously. You will only need to worry about triple intersections. *Approach 2.* (i) Use the data of the ideals I_B to define a sheaf of ideals $\mathcal{I} \hookrightarrow \mathcal{O}$. (ii) For each affine open subscheme $\operatorname{Spec} B$, show that $\mathcal{I}(\operatorname{Spec} B)$ is indeed I_B, and $(\mathcal{O}/\mathcal{I})(\operatorname{Spec} B)$ is indeed B/I_B, so the data of $\mathcal{I}$ recovers the closed subscheme on each $\operatorname{Spec} B$ as desired. *Approach 3.* (i) Describe X first as a subset of Y. (ii) Check that X is closed. (iii) Define the sheaf of functions $\mathcal{O}_X$ on this subset, perhaps using compatible stalks. (iv) Check that this resulting ringed space is indeed locally the closed subscheme given by $\operatorname{Spec} B/I \hookrightarrow \operatorname{Spec} B$.)

We will see later (§14.5.5) that closed subschemes correspond to *quasicoherent* sheaves of ideals; the mathematical content of this statement will turn out to be precisely Exercise 9.1.F.

9.1.G. IMPORTANT EXERCISE. (a) In analogy with closed subsets, define the notion of a finite union of closed subschemes of X, and an arbitrary (not necessarily finite)

intersection of closed subschemes of X.
(b) Describe the scheme-theoretic intersection of $V(y-x^2)$ and $V(y)$ in $\mathbb{A}^2$. See Figure 5.3 for a picture. (For example, explain informally how this corresponds to two curves meeting at a single point with multiplicity 2 — notice how the 2 is visible in your answer. Alternatively, what is the nonreducedness telling you — both its "size" and its "direction"?) Describe their scheme-theoretic union.
(c) Show that the underlying set of a finite union of closed subschemes is the finite union of the underlying sets, and similarly for arbitrary intersections.
(d) Describe the scheme-theoretic intersection of $V(y^2-x^2)$ and $V(y)$ in $\mathbb{A}^2$. Draw a picture. (Did you expect the intersection to have multiplicity one or multiplicity two?) Hence show that if X, Y, and Z are closed subschemes of W, then $(X \cap Z) \cup (Y \cap Z) \neq (X \cup Y) \cap Z$ in general.

9.1.H. IMPORTANT EXERCISE/DEFINITION: THE VANISHING SCHEME. (a) Suppose Y is a scheme, and $s \in \Gamma(\mathcal{O}_Y, Y)$. Define the closed scheme **cut out by** s. We call this the **vanishing scheme** $V(s)$ of s, as it is the scheme theoretical version of our earlier (set-theoretical) version of $V(s)$. (Hint: on affine open $\operatorname{Spec} B$, we just take $\operatorname{Spec} B/(s_B)$, where s_B is the restriction of s to $\operatorname{Spec} B$. Use Exercise 9.1.F to show that this yields a well-defined closed subscheme.)
(b) If u is an invertible function, show that $V(s) = V(su)$.
(c) If S is a set of functions, define $V(S)$. In Exercise 9.1.G(b), you are computing $V(y-x^2, y)$.

9.1.2. *Locally principal closed subschemes, and effective Cartier divisors.* (This section is just an excuse to introduce some notation, and is not essential to the current discussion.) A closed subscheme is **locally principal** if on each open set in a small enough open cover it is cut out by a single equation. Thus each homogeneous polynomial in $n+1$ variables defines a locally principal closed subscheme. (Warning: this is not an affine-local condition, see Exercise 6.4.M! Also, the example of a projective hypersurface given soon in §9.2.1 shows that a locally principal closed subscheme need not be cut out by a (global) function.) A case that will be important repeatedly later is when the ideal sheaf is not just locally generated by a function, but is generated by a function that is not a zerodivisor. For reasons that may become clearer later, we call such a closed subscheme an **effective Cartier divisor**. (To see how useful this notion is, see how often it appears in the index.) Warning: We will use this terminology before we explain where it came from!

9.1.I. EXERCISE (FOR THOSE FUZZILY VISUALIZING SCHEMES, CF. §6.5). Suppose X is a locally Noetherian scheme, and $t \in \Gamma(X, \mathcal{O}_X)$ is a function on it. Show that t (or more precisely $V(t)$) is an effective Cartier divisor if and only if it doesn't vanish on any associated point of X.

9.1.J. UNIMPORTANT EXERCISE. Suppose $V(s) = V(s') \subset \operatorname{Spec} A$ is an effective Cartier divisor, with s and s' non-zerodivisors in A. Show that s is a unit times s'.

9.1.K. $\star$ HARD EXERCISE (NOT USED LATER). In the literature, the usual definition of a closed immersion is a morphism $f : X \to Y$ such that f induces a homeomorphism of the underlying topological space of X onto a closed subset of the topological space of Y, and the induced map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on Y is surjective. Show that this definition agrees with the one given above. (To show that our definition involving surjectivity on the level of affine open sets implies this definition, you can use the fact that surjectivity of a morphism of sheaves can be checked on a base, Exercise 3.7.E.)

We have now defined the analogue of open subsets and closed subsets in the land of schemes. Their definition is slightly less "symmetric" than in the classical topological setting: the "complement" of a closed subscheme is a unique open subscheme, but there are many "complementary" closed subschemes to a given open subscheme in general. (We will soon define one that is "best", that has a reduced structure, §9.3.8.)

9.1.3. Locally closed immersions and locally closed subschemes.
Now that we have defined analogues of open and closed subsets, it is natural to define the analogue of locally closed subsets. Recall that locally closed subsets are intersections of open subsets and closed subsets. Hence they are closed subsets of open subsets, or equivalently open subsets of closed subsets. The analog of these equivalences will be a little problematic in the land of schemes.

We say a morphism $h : X \to Y$ is a **locally closed immersion** if h can factored into $X \xrightarrow{f} Z \xrightarrow{g} Y$ where f is a closed immersion and g is an open immersion. If X is a subset of Y (and h on the level of sets is the inclusion), we say X is a **locally closed subscheme** of Y. (Warning: The term *immersion* is often used instead of *locally closed immersion*, but this is unwise terminology. The differential geometric notion of immersion is closer to what algebraic geometers call unramified, which we will define in §23.4.5. The algebro-geometric notion of locally closed immersion is closer to the differential geometric notion of *embedding*.)

For example, $\operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[x, y]$ where $(x, y) \mapsto (t, 0)$ is a locally closed immersion (see Figure 9.1).

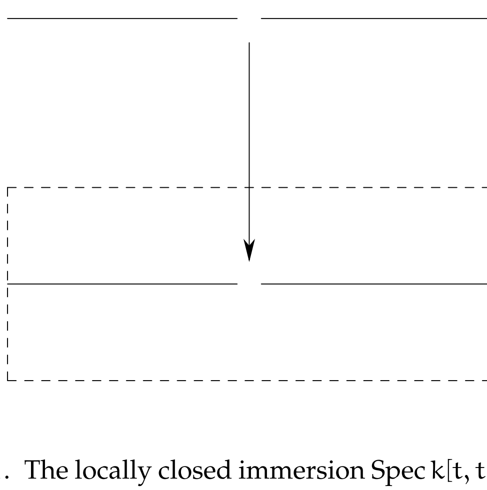


FIGURE 9.1. The locally closed immersion $\operatorname{Spec} k[t, t^{-1}] \to k[x, y]$ $(t \mapsto (t, 0) = (x, y)$, i.e. $(x, y) \to (t, 0))$

At this point, you could define the intersection of two locally closed immersions in a scheme X (which will also be a locally closed immersion in X). But it would be awkward, as you would have to show that your construction is independent of the factorizations of each locally closed immersion into a closed immersion and an open immersion. Instead, we wait until Exercise 10.2.C, when recognizing the intersection as a fibered product will make this easier.

Clearly an open subscheme U of a closed subscheme V of X can be interpreted as a closed subscheme of an open subscheme: as the topology on V is induced from the topology on X, the underlying set of U is the intersection of some open subset U' on X with V. We can take $V' = V \cap U$, and then $V' \to U'$ is a closed immersion, and $U' \to X$ is an open immersion.

It is not clear that a closed subscheme V' of an open subscheme U' can be expressed as an open subscheme U of a closed subscheme V. In the category of topological spaces, we would take V as the closure of V', so we are now motivated to define the analogous construction, which will give us an excuse to introduce several related ideas, in the next section. We will then resolve this issue in good cases (e.g. if X is Noetherian) in Exercise 9.3.C.

We formalize our discussion in an exercise.

9.1.L. EXERCISE. Suppose $V \to X$ is a morphism. Consider three conditions:

(i) V is an open subscheme of X intersect a closed subscheme of X (which you will have to define, see Exercise 8.1.A, or else see below).
(ii) V is an open subscheme of a closed subscheme of X (i.e. it factors into an open immersion followed by a closed immersion).
(iii) V is a closed subscheme of an open subscheme of X, i.e. V is a locally closed immersion.

Show that (i) and (ii) are equivalent, and both imply (iii). (Remark: (iii) does *not* always imply (i) and (ii), see [**Stacks**, tag 01QW].) Hint: It may be helpful to think of the problem as follows. You might hope to think of a locally closed immersion as a fibered diagram

$$\begin{tikzcd} V \arrow[r, hook, "\text{open imm.}"] \arrow[d, hook, "\text{closed imm.}"'] & Y \arrow[d, hook, "\text{closed imm.}"] \\ K \arrow[r, hook, "\text{open imm.}"'] & X. \end{tikzcd}$$

Interpret (i) as the existence of the diagram. Interpret (ii) as this diagram minus the lower left corner. Interpret (iii) as the diagram minus the upper right corner.

9.1.M. EXERCISE. Show that the composition of two locally closed immersions is a locally closed immersion. (Hint: you might use (ii) implies (iii) in the previous exercise.)

9.1.4. *Unimportant remark.* It may feel odd that in the definition of a locally closed immersions, we had to make a choice (as a composition of a closed followed by an open, rather than vice versa), but this type of issue comes up earlier: a subquotient of a group can be defined as the quotient of a subgroup, or a subgroup of a quotient. Which is the right definition? Or are they the same? (Hint: compositions of two subquotients should certainly be a subquotient, cf. Exercise 9.1.M.)

9.2 Closed immersions of projective schemes, and more projective geometry

9.2.1. Example: Closed immersions in projective space $\mathbb{P}^n_A$. Recall the definition of projective space $\mathbb{P}^n_A$ given in §5.4.10 (and the terminology defined there). Any *homogeneous* polynomial f in $x_0, \dots, x_n$ defines a closed subscheme. (Thus even if f doesn't make sense as a function, its vanishing scheme still makes sense.) On the open set U_i, the closed subscheme is $V(f(x_{0/i}, \dots, x_{n/i}))$, which we think of as $V(f(x_0, \dots, x_n)/x_i^{\deg f})$. On the overlap

$$U_i \cap U_j = \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1),$$

these functions on U_i and U_j don't exactly agree, but they agree up to a non-vanishing scalar, and hence cut out the same closed subscheme of $U_i \cap U_j$ (Exercise 9.1.H(b)):

$$f(x_{0/i}, \dots, f_{n/i}) = x_{j/i}^{\deg f} f(x_{0/j}, \dots, x_{n/j}).$$

Similarly, a collection of homogeneous polynomials in $A[x_0, \dots, x_n]$ cuts out a closed subscheme of $\mathbb{P}^n_A$.

9.2.2. *Definition.* A closed subscheme cut out by a single (homogeneous) equation is called a **hypersurface** in $\mathbb{P}^n_A$. A hypersurface is locally principal. Notice that a hypersurface is not in general cut out by a single global function on $\mathbb{P}^n_A$. For example, if $A = k$, there *are* no nonconstant global functions (Exercise 5.4.E). The **degree of a hypersurface** is the degree of the polynomial. (Implicit in this is that this notion can be determined from the subscheme itself; we won't really know this until Exercise 20.5.J.) A hypersurface of degree 1 (resp. degree 2, 3, ...) is called a **hyperplane** (resp. **quadric, cubic, quartic, quintic, sextic, septic, octic, ... hypersurface**). If $n = 2$, a degree 1 hypersurface is called a **line**, and a degree 2 hypersurface is called a **conic curve**, or a **conic** for short. If $n = 3$, a hypersurface is called a **surface**. (In Chapter 12, we will justify the terms *curve* and *surface*.)

9.2.A. EXERCISE. (a) Show that $wz = xy, x^2 = wy, y^2 = xz$ describes an irreducible subscheme in $\mathbb{P}^3_k$. In fact it is a curve, a notion we will define once we know what dimension is. This curve is called the **twisted cubic**. (The twisted cubic is a good non-trivial example of many things, so you should make friends with it as soon as possible. It implicitly appeared earlier in Exercise 4.6.H.)
(b) Show that the twisted cubic is isomorphic to $\mathbb{P}^1_k$.

We now extend this discussion to projective schemes in general.

9.2.B. EXERCISE. Suppose that $S_\bullet \twoheadrightarrow R_\bullet$ is a surjection of finitely-generated graded rings. Show that the induced morphism $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$ (Exercise 7.4.A) is a closed immersion.

9.2.C. EXERCISE. Suppose $X \hookrightarrow \operatorname{Proj} S_\bullet$ is a closed immersion in a projective A-scheme. Show that X is projective by describing it as $\operatorname{Proj} S_\bullet/I$, where I is a homogeneous prime ideal, of "projective functions" vanishing on X.

9.2.D. EXERCISE. Show that an injective linear map of k-vector spaces $V \hookrightarrow W$ induces a closed immersion $\mathbb{P}V \hookrightarrow \mathbb{P}W$. (This is another justification for the definition of $\mathbb{P}V$ in Example 5.5.8 in terms of the *dual* of V.)

9.2.3. *Definition.* This closed subscheme is called a **linear space**. Once we know about dimension, we will call this a linear space of dimension $\dim V - 1 = \dim \mathbb{P}V$. A linear space of dimension 1 (resp. 2, n, $\dim \mathbb{P}W - 1$) is called a **line** (resp. **plane**, n**-plane**, **hyperplane**). (If the linear map in the previous exercise is not injective, then the hypothesis (7.4.0.1) of Exercise 7.4.A fails.)

9.2.E. EXERCISE (A SPECIAL CASE OF BÉZOUT'S THEOREM). Suppose $X \subset \mathbb{P}^n_k$ is a degree d hypersurface cut out by $f = 0$, and L is a line not contained in X. A very special case of Bézout's theorem (Exercise 20.5.M) implies that X and L meet with multiplicity d, "counted correctly". Make sense of this, by restricting the degree d form f to the line L, and using the fact that a degree d polynomial in $k[x]$ has d roots, counted properly.

9.2.F. EXERCISE. Show that the map of graded rings $k[w, x, y, z] \to k[s, t]$ given by $w \mapsto s^3$, $x \mapsto s^2 t$, $y \mapsto st^2$, $z \mapsto t^3$ induces a closed immersion $\mathbb{P}^1_k \hookrightarrow \mathbb{P}^3_k$, which yields an isomorphism of $\mathbb{P}^1_k$ with the twisted cubic (defined in Exercise 9.2.A — in fact, this will solve Exercise 9.2.A(b)).

9.2.4. A particularly nice case: when $S_\bullet$ is generated in degree 1.

Suppose $S_\bullet$ is a finitely generated graded ring generated in degree 1. Then S_1 is a finitely-generated $S_\bullet$-module, and the irrelevant ideal S_+ is generated in degree 1 (cf. Exercise 5.5.A).

9.2.G. EXERCISE. Show that if $S_\bullet$ is generated (as an A-algebra) in degree 1 by $n+1$ elements $x_0, \dots, x_n$, then $\operatorname{Proj} S_\bullet$ may be described as a closed subscheme of $\mathbb{P}^n_A$ as follows. Consider A^{n+1} as a free module with generators $t_0, \dots, t_n$ associated to $x_0, \dots, x_n$. The surjection of

$$\begin{array}{rcl} \operatorname{Sym}^\bullet A^{n+1} = A[t_0, t_1, \dots, t_n] & \twoheadrightarrow & S_\bullet \\ \\ t_i & \longmapsto & x_i \end{array}$$

implies $S_\bullet = A[t_0, t_1, \dots t_n]/I$, where I is a homogeneous ideal. (In particular, by Exercise 7.4.G, $\operatorname{Proj} S_\bullet$ can always be interpreted as a closed subscheme of some $\mathbb{P}^n_A$.)

This is analogous to the fact that if R is a finitely-generated A-algebra, then choosing n generators of R as an algebra is the same as describing $\operatorname{Spec} R$ as a closed subscheme of $\mathbb{A}^n_A$. In the affine case this is "choosing coordinates"; in the projective case this is "choosing projective coordinates".

For example, $\operatorname{Proj} k[x, y, z]/(z^2 - x^2 - y^2)$ is a closed subscheme of $\mathbb{P}^2_k$. (A picture is shown in Figure 9.3.)

Recall (Exercise 5.4.F) that if k is algebraically closed, then we can interpret the closed points of $\mathbb{P}^n$ as the lines through the origin in $(n+1)$-space. The following exercise states this more generally.

9.2.H. EXERCISE. Suppose $S_\bullet$ is a finitely-generated graded ring over an algebraically closed field k, generated in degree 1 by $x_0, \dots, x_n$, inducing closed immersions $\operatorname{Proj} S_\bullet \hookrightarrow \mathbb{P}^n$ and $\operatorname{Spec} S_\bullet \hookrightarrow \mathbb{A}^n$. Give a bijection between the closed points of $\operatorname{Proj} S_\bullet$ and the "lines through the origin" in $\operatorname{Spec} S_\bullet \subset \mathbb{A}^n$.

9.2.5. *A second proof that finite morphisms are closed.* This interpretation of $\operatorname{Proj} S_\bullet$ as a closed subscheme of projective space (when it is generated in degree 1) yields the following second proof of the fact (shown in Exercise 8.3.M) that finite morphisms are closed. Suppose $\phi : X \to Y$ is a finite morphism. The question is local on the target, so it suffices to consider the affine case $Y = \operatorname{Spec} B$. It suffices to show that $\phi(X)$ is closed. Then by Exercise 8.3.J, X is a projective B-scheme, and hence by the Fundamental Theorem of Elimination Theory 8.4.5, its image is closed.

9.2.6. The Veronese embedding.

Suppose $S_\bullet = k[x,y]$, so $\operatorname{Proj} S_\bullet = \mathbb{P}^1_k$. Then $S_{2\bullet} = k[x^2, xy, y^2] \subset k[x,y]$ (see §7.4.2 on the Veronese subring). We identify this subring as follows.

9.2.I. EXERCISE. Let $u = x^2$, $v = xy$, $w = y^2$. Show that $S_{2\bullet} = k[u,v,w]/(uw - v^2)$.

We have a graded ring generated by three elements in degree 1. Thus we think of it as sitting "in" $\mathbb{P}^2$, via the construction of §9.2.G. This can be interpreted as "$\mathbb{P}^1$ as a conic in $\mathbb{P}^2$".

9.2.7. Thus if k is algebraically closed of characteristic not 2, using the fact that we can diagonalize quadrics (Exercise 6.4.J), the conics in $\mathbb{P}^2$, up to change of coordinates, come in only a few flavors: sums of 3 squares (e.g. our conic of the previous exercise), sums of 2 squares (e.g. $y^2 - x^2 = 0$, the union of 2 lines), a single square (e.g. $x^2 = 0$, which looks set-theoretically like a line, and is nonreduced), and 0 (perhaps not a conic at all). Thus we have proved: any plane conic (over an algebraically closed field of characteristic not 2) that can be written as the sum of three squares is isomorphic to $\mathbb{P}^1$. (See Exercise 7.5.G for a closely related fact.)

We now soup up this example.

9.2.J. EXERCISE. Show that $\operatorname{Proj} S_{d\bullet}$ is given by the equations that

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}$$

is rank 1 (i.e. that all the 2×2 minors vanish). This is called the **degree d rational normal curve** "in" $\mathbb{P}^d$. You did the *twisted cubic* case $d = 3$ in Exercises 9.2.A and 9.2.F.

9.2.8. *Remark.* More generally, if $S_\bullet = k[x_0, \dots, x_n]$, then $\operatorname{Proj} S_{d\bullet} \subset \mathbb{P}^{N-1}$ (where N is the number of degree d polynomials in $x_0, \dots, x_n$) is called the **d-uple embedding** or **d-uple Veronese embedding**. The reason for the word "embedding" is historical; we really mean closed immersion. (Combining Exercise 7.4.E with Exercise 9.2.G shows that $\operatorname{Proj} S_\bullet \to \mathbb{P}^{n-1}$ is a closed immersion.)

9.2.K. COMBINATORIAL EXERCISE. Show that $N = \binom{n+d}{d}$.

9.2.L. UNIMPORTANT EXERCISE. Find five linearly independent quadric equations vanishing on the **Veronese surface** $\operatorname{Proj} S_{2\bullet}$ where $S_\bullet = k[x_0, x_1, x_2]$, which sits naturally in $\mathbb{P}^5$. (You needn't show that these equations generate all the equations cutting out the Veronese surface, although this is in fact true.)

9.2.9. Rulings on the quadric surface. We return to rulings on the quadric surface, which first appeared in the optional section §5.4.12.

9.2.M. USEFUL GEOMETRIC EXERCISE: THE RULINGS ON THE QUADRIC SURFACE $wz = xy$. This exercise is about the lines on the quadric surface $wz - xy = 0$ in $\mathbb{P}^3_k$ (where the ordering of the coordinates in $\mathbb{P}^3_k$ is are ordered $[w; x; y; z]$). This construction arises all over the place in nature.
(a) Suppose a_0 and b_0 are elements of k, not both zero. Make sense of the statement: as $[c; d]$ varies in $\mathbb{P}^1$, $[a_0 c; b_0 c; a_0 d; b_0 d]$ is a line in the quadric surface. (This describes "a family of lines parametrized by $\mathbb{P}^1$", although we can't yet make this precise.) Find another family of lines. These are the two **rulings** of the quadric surface.
(b) Show there are no other lines. (There are many ways of proceeding. At risk of predisposing you to one approach, here is a germ of an idea. Suppose L is a line on the quadric surface, and $[1; x; y; z]$ and $[1; x'; y'; z']$ are distinct points on it. Because they are both on the quadric, $z = xy$ and $z' = x'y'$. Because all of L is on the quadric, $(1 + t)(z + tz') - (x + tx')(y + ty') = 0$ for all t. After some algebraic manipulation, this translates into $(x - x')(y - y') = 0$. How can this be made watertight? Another possible approach uses Bézout's theorem, in the form of Exercise 9.2.E.)

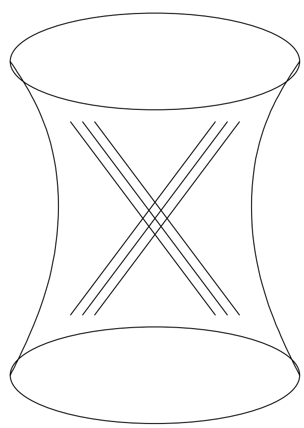


FIGURE 9.2. The two rulings on the quadric surface $V(wz-xy) \subset \mathbb{P}^3$. One ruling contains the line $V(w, x)$ and the other contains the line $V(w, y)$.

Hence by Exercise 6.4.J, if we are working over an algebraically closed field of characteristic not 2, we have shown that all rank 4 quadric surfaces have two rulings of lines. (In Example 10.5.2, we will recognize this quadric as $\mathbb{P}^1 \times \mathbb{P}^1$.)

9.2.10. Weighted projective space. If we put a non-standard weighting on the variables of $k[x_1, \dots, x_n]$ — say we give x_i degree d_i — then $\operatorname{Proj} k[x_1, \dots, x_n]$ is called **weighted projective space** $\mathbb{P}(d_1, d_2, \dots, d_n)$.

9.2.N. EXERCISE. Show that $\mathbb{P}(m, n)$ is isomorphic to $\mathbb{P}^1$. Show that $\mathbb{P}(1, 1, 2) \cong \operatorname{Proj} k[u, v, w, z]/(uw - v^2)$. Hint: do this by looking at the even-graded parts of $k[x_0, x_1, x_2]$, cf. Exercise 7.4.D. (This is a projective cone over a conic curve. Over an algebraically closed field of characteristic not 2, it is isomorphic to the traditional cone $x^2 + y^2 = z^2$ in $\mathbb{P}^3$, Figure 9.3.)

9.2.11. Affine and projective cones.
If $S_\bullet$ is a finitely-generated graded ring, then the **affine cone** of $\operatorname{Proj} S_\bullet$ is $\operatorname{Spec} S_\bullet$. Note that this construction depends on $S_\bullet$, not just of $\operatorname{Proj} S_\bullet$. As motivation, consider the graded ring $S_\bullet = \mathbb{C}[x, y, z]/(z^2 - x^2 - y^2)$. Figure 9.3 is a sketch of $\operatorname{Spec} S_\bullet$. (Here we draw the "real picture" of $z^2 = x^2 + y^2$ in $\mathbb{R}^3$.) It is a cone in the traditional sense; the origin $(0, 0, 0)$ is the "cone point".

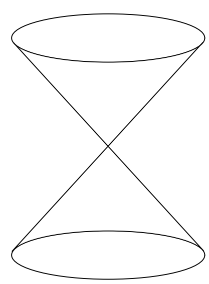


FIGURE 9.3. The cone $\operatorname{Spec} k[x, y, z]/(z^2 - x^2 - y^2)$.

This gives a useful way of picturing Proj (even over arbitrary rings, not just $\mathbb{C}$). Intuitively, you could imagine that if you discarded the origin, you would get something that would project onto $\operatorname{Proj} S_\bullet$. The following exercise makes that precise.

9.2.O. EXERCISE (CF. EXERCISE 7.3.E). If $\operatorname{Proj} S_\bullet$ is a projective scheme over a field k, describe a natural morphism $\operatorname{Spec} S_\bullet \setminus V(S_+) \to \operatorname{Proj} S_\bullet$. (Can you see why $V(S_+)$ is a single point, and should reasonably be called the origin?)

This readily generalizes to the following exercise, which again motivates the terminology "irrelevant".

9.2.P. EXERCISE. If $S_\bullet$ is a finitely generated graded ring, describe a natural morphism $\operatorname{Spec} S_\bullet \setminus V(S_+) \to \operatorname{Proj} S_\bullet$.

In fact, it can be made precise that $\operatorname{Proj} S_\bullet$ is the quotient (by the multiplicative group of scalars) of the affine cone minus the origin.

9.2.12. *Definition.* The **projective cone** of $\operatorname{Proj} S_\bullet$ is $\operatorname{Proj} S_\bullet[T]$, where T is a new variable of degree 1. For example, the cone corresponding to the conic $\operatorname{Proj} k[x, y, z]/(z^2-$

$x^2 - y^2$) is $\operatorname{Proj} k[x, y, z, T]/(z^2 - x^2 - y^2)$. The projective cone is sometimes called the **projective completion** of $\operatorname{Spec} S_\bullet$.

9.2.Q. EXERCISE (CF. §5.5.1). Show that the projective cone of $\operatorname{Proj} S_\bullet[T]$ has a closed subscheme isomorphic to $\operatorname{Proj} S_\bullet$ (corresponding to $T = 0$), whose complement (the distinguished open set $D(T)$) is isomorphic to the affine cone $\operatorname{Spec} S_\bullet$.

You can also check that $\operatorname{Proj} S_\bullet$ is a locally principal closed subscheme of the projective cone $\operatorname{Proj} S_\bullet[T]$, and is also locally not a zerodivisor (an *effective Cartier divisor*, §9.1.2).

This construction can be usefully pictured as the affine cone union some points "at infinity", and the points at infinity form the Proj. The reader may wish to ponder Figure 9.3, and try to visualize the conic curve "at infinity".

We have thus completely described the algebraic analogue of the classical picture of 5.5.1.

9.3 "Smallest closed subschemes such that ..."

We now define a series of notions that are all of the form "the smallest closed subscheme such that something or other is true". One example will be the notion of scheme-theoretic closure of a locally closed immersion, which will allow us to interpret locally closed immersions in three equivalent ways (open subscheme intersect closed subscheme; open subscheme of closed subscheme; and closed subscheme of open subscheme).

9.3.1. Scheme-theoretic image.

We start with the notion of scheme-theoretic image. Set-theoretic images are badly behaved in general (§8.4.1), and even with reasonable hypotheses such as those in Chevalley's theorem 8.4.2, things can be confusing. For example, there is no reasonable way to impose a scheme structure on the image of $\mathbb{A}^2_k \to \mathbb{A}^2_k$ given by $(x, y) \mapsto (x, xy)$. It will be useful (e.g. Exercise 9.3.C) to define a notion of a closed subscheme of the target that "best approximates" the image. This will incorporate the notion that the image of something with nonreduced structure ("fuzz") can also have nonreduced structure. As usual, we will need to impose reasonable hypotheses to make this notion behave well (see Theorem 9.3.4 and Corollary 9.3.5).

9.3.2. *Definition.* Suppose $i : Z \hookrightarrow Y$ is a closed subscheme, giving an exact sequence $0 \to \mathcal{I}_{Z/Y} \to \mathcal{O}_Y \to i_*\mathcal{O}_Z \to 0$. We say that *the image of* $f : X \to Y$ *lies in* Z if the composition $\mathcal{I}_{Z/Y} \to \mathcal{O}_Y \to f_*\mathcal{O}_X$ is zero. Informally, locally functions vanishing on Z pull back to the zero function on X. If the image of f lies in some subschemes Z_i (as i runs over some index set), it clearly lies in their intersection (cf. Exercise 9.1.G(a) on intersections of closed subschemes). We then define the **scheme-theoretic image of** f, a closed subscheme of Y, as the "smallest closed subscheme containing the image", i.e. the intersection of all closed subschemes containing the image.

Example 1. Consider $\operatorname{Spec} k[\epsilon]/\epsilon^2 \to \operatorname{Spec} k[x] = \mathbb{A}^1_k$ given by $x \mapsto \epsilon$. Then the scheme-theoretic image is given by $k[x]/x^2$ (the polynomials pulling back to 0 are precisely multiples of x^2). Thus the image of the fuzzy point still has some fuzz.

Example 2. Consider $f : \operatorname{Spec} k[\epsilon]/\epsilon^2 \to \operatorname{Spec} k[x] = \mathbb{A}^1_k$ given by $x \mapsto 0$. Then the scheme-theoretic image is given by $k[x]/x$: the image is reduced. In this picture, the fuzz is "collapsed" by f.

Example 3. Consider $f : \operatorname{Spec} k[t, t^{-1}] = \mathbb{A}^1 - \{0\} \to \mathbb{A}^1 = \operatorname{Spec} k[u]$ given by $u \mapsto t$. Any function $g(u)$ which pulls back to 0 as a function of t must be the zero-function. Thus the scheme-theoretic image is everything. The set-theoretic image, on the other hand, is the distinguished open set $\mathbb{A}^1 - \{0\}$. Thus in not-too-pathological cases, the underlying set of the scheme-theoretic image is not the set-theoretic image. But the situation isn't terrible: the underlying set of the scheme-theoretic image must be closed, and indeed it is the closure of the set-theoretic image. We might imagine that in reasonable cases this will be true, and in even nicer cases, the underlying set of the scheme-theoretic image will be set-theoretic image. We will later see that this is indeed the case (§9.3.6).

But sadly pathologies can sometimes happen.

Example 4. Let $X = \coprod \operatorname{Spec} k[\epsilon_n]/((\epsilon_n)^n)$ and $Y = \operatorname{Spec} k[x]$, and define $X \to Y$ by $x \to \epsilon_n$ on the nth component of X. Then if a function $g(x)$ on Y pulls back to 0 on X, then its Taylor expansion is 0 to order n (by examining the pullback to the nth component of X) for all n, so $g(x)$ must be 0. Thus the scheme-theoretic image is $V(0)$ on Y, i.e. Y itself, while the set-theoretic image is easily seen to be just the origin.

9.3.3. *Criteria for computing scheme-theoretic images affine-locally.* Example 4 clearly is weird though, and we can show that in "reasonable circumstances" such pathology doesn't occur. It would be great to compute the scheme-theoretic image affine-locally. On the affine open set $\operatorname{Spec} B \subset Y$, define the ideal $I_B \subset B$ of functions which pull back to 0 on X. Formally, $I_B := \ker(B \to \Gamma(\operatorname{Spec} B, f_*(\mathcal{O}_X))$. Then if for each such B, and each $g \in B$, $I_B \otimes_B B_g \to I_{B_g}$ is an isomorphism, then we will have defined the scheme-theoretic image as a closed subscheme (see Exercise 9.1.F). Clearly each function on $\operatorname{Spec} B$ that vanishes when pulled back to $f^{-1}(\operatorname{Spec} B)$ also vanishes when restricted to $D(g)$ and then pulled back to $f^{-1}(D(g))$. So the question is: given a function r/g^n on $D(g)$ that pulls back to $f^{-1}D(g)$, is it true that for some m, $rg^m = 0$ when pulled back to $f^{-1}(\operatorname{Spec} B)$? Here are three cases where the answer is "yes". (I would like to add a picture here, but I can't think of one that would enlighten more people than it would confuse. So you should try to draw one that suits you.) In a nutshell, for each affine in the source, there is an m which works. There is one that works for all affines in a cover if (i) if $m = 1$ always works, or (ii) or (iii) if there are only a finite number of affines in the cover.

(i) The answer is yes if $f^{-1}(\operatorname{Spec} B)$ is reduced: we simply take $m = 1$ (as r vanishes on $\operatorname{Spec} B_g$ and g vanishes on $V(g)$, so rg vanishes on $\operatorname{Spec} B = \operatorname{Spec} B_g \cup V(g)$.)

(ii) The answer is also yes if $f^{-1}(\operatorname{Spec} B)$ is affine, say $\operatorname{Spec} A$: if $r' = f^\sharp r$ and $g' = f^\sharp g$ in A, then if $r' = 0$ on $D(g')$, then there is an m such that $r'(g')^m = 0$ (as the statement $r' = 0$ in $D(g')$ means precisely this fact — the functions on $D(g')$ are $A_{g'}$).

(iii) More generally, the answer is yes if $f^{-1}(\operatorname{Spec} B)$ is quasicompact: cover $f^{-1}(\operatorname{Spec} B)$ with finitely many affine open sets. For each one there will be some m_i so that $rg^{m_i} = 0$ when pulled back to this open set. Then let $m = \max(m_i)$. (We see again that quasicompactness is our friend!)

In conclusion, we have proved the following (subtle) theorem.

9.3.4. Theorem. — *Suppose $f : X \to Y$ is a morphism of schemes. If X is reduced or f is quasicompact, then the scheme-theoretic image of f may be computed affine-locally: on* $\operatorname{Spec} A$, *it is cut out by the functions that pull back to 0.*

9.3.5. Corollary. — *Under the hypotheses of Theorem 9.3.4, the closure of the set-theoretic image of f is the underlying set of the scheme-theoretic image.*

(Example 4 above shows that we cannot excise these hypotheses.)

9.3.6. In particular, if the set-theoretic image is closed (e.g. if f is finite or projective), the set-theoretic image is the underlying set of the scheme-theoretic image, as promised in Example 3 above.

Proof. The set-theoretic image is in the underlying set of the scheme-theoretic image. (Check this!) The underlying set of the scheme-theoretic image is closed, so the closure of the set-theoretic image is contained in underlying set of the scheme-theoretic image. On the other hand, if U is the complement of the closure of the set-theoretic image, $f^{-1}(U) = \varnothing$. As under these hypotheses, the scheme theoretic image can be computed locally, the scheme-theoretic image is the empty set on U. □

We conclude with a few stray remarks.

9.3.A. Easy Exercise. If X is reduced, show that the scheme-theoretic image of $f : X \to Y$ is also reduced.

More generally, you might expect there to be no unnecessary nonreduced structure on the image not forced by nonreduced structure on the source. We make this precise in the locally Noetherian case, when we can talk about associated points.

9.3.B. ⋆ Unimportant exercise. If $f : X \to Y$ is a *quasicompact* morphism of locally Noetherian schemes, show that the associated points of the image subscheme are a subset of the image of the associated points of X. (The example of $\coprod_{a \in \mathbb{C}} \operatorname{Spec} \mathbb{C}[t]/(t-a) \to \operatorname{Spec} \mathbb{C}[t]$ shows what can go wrong if you give up quasicompactness — note that reducedness of the source doesn't help.) Hint: reduce to the case where X and Y are affine. (Can you develop your geometric intuition so that this is geometrically plausible?)

9.3.7. Scheme-theoretic closure of a locally closed subscheme.

We define the **scheme-theoretic closure** of a locally closed immersion $f : X \to Y$ as the scheme-theoretic image of X.

9.3.C. EXERCISE. If $V \to X$ is quasicompact (e.g. if V is Noetherian, Exercise 8.3.B(a)), or if V is reduced, show that (iii) implies (i) and (ii) in Exercise 9.1.L. Thus in this fortunate situation, a locally closed immersion can be thought of in three different ways, whichever is convenient.

9.3.D. UNIMPORTANT EXERCISE, USEFUL FOR INTUITION. If $f : X \to Y$ is a locally closed immersion into a locally Noetherian scheme (so X is also locally Noetherian), then the associated points of the scheme-theoretic closure are (naturally in bijection with) the associated points of X. (Hint: Exercise 9.3.B.) Informally, we get no nonreduced structure on the scheme-theoretic closure not "forced by" that on X.

9.3.8. The (reduced) subscheme structure on a closed subset.

Suppose X^{set} is a closed sub*set* of a scheme Y. Then we can define a canonical scheme structure X on X^{set} that is reduced. We could describe it as being cut out by those functions whose values are zero at all the points of X^{set}. On the affine open set $\operatorname{Spec} B$ of Y, if the set X^{set} corresponds to the radical ideal $I = I(X^{\text{set}})$ (recall the $I(\cdot)$ function from §4.7), the scheme X corresponds to $\operatorname{Spec} B/I$. You can quickly check that this behaves well with respect to any distinguished inclusion $\operatorname{Spec} B_f \hookrightarrow \operatorname{Spec} B$. We could also consider this construction as an example of a scheme-theoretic image in the following crazy way: let W be the scheme that is a disjoint union of all the points of X^{set}, where the point corresponding to p in X^{set} is Spec of the residue field of $\mathcal{O}_{Y,p}$. Let $f : W \to Y$ be the "canonical" map sending "p to p", and giving an isomorphism on residue fields. Then the scheme structure on X is the scheme-theoretic image of f. A third definition: it is the smallest closed subscheme whose underlying set contains X^{set}.

This construction is called the (induced) **reduced subscheme structure** on the closed subset X^{set}. (Vague exercise: Make a definition of the reduced subscheme structure precise and rigorous to your satisfaction.)

9.3.E. EXERCISE. Show that the underlying set of the induced reduced subscheme $X \to Y$ is indeed the closed subset X^{set}. Show that X is reduced.

9.3.9. Reduced version of a scheme.

In the main interesting case where X^{set} is all of Y, we obtain a *reduced closed subscheme* $Y^{\text{red}} \to Y$, called the **reduction** of Y. On the affine open subset $\operatorname{Spec} B \hookrightarrow Y$, $Y^{\text{red}} \hookrightarrow Y$ corresponds to the nilradical $\mathfrak{N}(B)$ of B. The *reduction* of a scheme is the "reduced version" of the scheme, and informally corresponds to "shearing off the fuzz".

An alternative equivalent definition: on the affine open subset $\operatorname{Spec} B \hookrightarrow Y$, the reduction of Y corresponds to the ideal $\mathfrak{N}(B) \subset Y$. As for any $f \in B$, $\mathfrak{N}(B)_f = \mathfrak{N}(B_f)$, by Exercise 9.1.F this defines a closed subscheme.

9.3.F. EXERCISE (USEFUL FOR VISUALIZATION). Show that if Y is a locally Noetherian scheme, the "reduced locus" of Y (the points of Y where $Y^{\text{red}} \to Y$ induces an isomorphism of stalks of the structure sheaves) is an open subset of Y. (Hint: if Y is affine, show that it is the complement of the closure of the embedded associated points.)

9.3.10. Scheme-theoretic support of a quasicoherent sheaf. Similar ideas are used in the definition of the scheme-theoretic support of a quasicoherent sheaf, see Exercise 20.8.B.

CHAPTER 10

Fibered products of schemes

10.1 They exist

Before we get to products, we note that coproducts exist in the category of schemes: just as with the category of sets (Exercise 2.3.S), coproduct is disjoint union. The next exercise makes this precise (and directly extends to coproducts of an infinite number of schemes).

10.1.A. EASY EXERCISE. Suppose X and Y are schemes. Let $X \coprod Y$ be the scheme whose underlying topological space is the disjoint union of the topological spaces of X and Y, and with structure sheaf on (the part corresponding to) X given by $\mathcal{O}_X$, and similarly for Y. Show that $X \coprod Y$ is the coproduct of X and Y (justifying the use of the symbol $\coprod$).

We will now construct the fibered product in the category of schemes.

10.1.1. Theorem: Fibered products exist. — *Suppose* $f : X \to Z$ *and* $g : Y \to Z$ *are morphisms of schemes. Then the fibered product*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

exists in the category of schemes.

Note: if A is a ring, people often write $\times_A$ for $\times_{\operatorname{Spec} A}$. If B is an A-algebra, and X is an A-scheme, people often write X_B or $X \times_A B$ for $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$.

10.1.2. *Warning: products of schemes aren't products of sets.* Before showing existence, here is a warning: the product of schemes isn't a product of sets (and more generally for fibered products). We have made a big deal about schemes being *sets*, endowed with a *topology*, upon which we have a *structure sheaf*. So you might think that we will construct the product in this order. But we won't, because products behave oddly on the level of sets. You may have checked (Exercise 7.6.C(a)) that the product of two affine lines over your favorite algebraically closed field $\overline{k}$ is the affine plane: $\mathbb{A}^1_{\overline{k}} \times_{\overline{k}} \mathbb{A}^1_{\overline{k}} \cong \mathbb{A}^2_{\overline{k}}$. But the underlying set of the latter is *not* the underlying set of the former —- we get additional points, corresponding to curves in $\mathbb{A}^2$ that are not lines parallel to the axes!

10.1.3. On the other hand, S-valued points (where S is a scheme, Definition 7.3.6) *do* behave well under (fibered) products. This is just the *definition* of fibered product: an S-valued point of a scheme X is defined as $\mathrm{Hom}(S, X)$, and the fibered product is defined by

$$\mathrm{Hom}(S, X \times_Z Y) = \mathrm{Hom}(S, X) \times_{\mathrm{Hom}(S,Z)} \mathrm{Hom}(S, Y). \tag{10.1.3.1}$$

This is one justification for making the definition of S-valued point. For this reason, those classical people preferring to think only about varieties over an algebraically closed field $\overline{k}$ (or more generally, finite-type schemes over $\overline{k}$), and preferring to understand them through their closed points — or equivalently, the $\overline{k}$-valued points, by the Nullstellensatz (Exercise 6.3.E) — needn't worry: the closed points of the product of two finite type $\overline{k}$-schemes over $\overline{k}$ are (naturally identified with) the product of the closed points of the factors. This will follow from the fact that the product is also finite type over $\overline{k}$, which we verify in Exercise 10.2.D. This is one of the reasons that varieties over algebraically closed fields can be easier to work with. But over a nonalgebraically closed field, things become even more interesting; Example 10.2.2 is a first glimpse.

(Fancy remark: You may feel that (i) "products of topological spaces are products on the underlying sets" is natural, while (ii) "products of schemes are not necessarily are products on the underlying sets" is weird. But really (i) is the lucky consequence of the fact that the underlying set of a topological space can be interpreted as set of p-valued points, where p is a point, so it is best seen as a consequence of paragraph 10.1.3, which is the "more correct" — i.e. more general — fact.)

10.1.B. EXERCISE (TO GET PRACTICE WITH THE CONCEPT). Suppose $K \subset L$ is a field extension, and X is a K-scheme. Assume $X_L := X \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ exists (which it does by as-yet unproved Theorem fiberedproductsexist). Show that the L-valued points of X are in natural bijection with the L-valued points of X_L.

10.1.4. Philosophy behind the proof of Theorem 10.1.1. The proof of Theorem 10.1.1 can be confusing. The following comments may help a little.

We already basically know existence of fibered products in two cases: the case where X, Y, and Z is affine (stated explicitly below), and the case where $Y \to Z$ is an open immersion (Exercise 8.1.A).

10.1.C. EXERCISE. Use Exercise 7.3.F (that $\mathrm{Hom}_{Sch}(W, \mathrm{Spec}\, A) = \mathrm{Hom}_{Rings}(A, \Gamma(W, \mathcal{O}_W))$) to show that given ring maps $C \to B$ and $C \to A$,

$$\mathrm{Spec}(A \otimes_C B) \cong \mathrm{Spec}\, A \times_{\mathrm{Spec}\, C} \mathrm{Spec}\, B.$$

(Interpret tensor product as the "cofibered product" in the category of rings.) Hence the fibered product of affine schemes exists (in the category of schemes). (This generalizes the fact that the product of affine lines exist, Exercise 7.6.C(a).)

The main theme of the proof of Theorem 10.1.1 is that because schemes are built by gluing affine schemes along open subsets, these two special cases will be all that we need. The argument will repeatedly use the same ideas — roughly, that schemes glue (Exercise 5.4.A), and that morphisms of schemes glue (Exercise 7.3.A). This is a sign that something more structural is going on; §10.1.5 describes this for experts.

Proof of Theorem 10.1.1. The key idea is this: we cut everything up into affine open sets, do fibered products there, and show that everything glues nicely. The conceptually difficult part of the proof comes from the gluing, and the realization that we have to check almost nothing. We divide the proof up into a number of bite-sized pieces.

Step 1: fibered products of affine with almost-affine over affine. We begin by combining the affine case with the open immersion case as follows. Suppose X and Z are affine, and $Y \to Z$ factors as $Y \overset{i}{\hookrightarrow} Y' \xrightarrow{g} Z$ where i is an open immersion and Y' is affine. Then $X \times_Z Y$ exists. This is because if the two small squares of

$$\begin{tikzcd} W \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\ W' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

are fibered diagrams, then the "outside rectangle" is also a fibered diagram. (This was Exercise 2.3.P, although you should be able to see this on the spot.) It will be important to remember that "open immersions" are "preserved by fibered product": the fact that $Y \to Y'$ is an open immersion implies that $W \to W'$ is an open immersion.

Key Step 2: fibered product of affine with arbitrary over affine exists. We now come to the key part of the argument: if X and Z are affine, and Y is arbitrary. This is confusing when you first see it, so we first deal with a special case, when Y is the union of two affine open sets $Y_1 \cup Y_2$. Let $Y_{12} = Y_1 \cap Y_2$.

Now for $i = 1, 2$, $X \times_Z Y_i$ exists by the affine case, Exercise 10.1.C. Call this W_i. Also, $X \times_Z Y_{12}$ exists by Step 1 (call it W_{12}), and comes with open immersions into W_1 and W_2 (by construction of fibered products with open immersion). Thus we can glue W_1 to W_2 along W_{12}; call this resulting scheme W.

We check that this is the fibered product by verifying that it satisfies the universal property. Suppose we have maps $f'' : V \to X$, $g'' : V \to Y$ that compose (with f and g respectively) to the same map $V \to Z$. We need to construct a unique map $h : V \to W$, so that $f' \circ h = g''$ and $g' \circ h = f''$.

(10.1.4.1)
$$\begin{tikzcd} V \arrow[dr, "\exists!?"] \arrow[drr, bend left, "g''"] \arrow[ddr, bend right, "f''"'] & & \\ & W \arrow[r, "f'"'] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

For $i = 1, 2$, define $V_i := (g'')^{-1}(Y_i)$. Define $V_{12} := (g'')^{-1}(Y_{12}) = V_1 \cap V_2$. Then there is a unique map $V_i \to W_i$ such that the composed maps $V_i \to X$ and $V_i \to Y_i$ are as desired (by the universal product of the fibered product $X \times_Z Y_i = W_i$), hence a unique map $h_i : V_i \to W$. Similarly, there is a unique map $h_{12} : V_{12} \to W$ such that the composed maps $V_{12} \to X$ and $V_{12} \to Y$ are as desired. But the restriction of h_i to V_{12} is one such map, so it must be h_{12}. Thus the maps h_1 and h_2 agree on V_{12}, and glue together to a unique map $h : V \to W$. We have shown existence and uniqueness of the desired h.

We have thus shown that if Y is the union of two affine open sets, and X and Z are affine, then $X \times_Z Y$ exists.

We now tackle the general case. (You may prefer to first think through the case where "two" is replaced by "three".) We now cover Y with open sets Y_i, as i runs over some index set (not necessarily finite!). As before, we define W_i and W_{ij}. We can glue these together to produce a scheme W along with open sets we identify with W_i (Exercise 5.4.A — you should check the triple intersection "cocycle" condition).

As in the two-affine case, we show that W is the fibered product by showing that it satisfies the universal property. Suppose we have maps $f'' : V \to X$, $g'' : V \to Y$ that compose to the same map $V \to Z$. We construct a unique map $h : V \to W$, so that $f' \circ h = g''$ and $g' \circ h = f''$. Define $V_i = (g'')^{-1}(Y_i)$ and $V_{ij} := (g'')^{-1}(Y_{ij}) = V_i \cap V_j$. Then there is a unique map $V_i \to W_i$ such that the composed maps $V_i \to X$ and $V_i \to Y_i$ are as desired, hence a unique map $h_i : V_i \to W$. Similarly, there is a unique map $h_{ij} : V_{ij} \to W$ such that the composed maps $V_{ij} \to X$ and $V_{ij} \to Y$ are as desired. But the restriction of h_i to V_{ij} is one such map, so it must be h_{ij}. Thus the maps h_i and h_j agree on V_{ij}. Thus the h_i glue together to a unique map $h : V \to W$. We have shown existence and uniqueness of the desired h, completing this step.

Step 3: Z affine, X and Y arbitrary. We next show that if Z is affine, and X and Y are arbitrary schemes, then $X \times_Z Y$ exists. We just follow Step 2, with the roles of X and Y reversed, using the fact that by the previous step, we can assume that the fibered product with an affine scheme with an arbitrary scheme over an affine scheme exists.

Step 4: Z admits an open immersion into an affine scheme Z', X and Y arbitrary. This is akin to Step 1: $X \times_Z Y$ satisfies the universal property of $X \times_{Z'} Y$.

Step 5: the general case. We again employ the trick from Step 4. Say $f : X \to Z$, $g : Y \to Z$ are two morphisms of schemes. Cover Z with affine open subsets Z_i. Let $X_i = f^{-1} Z_i$ and $Y_i = g^{-1} Z_i$. Define $Z_{ij} = Z_i \cap Z_j$, and X_{ij} and Y_{ij} analogously. Then $W_i := X_i \times_{Z_i} Y_i$ exists for all i, and has as open sets $W_{ij} := X_{ij} \times_{Z_{ij}} Y_{ij}$ along with gluing information satisfying the cocycle condition (arising from the gluing information for Z from the Z_i and Z_{ij}). Once again, we show that this satisfies the universal property. Suppose V is any scheme, along with maps to X and Y that agree when they are composed to Z. We need to show that there is a unique morphism $V \to W$ completing the diagram (10.1.4.1). Now break V up into open sets $V_i = g'' \circ f^{-1}(Z_i)$. Then by the universal property for W_i, there is a unique map $V_i \to W_i$ (which we can interpret as $V_i \to W$). Thus we have already shown uniqueness of $V \to W$. These must agree on $V_i \cap V_j$, because there is only one map $V_i \cap V_j$ to W making the diagram commute. Thus all of these morphisms $V_i \to W$ glue together, so we are done. $\square$

10.1.5. $\star\star$ **Describing the existence of fibered products using the high-falutin' language of representable functors.** The proof above can be described more cleanly in the language of representable functors (§7.6). This will be enlightening only after you have absorbed the above argument and meditated on it for a long time. It may be most useful to shed light on representable functors, rather than on the existence of the fibered product.

Until the end of §10.1 only, by functor, *we mean contravariant functor from the category Sch of schemes to the category of Sets.* For each scheme X, we have a functor h_X, taking a scheme Y to $\mathrm{Mor}(Y, X)$ (§2.2.20). Recall (§2.3.10, §7.6) that a functor is *representable* if it is naturally isomorphic to some h_X. The existence of the fibered product can be reinterpreted as follows. Consider the functor $h_{X \times_Z Y}$ defined by $h_{X \times_Z Y}(W) = h_X(W) \times_{h_Z(W)} h_Y(W)$. (This isn't quite enough to define a functor; we have only described where objects go. You should work out where morphisms go too.) Then "$X \times_Z Y$ exists" translates to "$h_{X \times_Z Y}$ is representable".

If a functor is representable, then the representing scheme is unique up to unique isomorphism (Exercise 7.6.B). This can be usefully extended as follows:

10.1.D. EXERCISE (YONEDA'S LEMMA). If X and Y are schemes, describe a bijection between morphisms of schemes $X \to Y$ and natural transformations of functors $h_X \to h_Y$. Hence show that the category of schemes is a fully faithful subcategory of the "functor category" of all functors (contravariant, *Sch* $\to$ *Sets*). Hint: this has nothing to do with schemes; your argument will work in any category. This is the contravariant version of Exercise 2.3.Y(c).

One of Grothendieck's insights is that we should try to treat such functors as "geometric spaces", without worrying about representability. Many notions carry over to this more general setting without change, and some notions are easier. For example, fibered products of functors always exist: $h \times_{h''} h'$ may be defined by

$$(h \times_{h''} h')(W) = h(W) \times_{h''(W)} h'(W)$$

(where the fibered product on the right is a fibered product of sets, which always exists). We didn't use anything about schemes; this works with *Sch* replaced by any category.

10.1.6. *Representable functors are Zariski sheaves.* Because "morphisms to schemes glue" (Exercise 7.3.A), we have a necessary condition for a functor to be representable. We know that if $\{U_i\}$ is an open cover of Y, a morphism $Y \to X$ is determined by its restrictions $U_i \to X$, and given morphisms $U_i \to X$ that agree on the overlap $U_i \cap U_j \to X$, we can glue them together to get a morphism $Y \to X$. In the language of equalizer exact sequences (§3.2.7),

$$\cdot \longrightarrow \mathrm{Hom}(Y, X) \longrightarrow \prod \mathrm{Hom}(U_i, X) \rightrightarrows \prod \mathrm{Hom}(U_i \cap U_j, X)$$

is exact. Thus morphisms to X (i.e. the functor h_X) form a sheaf on every scheme Y. If this holds, we say that *the functor is a Zariski sheaf*. (You can impress your friends by telling them that this is a *sheaf on the big Zariski site*.) We can repeat this discussion with *Sch* replaced by the category Sch_S of schemes over a given base scheme S. We have proved (or observed) that *in order for a functor to be representable, it is necessary for it to be a Zariski sheaf*.

The fiber product passes this test:

10.1.E. EXERCISE. If $X, Y \to Z$ are schemes, show that $h_{X \times_Z Y}$ is a Zariski sheaf. (Do not use the fact that $X \times_Z Y$ is representable! The point of this section is to recover representability from a more sophisticated perspective.)

We can make some other definitions that extend notions from schemes to functors. We say that a map (i.e. natural transformation) of functors $h \to h'$ expresses h as an **open subfunctor** of h' if for all representable functors h_X and maps $h_X \to h'$, the fibered product $h_X \times_{h'} h$ is representable, by U say, and $h_U \to h_X$ corresponds to an open immersion of schemes $U \to X$. The following fibered square may help.

$$\begin{tikzcd} h_U \arrow[r] \arrow[d, "\text{open}"'] & h \arrow[d] \\ h_X \arrow[r] & h' \end{tikzcd}$$

Notice that a map of representable functors $h_W \to h_Z$ is an open subfunctor if and only if $W \to Z$ is an open immersion, so this indeed extends the notion of open immersion to (contravariant) functors ($\mathit{Sch} \to \mathit{Sets}$).

10.1.F. EXERCISE. Suppose $h \to h''$ and $h' \to h''$ are two open subfunctors of h''. Define the intersection of these two open subfunctors, which should also be an open subfunctor of h''.

10.1.G. EXERCISE. Suppose $X, Y \to Z$ are schemes, and $U \subset X$, $V \subset Y$, $W \subset Z$ are open subsets, where U and V map to W. Interpret $U \times_W V$ as an open subfunctor of $X \times_Z Y$. (Hint: given a map $h_T \to h_{X \times_Z Y}$, what open subset of T should correspond to $U \times_W V$?)

A collection h_i of open subfunctors of h' is said to **cover** h' if for *every* map $h_X \to h'$ from a representable subfunctor, the corresponding open subsets $U_i \hookrightarrow X$ cover X.

Given that functors do not have an obvious underlying set (let alone a topology), it is rather amazing that we are talking about when one is an "open subset" of another, or when some functors "cover" another!

10.1.H. EXERCISE. Suppose $\{Z_i\}_i$ is an affine cover of Z, $\{X_{ij}\}_j$ is an affine cover of the preimage of Z_i in X, and $\{Y_{ik}\}_k$ is an affine cover of the preimage of Z_i in Y. Show that $\{h_{X_{ij} \times_{Z_i} Y_{ik}}\}_{ijk}$ is an open cover of the functor $h_{X \times_Z Y}$. (Hint: consider a map $h_T \to h_{X \times_Z Y}$, and extend your solution to the Exercise 10.1.G.)

We now come to a key point: a Zariski sheaf that is "locally representable" must be representable:

10.1.I. KEY EXERCISE. If a functor h is a Zariski sheaf that has an open cover by representable functors ("is covered by schemes"), then h is representable. (Hint: use Exercise 5.4.A to glue together the schemes representing the open subfunctors.)

This immediately leads to the existence of fibered products as follows. Exercise 10.1.E shows that $h_{X \times_Z Y}$ is a Zariski sheaf. But $(h_{X_{ij} \times_{Z_i} Y_{ik}})_{ijk}$ is representable (fibered products of affines over an affine exist, Exercise 10.1.C), and these functors are an open cover of $h_{X \times_Z Y}$ by Exercise 10.1.H, so by Key Exercise 10.1.I we are done.

10.2 Computing fibered products in practice

Before giving some examples, we first see how to compute fibered products in practice. There are four types of morphisms **(1)–(4)** that it is particularly easy to take fibered products with, and all morphisms can be built from these four atomic components (see the last paragraph of **(1)**).

(1) Base change by open immersions.

We have already done this (Exercise 8.1.A), and we used it in the proof that fibered products of schemes exist.

I will describe the remaining three on the level of affine open sets, because we obtain general fibered products by gluing. Theoretically, only **(2)** and **(3)** are necessary, as any map of rings $\phi : B \to A$ can be interpreted by adding variables (perhaps infinitely many) to A, and then imposing relations. But in practice **(4)** is useful, as we will see in examples.

(2) Adding an extra variable.

10.2.A. EASY ALGEBRA EXERCISE. Show that $B \otimes_A A[t] \cong B[t]$, so the following is a fibered diagram. (Your argument might naturally extend to allow the addition of infinitely many variables, but we won't need this generality.) Hint: show that $B[t]$ satisfies an appropriate universal property.

$$\begin{tikzcd} \operatorname{Spec} B[t] \arrow[r] \arrow[d] & \operatorname{Spec} A[t] \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

(3) Base change by closed immersions

10.2.B. EXERCISE. Suppose $\phi : A \to B$ is a ring homomorphism, and $I \subset A$ is an ideal. Let $I^e := \langle \phi(i) \rangle_{i \in I} \subset B$ be the **extension of** I **to** B. Describe a natural isomorphism $B/I^e \cong B \otimes_A (A/I)$. (Hint: consider $I \to A \to A/I \to 0$, and use the right-exactness of $\otimes_A B$, Exercise 2.3.H.)

10.2.1. As an immediate consequence: the fibered product with a closed subscheme is a closed subscheme of the fibered product in the obvious way. We say that "closed immersions are preserved by base change".

10.2.C. EXERCISE. (a) Interpret the intersection of two closed immersions into X (cf. Exercise 9.1.G) as their fibered product over X.
(b) Show that "locally closed immersions" are preserved by base change.
(c) Define the intersection of a finite number of locally closed immersions in X.

As an application of Exercise 10.2.B, we can compute tensor products of finitely generated k algebras over k. For example, we have a canonical isomorphism

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

10.2.D. EXERCISE. Suppose X and Y are locally finite type k-schemes. Show that $X \times_k Y$ is also locally of finite type over k. Prove the same thing with "locally" removed from both the hypothesis and conclusion.

10.2.2. *Example.* We can use Exercise 10.2.B to compute $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$:

$$\begin{aligned}
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}[x]/(x^2+1)) \\
&\cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x])/(x^2+1) \qquad \text{by 10.2}\textbf{(3)} \\
&\cong \mathbb{C}[x]/(x^2+1) \qquad \text{by 10.2}\textbf{(2)} \\
&\cong \mathbb{C}[x]/((x-i)(x+i)) \\
&\cong \mathbb{C}[x]/(x-i) \times \mathbb{C}[x]/(x+i) \qquad \text{by the Chinese Remainder Theorem} \\
&\cong \mathbb{C} \times \mathbb{C}
\end{aligned}$$

Thus $\operatorname{Spec}\mathbb{C} \times_{\mathbb{R}} \operatorname{Spec}\mathbb{C} \cong \operatorname{Spec}\mathbb{C} \coprod \operatorname{Spec}\mathbb{C}$. This example is the first example of many different behaviors. Notice for example that two points somehow correspond to the Galois group of $\mathbb{C}$ over $\mathbb{R}$; for one of them, x (the "i" in one of the copies of $\mathbb{C}$) equals i (the "i" in the other copy of $\mathbb{C}$), and in the other, $x = -i$.

10.2.3. $\star$ *Remark.* Here is a clue that there is more going on. If L/K is a Galois extension with Galois group G, then $L \otimes_K L$ is isomorphic to L^G (the product of $|G|$ copies of L). This turns out to be a restatement of the classical form of linear independence of characters! In the language of schemes, $\operatorname{Spec} L \times_K \operatorname{Spec} L$ is a union of a number of copies of L that naturally form a torsor over the Galois group G.

10.2.E. $\star$ HARD BUT FASCINATING EXERCISE FOR THOSE FAMILIAR WITH $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Show that the points of $\operatorname{Spec}\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter. (Some hints: first do the case of finite Galois extensions. Relate the topology on Spec of a direct limit of rings to the inverse limit of Specs. Can you see which point corresponds to the identity of the Galois group?)

(4) Base change of affine schemes by localization.

10.2.F. EXERCISE. Suppose $\phi : A \to B$ is a ring homomorphism, and $S \subset A$ is a multiplicative subset of A, which implies that $\phi(S)$ is a multiplicative subset of B. Describe a natural isomorphism $\phi(S)^{-1}B \cong B \otimes_A (S^{-1}A)$.

Translation: the fibered product with a localization is the localization of the fibered product in the obvious way. We say that "localizations are preserved by base change". This is handy if the localization is of the form $A \hookrightarrow A_f$ (corresponding to taking distinguished open sets) or $A \hookrightarrow K(A)$ (from A to the fraction field of A, corresponding to taking generic points), and various things in between.

These four facts let you calculate lots of things in practice, and we will use them freely.

10.2.G. EXERCISE: THE THREE IMPORTANT TYPES OF MONOMORPHISMS OF SCHEMES. Show that the following are monomorphisms (Definition 2.3.9): open immersions, closed immersions, and localization of affine schemes. As monomorphisms are closed under composition, Exercise 2.3.U, compositions of the above are also monomorphisms (e.g. locally closed immersions, or maps from "Spec of stalks at points of X" to X).

10.2.H. EXERCISE. If $X, Y \hookrightarrow Z$ are two locally closed immersions, show that $X \times_Z Y$ is canonically isomorphic to $X \cap Y$.

10.2.I. EXERCISE. Prove that $\mathbb{A}^n_A \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A$. Prove that $\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A$. Thus affine space and projective space are pulled back from their universal manifestation over the final object $\operatorname{Spec}\mathbb{Z}$.

10.2.4. *Extending the base field.* One special case of base change is called **extending the base field**: if X is a k-scheme, and k' is a field extension (often k' is the algebraic closure of k), then $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ (sometimes informally written $X \times_k k'$ or $X_{k'}$) is a k'-scheme. Often properties of X can be checked by verifying them instead on $X_{k'}$. This is the subject of *descent* — certain properties "descend" from $X_{k'}$ to X. We have already seen that the property of being *normal* descends in this way in characteristic 0 (Exercise 6.4.L — but note that this holds even in positive characteristic). The following two exercises is another example of this: the property of two morphisms being equal, and the property of a(n affine) morphism begin a closed immersion, both descend in this way. Those interested in schemes over non-algebraically closed fields will use this repeatedly, to reduce results to the algebraically closed case.

10.2.J. EXERCISE. Suppose $\pi : X \to Y$ and $\rho : X \to Y$ are morphisms of k-schemes, ℓ/k is a field extension, and $\pi_\ell : X \times_{\operatorname{Spec} k} \operatorname{Spec} \ell \to Y \times_{\operatorname{Spec} k} \operatorname{Spec} \ell$ and $\rho_\ell : X \times_{\operatorname{Spec} k} \operatorname{Spec} \ell \to Y \times_{\operatorname{Spec} k} \operatorname{Spec} \ell$ are the induced maps of ℓ-schemes. (Be sure you understand what this means!) Show that if $\pi_\ell = \rho_\ell$ then $\pi = \rho$. (Hint: show that π and ρ are the same on the level of sets. Then reduce to the case where X and Y are affine.)

10.2.K. EASY EXERCISE. Suppose $f : X \to Y$ is an affine morphism over k. Show that f is a closed immersion if and only if $f \times_k \overline{k} : X \times_k \overline{k} \to Y \times_k \overline{k}$ is. (The affine hypothesis is not necessary for this result, but it makes the proof easier, and this is the situation in which we will most need it.)

10.2.L. UNIMPORTANT BUT FUN EXERCISE. Show that $\operatorname{Spec}\mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C}$ has closed points in natural correspondence with the transcendental complex numbers. (If the description $\operatorname{Spec}\mathbb{Q}(t) \otimes_{\mathbb{Q}[t]} \mathbb{C}[t]$ is more striking, you can use that instead.) This scheme doesn't come up in nature, but it is certainly neat!

10.2.5. *A first view of a blow-up.*

10.2.M. IMPORTANT CONCRETE EXERCISE. (The discussion here immediately generalizes to $\mathbb{A}^n_A$.) Define a closed subscheme $\operatorname{Bl}_{(0,0)} \mathbb{A}^2_k$ of $\mathbb{A}^2_k \times \mathbb{P}^1_k$ as follows (see Figure 10.1). If the coordinates on $\mathbb{A}^2_k$ are x, y, and the projective coordinates on $\mathbb{P}^1_k$ are u, v, this subscheme is cut out in $\mathbb{A}^2_k \times \mathbb{P}^1_k$ by the single equation $xv = yu$. (You may wish to interpret $\operatorname{Bl}_{(0,0)} \mathbb{A}^2_k$ as follows. The $\mathbb{P}^1_k$ parametrizes lines through the origin. The blow-up corresponds to ordered pairs of (point p, line ℓ) such that $(0,0), p \in \ell$.) Describe the fiber of the morphism $\operatorname{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$. Show that the morphism $\operatorname{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$ is an isomorphism away from $(0,0) \in \mathbb{A}^2_k$. Show that the fiber over $(0,0)$ is a closed subscheme that is locally principal and not locally a zerodivisor (an *effective Cartier divisor*, §9.1.2). It is called the **exceptional divisor**. We will discuss blow-ups in Chapter 19. This particular example will come up in the motivating example of §19.1, and in Exercise 22.2.D.

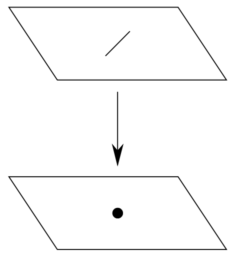


FIGURE 10.1. A first example of a blow-up

We haven't yet discussed nonsingularity, but here is a hand-waving argument suggesting that the $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$ is "smooth": the preimage above either standard open set $U_i \subset \mathbb{P}^1$ is isomorphic to $\mathbb{A}^2$. Thus "the blow-up is a surgery that takes the smooth surface $\mathbb{A}^2_k$, cuts out a point, and glues back in a $\mathbb{P}^1$, in such a way that the outcome is another smooth surface."

10.2.6. *The graph of a rational map.*

Define the **graph** Γ_f of a rational map $f : X \dashrightarrow Y$ as follows. Let (U, f') be any representative of this rational map (so $f' : U \to Y$ is a morphism). Let Γ_f be the scheme-theoretic closure of $\Gamma_{f'} \hookrightarrow U \times Y \hookrightarrow X \times Y$, where the first map is a closed immersion, and the second is an open immersion. Equivalently, it is the scheme-theoretic image of the morphism $U \xrightarrow{(i,f')} X \times Y$. The product here should be taken in the category you are working in. For example, if you are working with k-schemes, the fibered product should be taken over k.

10.2.N. EXERCISE. Show that these definitions are indeed equivalent. Show that the graph of a rational map is independent of the choice of representative of the rational map.

In analogy with graphs of morphisms (e.g. Figure 11.3), the following diagram of a graph of a rational map can be handy.

$$\begin{tikzcd}
\Gamma \arrow[r, hook, "\text{cl. imm.}"] & X \times Y \arrow[dl] \arrow[dr] & \\
X \arrow[u, dashed] & & Y.
\end{tikzcd}$$

10.2.O. EXERCISE (THE BLOW-UP OF THE PLANE AS THE GRAPH OF A RATIONAL MAP). Consider the rational map $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$. Show that this rational map cannot be extended over the origin. (A similar argument arises in Exercise 7.5.J on the Cremona transformation.) Show that the graph of the rational map is the morphism (the blow-up) described in Exercise 10.2.M. (When we defined blow ups in general, we will see that they are often graphs of rational maps.)

10.3 Pulling back families and fibers of morphisms

10.3.1. Pulling back families.
We can informally interpret fibered product in the following geometric way. Suppose $Y \to Z$ is a morphism. We interpret this as a "family of schemes parametrized by a **base scheme** (or just plain **base**) Z." Then if we have another morphism $f : X \to Z$, we interpret the induced map $X \times_Z Y \to X$ as the "pulled back family" (see Figure 10.2).

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d, "\text{pulled back family}"'] & Y \arrow[d, "\text{family}"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

We sometimes say that $X \times_Z Y$ is the **scheme-theoretic pullback of** Y, **scheme-theoretic inverse image**, or **inverse image scheme of** Y. (Our forthcoming discussion of fibers may give some motivation for this.) For this reason, fibered product is often called **base change** or **change of base** or **pullback**. In addition to the various names for a Cartesian diagram given in §2.3.6, in algebraic geometry it is often called a **base change diagram** or a **pullback diagram**, and $X \times_Z Y \to X$ is called the **pullback** of $Y \to Z$ by f, and $X \times_Z Y$ is called the **pullback** of Y by f. (Random side remark: scheme-theoretic pullback always makes sense, while the notion of scheme-theoretic image is somehow problematic, as discussed in §s:stimage.)

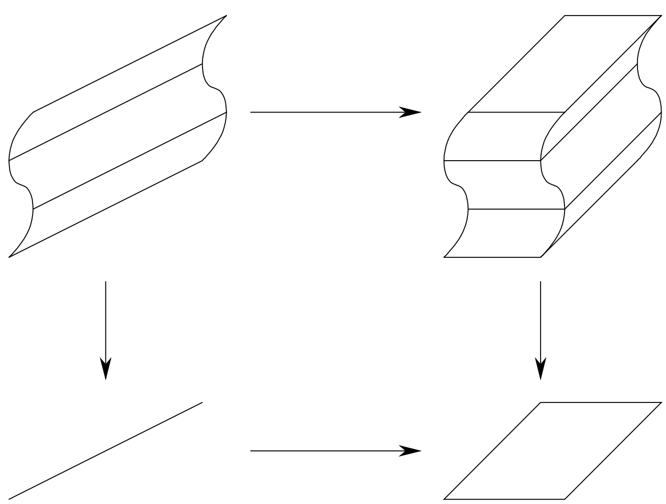


FIGURE 10.2. A picture of a pulled back family

Before making any definitions, we give a motivating informal example. Consider the "family of curves" $y^2 = x^3 + tx$ in the xy-plane parametrized by t. Translation: consider $\operatorname{Spec} k[x, y, t]/(y^2 - x^3 - tx) \to \operatorname{Spec} k[t]$. If we pull back to a family parametrized by the uv-plane via $uv = t$ (i.e. $\operatorname{Spec} k[u, v] \to \operatorname{Spec} k[t]$ given by

$t \mapsto uv$), we get $y^2 = x^3 + uvx$, i.e. $\operatorname{Spec} k[x,y,u,v]/(y^2 - x^3 - uvx) \to \operatorname{Spec} k[u,v]$. If instead we set t to 3 (i.e. pull back by $\operatorname{Spec} k[t]/(t-3) \to \operatorname{Spec} k[t]$, we get the curve $y^2 = x^3 + 3x$ (i.e. $\operatorname{Spec} k[x,y]/(y^2 - x^3 - 3x) \to \operatorname{Spec} k$), which we interpret as the fiber of the original family above $t = 3$. We will soon be able to interpret these constructions in terms of fiber products.

10.3.2. Fibers of morphisms.
A special case of pullback is the notion of a fiber of a morphism. We motivate this with the notion of fiber in the category of topological spaces.

10.3.A. EXERCISE. Show that if $Y \to Z$ is a continuous map of topological spaces, and X is a point p of Z, then the fiber of Y over p (the set-theoretic fiber, with the induced topology) is naturally identified with $X \times_Z Y$.

More generally, for general $X \to Z$, the fiber of $X \times_Z Y \to X$ over a point p of X is naturally identified with the fiber of $Y \to Z$ over $f(p)$.

Motivated by topology, we return to the category of schemes. Suppose $p \to Z$ is the inclusion of a point (not necessarily closed). More precisely, if p is a K-valued point, consider the map $\operatorname{Spec} K \to Z$ sending $\operatorname{Spec} K$ to p, with the natural isomorphism of residue fields. Then if $g : Y \to Z$ is any morphism, the base change with $p \to Z$ is called the (scheme-theoretic) **fiber of** g **above** p or the (scheme-theoretic) **preimage of** p, and is denoted $g^{-1}(p)$. If Z is irreducible, the fiber above the generic point is called the **generic fiber**. In an affine open subscheme $\operatorname{Spec} A$ containing p, p corresponds to some prime ideal $\mathfrak{p}$, and the morphism $\operatorname{Spec} K \to Z$ corresponds to the ring map $A \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. This is the composition of localization and closed immersion, and thus can be computed by the tricks above. (Note that $p \to Z$ is a monomorphism, by Exercise 10.2.G.)

10.3.B. EXERCISE. Show that the underlying topological space of the (scheme-theoretic) fiber $X \to Y$ above a point p is naturally identified with the topological fiber of $X \to Y$ above p.

10.3.C. EXERCISE (ANALOG OF EXERCISE 10.3.A). Suppose that $\pi : Y \to Z$ and $f : X \to Z$ are morphisms, and $x \in X$ is a point. Show that the fiber of $X \times_Z Y \to X$ over x is (isomorphic to) the base change to x of the fiber of $\pi : Y \to Z$ over $f(x)$.

10.3.3. *Example (enlightening in several ways).* Consider the projection of the parabola $y^2 = x$ to the x-axis over $\mathbb{Q}$, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. If $\mathbb{Q}$ alarms you, replace it with your favorite field and see what happens. (You should look at Figure 4.6, and figure out how to edit it to reflect what we glean here.) Writing $\mathbb{Q}[y]$ as $\mathbb{Q}[x,y]/(y^2 - x)$ helps us interpret the morphism conveniently.

(i) Then the preimage of 1 is two points:

$$\begin{aligned}
\operatorname{Spec} \mathbb{Q}[x,y]/(y^2 - x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x-1) &\cong \operatorname{Spec} \mathbb{Q}[x,y]/(y^2 - x, x-1) \\
&\cong \operatorname{Spec} \mathbb{Q}[y]/(y^2 - 1) \\
&\cong \operatorname{Spec} \mathbb{Q}[y]/(y-1) \coprod \operatorname{Spec} \mathbb{Q}[y]/(y+1).
\end{aligned}$$

(ii) The preimage of 0 is one nonreduced point:

$$\operatorname{Spec} \mathbb{Q}[x,y]/(y^2 - x, x) \cong \operatorname{Spec} \mathbb{Q}[y]/(y^2).$$

(iii) The preimage of -1 is one reduced point, but of "size 2 over the base field".

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x,x+1)\cong\operatorname{Spec}\mathbb{Q}[y]/(y^2+1)\cong\operatorname{Spec}\mathbb{Q}[i]=\operatorname{Spec}\mathbb{Q}(i).$$

(iv) The preimage of the generic point is again one reduced point, but of "size 2 over the residue field", as we verify now.

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x)\otimes_{\mathbb{Q}[x]}\mathbb{Q}(x)\cong\operatorname{Spec}\mathbb{Q}[y]\otimes\mathbb{Q}(y^2)$$

i.e. (informally) the Spec of the ring of polynomials in y divided by polynomials in y^2. A little thought shows you that in this ring you may invert *any* polynomial in y, as if $f(y)$ is any polynomial in y, then

$$\frac{1}{f(y)}=\frac{f(-y)}{f(y)f(-y)},$$

and the latter denominator is a polynomial in y^2. Thus

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x)\otimes\mathbb{Q}(x)\cong\mathbb{Q}(y)$$

which is a degree 2 field extension of $\mathbb{Q}(x)$ (note that $\mathbb{Q}(x)=\mathbb{Q}(y^2)$).

Notice the following interesting fact: in each of the four cases, the number of preimages can be interpreted as 2, where you count to two in several ways: you can count points (as in the case of the preimage of 1); you can get nonreduced behavior (as in the case of the preimage of 0); or you can have a field extension of degree 2 (as in the case of the preimage of -1 or the generic point). In each case, the fiber is an affine scheme whose dimension as a vector space over the residue field of the point is 2. Number theoretic readers may have seen this behavior before. We will discuss this example again in §18.4.8. This is going to be symptomatic of a very special and important kind of morphism (a finite flat morphism).

Try to draw a picture of this morphism if you can, so you can develop a pictoral shorthand for what is going on. A good first approximation is the parabola of Figure 4.6, but you will want to somehow depict the peculiarities of (iii) and (iv).

10.3.D. EXERCISE (IMPORTANT FOR THOSE WITH MORE ARITHMETIC BACKGROUND). What is the scheme-theoretic fiber of $\operatorname{Spec}\mathbb{Z}[i]\to\operatorname{Spec}\mathbb{Z}$ over the prime (p)? Your answer will depend on p, and there are four cases, corresponding to the four cases of Example 10.3.3. (Can you draw a picture?)

10.3.E. EXERCISE. Consider the morphism of schemes $X=\operatorname{Spec}k[t]\to Y=\operatorname{Spec}k[u]$ corresponding to $k[u]\to k[t]$, $u\mapsto t^2$, where $\operatorname{char}k\neq 2$. Show that $X\times_Y X$ has 2 irreducible components. (This exercise will give you practice in computing a fibered product over something that is not a field.)

(What happens if $\operatorname{char}k=2$? See Exercise 10.4.H for a clue.)

10.4 Properties preserved by base change

All reasonable properties of morphisms are preserved under base change. (In fact, one might say that a property of morphisms cannot be reasonable if it is not preserved by base change!) We discuss this, and explain how to fix those that don't fit this pattern.

We have already shown that the notion of "open immersion" is preserved by base change (Exercise 8.1.A). We did this by explicitly describing what the fibered product of an open immersion is: if $Y \hookrightarrow Z$ is an open immersion, and $f : X \to Z$ is any morphism, then we checked that the open subscheme $f^{-1}(Y)$ of X satisfies the universal property of fibered products.

We have also shown that the notion of "closed immersion" is preserved by base change (§10.2 **(3)**). In other words, given a fiber diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r, hook, "\text{cl. imm.}"] & Z \end{tikzcd}$$

where $Y \hookrightarrow Z$ is a closed immersion, $W \to X$ is as well.

10.4.A. EASY EXERCISE. Show that locally principal closed subschemes pull back to locally principal closed subschemes.

Similarly, other important properties are preserved by base change.

10.4.B. EXERCISE. Show that the following properties of morphisms are preserved by base change.

(a) quasicompact
(b) quasiseparated
(c) affine morphism
(d) finite
(e) locally of finite type
(f) finite type
⋆⋆ (g) locally of finite presentation
⋆⋆ (h) finite presentation

10.4.C. ⋆ HARD EXERCISE. Show that the notion of "quasifinite morphism" (finite type + finite fibers, Definition 8.3.11) is preserved by base change. (Warning: the notion of "finite fibers" is not preserved by base change. $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ has finite fibers, but $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, see Exercise 10.2.E.) Hint: reduce to the case $\operatorname{Spec} A \to \operatorname{Spec} B$. Reduce to the case $\phi : \operatorname{Spec} A \to \operatorname{Spec} k$. Show that if ϕ is quasifinite then ϕ is finite.

10.4.D. EXERCISE. Show that surjectivity is preserved by base change. (**Surjectivity** has its usual meaning: surjective as a map of sets.) You may end up showing that for any fields k_1 and k_2 containing k_3, $k_1 \otimes_{k_3} k_2$ is non-zero, and using the axiom of choice to find a maximal ideal in $k_1 \otimes_{k_3} k_2$.

10.4.1. On the other hand, injectivity is not preserved by base change — witness the bijection $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, which loses injectivity upon base change by $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ (see Example 10.2.2). This can be rectified (§10.4.5).

10.4.E. EXERCISE. Suppose X and Y are integral finite type $\overline{k}$-schemes. Show that $X \times_{\overline{k}} Y$ is an integrable finite type $\overline{k}$-scheme. (Once we define "variety", this will become the important fact that the product of irreducible varieties over an algebraically closed field is an irreducible variety, Exercise 11.1.E. The hypothesis that k is algebraically closed is essential, see §10.4.2.) Hint: reduce to the case where X and Y are both affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ with A and B integral domains. Suppose $(\sum a_i \otimes b_i)\left(a'_j \otimes b'_j\right) = 0$ in $A \otimes_{\overline{k}} B$ with $a_i, a'_j \in A$, $b_i, b'_j \in B$, where both $\{b_i\}$ and $\{b'_j\}$ are linearly independent over $\overline{k}$, and a_1 and a'_1 are nonzero. Show that $D(a_1 a'_1) \subset \operatorname{Spec} A$ is nonempty. By the Weak Nullstellensatz 4.2.2, there is a maximal $\mathfrak{m} \subset A$ in $D(a_1 a'_1)$ with $A/\mathfrak{m} = \overline{k}$. By reducing modulo $\mathfrak{m}$, deduce $(\sum \overline{a}_i \otimes b_i)\left(\overline{a'}_j \otimes b'_j\right) = 0$ in B, where the overline indicates residue modulo $\mathfrak{m}$. Show that this contradicts the fact that B is a domain.

10.4.F. EXERCISE. If P is a property of morphisms preserved by base change and composition, and $X \to Y$ and $X' \to Y'$ are two morphisms of S-schemes with property P, show that $X \times_S X' \to Y \times_S Y'$ has property P as well.

10.4.G. $\star\star$ EXERCISE. Suppose $\pi : X \to \operatorname{Spec} B$ is a finitely presented morphism. Show that there exists a base change diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"'] & X' \arrow[d, "\pi'"] \\ \operatorname{Spec} B \arrow[r, "\rho"] & \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_N]/I \end{tikzcd}$$

where N is some integer, $I \subset \mathbb{Z}[x_1, \dots, x_N]$, and π' is finitely presented (= finite type as the target is Noetherian, see §8.3.13). Thus each finitely presented morphism is locally (on the base) a pullback of a finite type morphism to a Noetherian scheme. Hence any result proved for Noetherian schemes and stable under base change is automatically proved for finitely presented morphisms to arbitrary schemes. (One example will be the Cohomology and Base Change Theorem 25.7.5.) Hint: Try to deal with the case where X is affine first. If $X = \operatorname{Spec} A$, then $A = B[y_1, \dots, y_n]/(f_1, \dots, f_r)$. Choose one variable x_i for each y_i, and one for each coefficient of $f_i \in B[y_1, \dots, y_n]$. What is ρ in this case? Then consider the case where X is the union of two affine open sets, that intersect in an affine open set. Then consider more general cases until you solve the full problem. You will need to use every part of the definition of finite presentation.

10.4.2. $\star$ **Properties not preserved by base change, and how to fix (some of) them.**

There are some notions that you should reasonably expect to be preserved by pullback based on your geometric intuition. Given a family in the topological category, fibers pull back in reasonable ways. So for example, any pullback of a family in which all the fibers are irreducible will also have this property; ditto for connected. Unfortunately, both of these fail in algebraic geometry, as Example 10.2.2 shows:

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C} \coprod \operatorname{Spec} \mathbb{C} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{C} \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{R} \end{tikzcd}$$

The family on the right (the vertical map) has irreducible and connected fibers, and the one on the left doesn't. The same example shows that the notion of "integral fibers" also doesn't behave well under pullback.

10.4.H. EXERCISE. Suppose k is a field of characteristic p, so $k(u)/k(u^p)$ is an inseparable extension. By considering $k(u) \otimes_{k(u^p)} k(u)$, show that the notion of "reduced fibers" does not necessarily behave well under pullback. (The fact that I am giving you this example should show that this happens only in characteristic p, in the presence of something as strange as inseparability.)

We rectify this problem as follows.

10.4.3. A **geometric point** of a scheme X is defined to be a morphism $\operatorname{Spec} k \to X$ where k is an algebraically closed field. Awkwardly, this is now the third kind of "point" of a scheme! There are just plain points, which are elements of the underlying set; there are S-valued points, which are maps $S \to X$, §7.3.6; and there are geometric points. Geometric points are clearly a flavor of an S-valued point, but they are also an enriched version of a (plain) point: they are the data of a point with an inclusion of the residue field of the point in an algebraically closed field.

A **geometric fiber** of a morphism $X \to Y$ is defined to be the fiber over a geometric point of Y. A morphism has **connected** (resp. **irreducible**, **integral**, **reduced**) **geometric fibers** if all its geometric fibers are connected (resp. irreducible, integral, reduced). One usually says that the morphism has **geometrically connected (resp. irreducible, integral, reduced) fibers**. A k-scheme X is **geometrically connected (resp. irreducible, integral, reduced)** if the structure morphism $X \to \operatorname{Spec} k$ has geometrically connected (resp. irreducible, integral, reduced) fibers.

10.4.I. EXERCISE. Show that the notion of "connected (resp. irreducible, integral, reduced) geometric fibers" behaves well under base change.

10.4.J. EXERCISE FOR THE ARITHMETICALLY-MINDED. Show that for the morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, all geometric fibers consist of two reduced points. (Cf. Example 10.2.2.) Thus $\operatorname{Spec} \mathbb{C}$ is a geometrically reduced but not geometrically irreducible $\mathbb{R}$-scheme.

10.4.K. EXERCISE. Recall Example 10.3.3, the projection of the parabola $y^2 = x$ to the x-axis, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. Show that the geometric fibers of this map are always two points, except for those geometric fibers "over $0 = [(x)]$". (Note that $\operatorname{Spec} \mathbb{C} \to \mathbb{Q}[x]$ and $\operatorname{Spec} \overline{\mathbb{Q}} \to \mathbb{Q}[x]$, both with $x \mapsto 0$, are both geometric points "above 0".)

Checking whether a k-scheme is geometrically connected etc. seems annoying: you need to check every single algebraically closed field containing k. However, in each of these four cases, the failure of nice behavior of geometric fibers can already be detected after a finite field extension. For example, $\operatorname{Spec} \mathbb{Q}(i) \to \operatorname{Spec} \mathbb{Q}$ is not geometrically connected, and in fact you only need to base change by $\operatorname{Spec} \mathbb{Q}(i)$ to see this. We make this precise as follows.

Suppose X is a k-scheme. If K/k is a field extension, define $X_K = X \times_k \operatorname{Spec} K$. Consider the following twelve statements.

- X_K is reduced:
  (R_a) for all fields K,

(R_b) for all algebraically closed fields K (X is geometrically reduced),
(R_c) for $K = \overline{k}$,
(R_d) for $K = k^p$ (k^p is the perfect closure of k)

- X_K is irreducible:
  (I_a) for all fields K,
  (I_b) for all algebraically closed fields K (X is geometrically irreducible),
  (I_c) for $K = \overline{k}$,
  (I_d) for $K = k^s$ (k^s is the separable closure of k).
- X_K is connected:
  (C_a) for all fields K,
  (C_b) for all algebraically closed fields K (X is geometrically connected),
  (C_c) for $K = \overline{k}$,
  (C_d) for $K = k^s$.

Trivially *(R_a)* implies *(R_b)* implies *(R_c)*, and *(R_a)* implies *(R_d)*, and similarly with "reduced" replaced by "irreducible" and "connected".

10.4.L. EXERCISE. (a) Suppose that E/F is a field extension, and A is an F-algebra. Show that A is a subalgebra of $A \otimes_F E$. (Hint: think of these as vector spaces over F.)
(b) Show that: *(R_b)* implies *(R_a)* and *(R_c)* implies *(R_d)*.
(c) Show that: *(I_b)* implies *(I_a)* and *(I_c)* implies *(I_d)*.
(d) Show that: *(C_b)* implies *(C_a)* and *(C_c)* implies *(C_d)*.
Notice: you may use the fact that if Y is a nonempty F-scheme, then $Y \times_F \operatorname{Spec} E$ is nonempty, cf. Exercise 10.4.D.

Thus for example a k-scheme is geometrically integral if and only if it remains integral under any field extension.

10.4.4. $\star\star$ **Hard fact.** In fact, *(R_d)* implies *(R_a)*, and thus *(R_a)* through *(R_d)* are all equivalent, and similarly for the other two rows.

10.4.5. $\star$ **Universally injective (radicial) morphisms.** As remarked in §10.4.1, injectivity is not preserved by base change. A better notion is that of **universally injective** morphisms: morphisms that are injections of sets after any base change. In keeping with the traditional agricultural terminology (sheaves, germs, ..., cf. Remark 3.4.3), these morphisms were named **radicial** after one of the lesser vegetables. This notion is more useful in positive characteristic, as the following exercise makes clear.

10.4.M. EXERCISE. (a) Show that locally closed immersions (and in particular open and closed immersions) are universally injective. (a) Show that $f : X \to Y$ is universally injective only if f is injective, and for each $x \in X$, the field extension $\kappa(x)/\kappa(f(x))$ is purely inseparable.
(b) Show that the class of universally injective morphisms are stable under composition, products, and base change.
(c) If $g : Y \to Z$ is another morphism, show that if $g \circ f$ is radicial, then f is radicial.

10.5 Products of projective schemes: The Segre embedding

We next describe products of projective A-schemes over A. (The case of greatest initial interest is if $A = k$.) To do this, we need only describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, because any projective A-scheme has a closed immersion in some $\mathbb{P}^m_A$, and closed immersions behave well under base change, so if $X \hookrightarrow \mathbb{P}^m_A$ and $Y \hookrightarrow \mathbb{P}^n_A$ are closed immersions, then $X \times_A Y \hookrightarrow \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ is also a closed immersion, cut out by the equations of X and Y (§10.2**(3)**). We will describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, and see that it too is a projective A-scheme. (Hence if X and Y are projective A-schemes, then their product $X \times_A Y$ over A is also a projective A-scheme.)

Before we do this, we will get some motivation from classical projective spaces (non-zero vectors modulo non-zero scalars, Exercise 5.4.F) in a special case. Our map will send $[x_0; x_1; x_2] \times [y_0; y_1]$ to a point in $\mathbb{P}^5$, whose coordinates we think of as being entries in the "multiplication table"

$$\begin{array}{lllll} [& x_0y_0; & x_1y_0; & x_2y_0; & \\ & x_0y_1; & x_1y_1; & x_2y_1 &]. \end{array}$$

This is indeed a well-defined map of sets. Notice that the resulting matrix is rank one, and from the matrix, we can read off $[x_0; x_1; x_2]$ and $[y_0; y_1]$ up to scalars. For example, to read off the point $[x_0; x_1; x_2] \in \mathbb{P}^2$, we take the first row, unless it is all zero, in which case we take the second row. (They can't both be all zero.) In conclusion: in classical projective geometry, given a point of $\mathbb{P}^m$ and $\mathbb{P}^n$, we have produced a point in $\mathbb{P}^{mn+m+n}$, and from this point in $\mathbb{P}^{mn+m+n}$, we can recover the points of $\mathbb{P}^m$ and $\mathbb{P}^n$.

Suitably motivated, we return to algebraic geometry. We define a map

$$\mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$$

by

$$([x_0; \dots; x_m], [y_0; \dots; y_n]) \mapsto [z_{00}; z_{01}; \cdots; z_{ij}; \cdots; z_{mn}]$$

$$= [x_0y_0; x_0y_1; \cdots; x_iy_j; \cdots x_my_n].$$

More explicitly, we consider the map from the affine open set $U_i \times V_j$ (where $U_i = D(x_i)$ and $V_j = D(y_j)$ to the affine open set $W_{ij} = D(z_{ij})$ by

$$(x_{0/i}, \dots, x_{m/i}, y_{0/j}, \dots, y_{n/j}) \mapsto (x_{0/i}y_{0/j}; \dots; x_{i/i}y_{j/j}; \dots; x_{m/i}y_{n/j})$$

or, in terms of algebras, $z_{ab/ij} \mapsto x_{a/i}y_{b/j}$.

10.5.A. EXERCISE. Check that these maps glue to give a well-defined morphism $\mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$.

10.5.1. We next show that this morphism is a closed immersion. We can check this on an open cover of the target (the notion of being a closed immersion is affine-local, Exercise 9.1.C). Let's check this on the open set where $z_{ij} \neq 0$. The preimage of this open set in $\mathbb{P}^m_A \times \mathbb{P}^n_A$ is the locus where $x_i \neq 0$ and $y_j \neq 0$, i.e. $U_i \times V_j$. As described above, the map of rings is given by $z_{ab/ij} \mapsto x_{a/i}y_{b/j}$; this is clearly a surjection, as $z_{aj/ij} \mapsto x_{a/i}$ and $z_{ib/ij} \mapsto y_{b/j}$. (A generalization of this ad hoc description will be given in Exercise 17.4.D.)

This map is called the **Segre morphism** or **Segre embedding**. If A is a field, the image is called the **Segre variety**.

10.5.B. EXERCISE. Show that the Segre scheme (the image of the Segre morphism) is cut out (scheme-theoretically) by the equations corresponding to

$$\operatorname{rank}\begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} = 1,$$

i.e. that all 2×2 minors vanish. Hint: suppose you have a polynomial in the a_{ij} that becomes zero upon the substitution $a_{ij} = x_i y_j$. Give a recipe for subtracting polynomials of the form "monomial times 2×2 minor" so that the end result is 0. (The analogous question for the Veronese embedding in special cases is the content of Exercises 9.2.J and 9.2.L.)

10.5.2. *Important Example.* Let's consider the first non-trivial example, when $m = n = 1$. We get $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. We get a single equation

$$\operatorname{rank}\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = 1,$$

i.e. $a_{00}a_{11} - a_{01}a_{10} = 0$. We again meet our old friend, the quadric surface (§9.2.9)! Hence: the nonsingular quadric surface $wz - xy = 0$ (Figure 9.2) is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Recall from Exercise 9.2.M that the quadric has two families of lines. You may wish to check that one family of lines corresponds to the image of $\{x\} \times \mathbb{P}^1$ as x varies, and the other corresponds to the image $\mathbb{P}^1 \times \{y\}$ as y varies.

If we are working over an algebraically closed field of characteristic not 2, then by diagonalizability of quadratics (Exercise 6.4.J), all rank 4 ("full rank") quadratics are isomorphic, so all rank 4 quadric surfaces over an algebraically closed field of characteristic not 2 are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that this is not true over a field that is not algebraically closed. For example, over $\mathbb{R}$, $w^2 + x^2 + y^2 + z^2 = 0$ is not isomorphic to $\mathbb{P}^1_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}}$. Reason: the former has no real points, while the latter has lots of real points.

You may wish to do the next two exercises in either order.

10.5.C. EXERCISE: A COORDINATE-FREE DESCRIPTION OF THE SEGRE EMBEDDING. Show that the Segre embedding can be interpreted as $\mathbb{P}V \times \mathbb{P}W \to \mathbb{P}(V \otimes W)$ via the surjective map of graded rings

$$\operatorname{Sym}^\bullet(V^\vee \otimes W^\vee) \longrightarrow\!\!\!\!\!\rightarrow \sum_{i=0}^\infty \left(\operatorname{Sym}^i V^\vee\right) \otimes \left(\operatorname{Sym}^i W^\vee\right)$$

"in the opposite direction".

10.5.D. EXERCISE: A COORDINATE-FREE DESCRIPTION OF PRODUCTS OF PROJECTIVE A-SCHEMES IN GENERAL. Suppose that $S_\bullet$ and $T_\bullet$ are finitely-generated graded rings over A. Describe an isomorphism

$$(\operatorname{Proj} S_\bullet) \times_A (\operatorname{Proj} T_\bullet) \cong \operatorname{Proj} \oplus_{n=0}^\infty (S_n \otimes_A T_n)$$

(where hopefully the definition of multiplication in the graded ring $\oplus_{n=0}^\infty S_n \otimes_A T_n$ is clear).

10.6 Normalization

Normalization is a means of turning a *reduced* scheme into a normal scheme. A *normalization* of a scheme X is a morphism $\nu : \tilde{X} \to X$ from a normal scheme, where ν induces a bijection of irreducible components of $\tilde{X}$ and X, and ν gives a birational morphism on each of the irreducible components. (We need the scheme to have irreducible components for this to make sense, so we will often impose hypotheses such as Noetherianness to keep our scheme from being pathological.) It will satisfy a universal property, and hence it is unique up to unique isomorphism. Figure 8.4 is an example of a normalization. We discuss normalization now because the argument for its existence follows that for the existence of the fibered product.

We begin with the case where X is irreducible, and hence integral. (We will then deal with a more general case, and also discuss normalization in a function field extension.) In this case of irreducible X, the **normalization** $\nu : \tilde{X} \to X$ is a dominant morphism from an irreducible normal scheme to X, such that any other such morphism factors through ν:

$$\begin{tikzcd} \text{normal} & Y \arrow[rr, dotted, "\exists !"] \arrow[dr, "f \text{ dominant}"'] & & \tilde{X} \arrow[dl, "\nu \text{ dominant}"] & \text{normal} \\ & & X & & \end{tikzcd}$$

Thus if the normalization exists, then it is unique up to unique isomorphism. We now have to show that it exists, and we do this in a way that will look familiar. We deal first with the case where X is affine, say $X = \operatorname{Spec} A$, where A is an integral domain. Then let $\tilde{A}$ be the *integral closure* of A in its fraction field $K(A)$. (Recall that the integral closure of A in its fraction field consists of those elements of $K(A)$ that are solutions to monic polynomials in $A[x]$. It is a ring extension by Exercise 8.2.D, and integrally closed by Exercise 8.2.K.)

10.6.A. EXERCISE. Show that $\nu : \operatorname{Spec} \tilde{A} \to \operatorname{Spec} A$ satisfies the universal property. (En route, you might show that the global sections of a normal scheme are also normal.)

10.6.B. IMPORTANT (BUT SURPRISINGLY EASY) EXERCISE. Show that normalizations of integral schemes exist in general. (Hint: Ideas from the existence of fiber products, §10.1, may help.)

10.6.C. EASY EXERCISE. Show that normalizations are integral and surjective. (Hint for surjectivity: the Lying Over Theorem, see §8.2.6.)

10.6.D. EXERCISE. Explain how to extend the notion of normalization to the case where X is a reduced Noetherian scheme, with possibly more than one component. (We add the Noetherian hypotheses to ensure that we have irreducible components, Proposition 4.6.6.) This basically requires defining a universal property. I'm not sure what the "perfect" definition is, but all reasonable universal properties should be equivalent.

Here are some examples.

10.6.E. EXERCISE. Show that $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^2(x+1))$ given by $(x,y) \mapsto (t^2-1, t(t^2-1))$ (see Figure 8.4) is a normalization. (Hint: show that $k[t]$

and $k[x,y]/(y^2-x^2(x+1))$ have the same fraction field. Show that $k[t]$ is integrally closed. Show that $k[t]$ is contained in the integral closure of $k[x,y]/(y^2-x^2(x+1))$.)

You will see from the previous exercise that once we guess what the normalization is, it isn't hard to verify that it is indeed the normalization. Perhaps a few words are in order as to where the polynomials t^2-1 and $t(t^2-1)$ arose in the previous exercise. The key idea is to guess $t = y/x$. (Then $t^2 = x+1$ and $y = xt$ quickly.) This idea comes from three possible places. We begin by sketching the curve, and noticing the node at the origin. (a) The function y/x is well-defined away from the node, and at the node, the two branches have "values" $y/x = 1$ and $y/x = -1$. (b) We can also note that if $t = y/x$, then t^2 is a polynomial, so we will need to adjoin t in order to obtain the normalization. (c) The curve is cubic, so we expect a general line to meet the cubic in three points, counted with multiplicity. (We will make this precise when we discuss Bézout's Theorem, Exercise 20.5.M, but in this case we have already gotten a hint of this in Exercise 7.5.H.) There is a $\mathbb{P}^1$ parametrizing lines through the origin (with coordinate equal to the slope of the line, y/x), and most such lines meet the curve with multiplicity two at the origin, and hence meet the curve at precisely one other point of the curve. So this "co-ordinatizes" most of the curve, and we try adding in this coordinate.

10.6.F. EXERCISE. Find the normalization of the cusp $y^2 = x^3$ (see Figure 10.3).

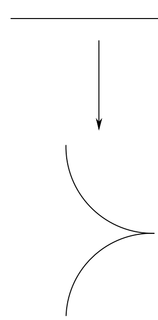


FIGURE 10.3. Normalization of a cusp

10.6.G. EXERCISE. Find the normalization of the tacnode $y^2 = x^4$, and draw a picture analogous to Figure 10.3.

(Although we haven't defined "singularity", "smooth", "curve", or "dimension", you should still read this.) Notice that in the previous examples, normalization "resolves" the singularities ("non-smooth" points) of the curve. In general, it will do so in dimension one (in reasonable Noetherian circumstances, as normal Noetherian integral domains of dimension one are all discrete valuation rings, §13.4), but won't do so in higher dimension (the cone $z^2 = x^2 + y^2$ over a field k of characteristic not 2 is normal, Exercise 6.4.I(b)).

10.6.H. EXERCISE. Suppose $X = \operatorname{Spec} \mathbb{Z}[15i]$. Describe the normalization $\tilde{X} \to X$. (Hint: $\mathbb{Z}[i]$ is a unique factorization domain, §6.4.5(0), and hence is integrally closed by Exercise 6.4.F.) Over what points of X is the normalization not an isomorphism?

Another exercise in a similar vein is the normalization of the "knotted plane", Exercise 13.4.I.

10.6.I. EXERCISE (NORMALIZATION IN A FUNCTION FIELD EXTENSION, AN IMPORTANT GENERALIZATION). Suppose X is an integral scheme. The **normalization of** X**,** $\nu : \tilde{X} \to X$**, in a given finite field extension** L **of the function field** $K(X)$ **of** X is a dominant morphism from a normal scheme $\tilde{X}$ with function field L, such that ν induces the inclusion $K(X) \hookrightarrow L$, and that is universal with respect to this property.

$$\begin{tikzcd}
\operatorname{Spec} L = K(Y) \arrow[r] \arrow[d] & Y \arrow[d, dotted, "\exists!"'] \arrow[dd, bend left] & \text{normal} \\
\operatorname{Spec} L = K(\tilde{X}) \arrow[r] \arrow[d] & \tilde{X} \arrow[d] & \text{normal} \\
K(X) \arrow[r] & X &
\end{tikzcd}$$

Show that the normalization in a finite field extension exists.

The following two examples, one arithmetic and one geometric, show that this is an interesting construction.

10.6.J. EXERCISE. Suppose $X = \operatorname{Spec} \mathbb{Z}$ (with function field $\mathbb{Q}$). Find its integral closure in the field extension $\mathbb{Q}(i)$. (There is no "geometric" way to do this; it is purely an algebraic problem, although the answer should be understood geometrically.)

10.6.1. *Remark: rings of integers in number fields.* A finite extension K of $\mathbb{Q}$ is called a **number field**, and the integral closure of $\mathbb{Z}$ in K the **ring of integers in** K, denoted $\mathcal{O}_K$. (This notation is awkward given our other use of the symbol $\mathcal{O}$.)

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} \mathcal{O}_K \arrow[d] \\
\operatorname{Spec} \mathbb{Q} \arrow[r] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

By the previous exercises, $\operatorname{Spec} \mathcal{O}_K$ is a Noetherian normal integral domain of dimension 1. This is an example of a *Dedekind domain*, see §13.4.14. We will think of it as a "smooth" curve as soon as we define what "smooth" (really, nonsingular) and "curve" mean.

10.6.K. EXERCISE. (a) Suppose $X = \operatorname{Spec} k[x]$ (with function field $k(x)$). Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Again we get a Dedekind domain.) Hint: this can be done without too much pain. Show that $\operatorname{Spec} k[x, y]/(x^2 + x - y^2)$ is normal, possibly by identifying it as an open subset of $\mathbb{P}^1_k$, or possibly using Exercise 6.4.H.
(b) Suppose $X = \mathbb{P}^1$, with distinguished open $\operatorname{Spec} k[x]$. Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Part (a) involves computing the

normalization over one affine open set; now figure out what happens over the "other" affine open set.)

10.6.2. Fancy fact: finiteness of integral closure.
The following fact is useful.

10.6.3. Theorem (finiteness of integral closure). — *Suppose A is a Noetherian integral domain, $K = K(A)$, L/K is a finite separable field extension, and B is the integral closure of A in L ("the integral closure of A in the field extension L/K", i.e. those elements of L integral over A).*
(a) If A is integrally closed, then B is a finitely generated A-module.
(b) If A is a finitely generated k-algebra and $L = K$, then B is a finitely generated A-module.

Eisenbud gives a proof in a page and a half: (a) is [**E**, Prop. 13.14] and (b) is [**E**, Cor. 13.13]. A sketch is given in §10.6.4.

Warning: (b) does *not* hold for Noetherian A in general. In fact, the integral closure of a Noetherian ring need not be Noetherian (see [**E**, p. 299] for some discussion). This is alarming. The existence of such an example is a sign that Theorem 10.6.3 is not easy.

10.6.L. EXERCISE. (a) Show that if X is an integral finite-type k-scheme, then its normalization $\nu : \tilde{X} \to X$ is a finite morphism.
(b) Suppose X is an integral scheme. Show that if either X is normal, or X is a finite type k-scheme, then the normalization in a finite field extension is a finite morphism. In particular, the normalization of a variety (including in a finite separable field extension) is a variety.

10.6.M. EXERCISE. Suppose that if X is an integral finite type k-scheme. Show that the normalization map of X is an isomorphism on an open dense subset of X. Hint: reduce to the case $X = \operatorname{Spec} A$. By Theorem 10.6.3, $\tilde{A}$ is generated over A by a finite number of elements of $K(A)$. Let I be the ideal generated by their denominators. Show that $\operatorname{Spec} \tilde{A} \to \operatorname{Spec} A$ is an isomorphism away from $V(I)$. (Alternatively, the ideas of Proposition 11.2.3 can also be applied.)

10.6.4. $\star\star$ *Sketch of proof of finiteness of integral closure, Theorem 10.6.3.* Here is a sketch to show the structure of the argument. It uses commutative algebra ideas from Chapter 12, so you should only glance at this to see that nothing fancy is going on. *Part (a)*: reduce to the case where L/K is Galois, with group $\{\sigma_1, \dots, \sigma_n\}$. Choose $b_1, \dots, b_n \in B$ forming a K-vector space basis of L. Let M be the matrix (familiar from Galois theory) with ijth entry $\sigma_i b_j$, and let $d = \det M$. Show that the entries of M lie in B, and that $d^2 \in K$ (as d^2 is Galois-fixed). Show that $d \neq 0$ using linear independence of characters. Then complete the proof by showing that $B \subset d^{-2}(Ab_1 + \cdots + Ab_n)$ (submodules of finitely generated modules over Noetherian rings are also Noetherian, §4.6.4) as follows. Suppose $b \in B$, and write $b = \sum c_i b_i$ ($c_i \in K$). If c is the column vector with entries c_i, show that the ith entry of the column vector Mc is $\sigma_i b \in B$. Multiplying Mc on the left by $\operatorname{adj} M$ (see the trick of the proof of Lemma 8.2.1), show that $dc_i \in B$. Thus $d^2 c_i \in B \cap K = A$ (as A is integrally closed), as desired.

For *(b)*, use the Noether Normalization Lemma 12.2.7 to reduce to the case $A = k[x_1, \dots, x_n]$. Reduce to the case where L is normally closed over K. Let L' be the subextension of L/K so that L/L' is Galois and L'/K is purely inseparable. Use part (a) to reduce to the case $L = L'$. If $L' \neq K$, then for some q, L' is generated over K by the qth root of a finite set of rational functions. Reduce to the case $L' = k'(x_1^{1/q}, \dots, x_n^{1/q})$ where k'/k is a finite purely inseparable extension. In this case, show that $B = k'[x_1^{1/q}, \dots, x_n^{1/q}]$, which is indeed finite over $k[x_1, \dots, x_n]$.

CHAPTER 11

Separated and proper morphisms, and (finally!) varieties

11.1 Separated morphisms (and quasiseparatedness done properly)

Separatedness is a fundamental notion. It is the analogue of the Hausdorff condition for manifolds (see Exercise 11.1.A), and as with Hausdorffness, this geometrically intuitive notion ends up being just the right hypothesis to make theorems work. Although the definition initially looks odd, in retrospect it is just perfect.

11.1.1. Motivation. Let's review why we like Hausdorffness. Recall that a topological space is *Hausdorff* if for every two points x and y, there are disjoint open neighborhoods of x and y. The real line is Hausdorff, but the "real line with doubled origin" is not (of which Figure 5.4 may be taken as a sketch). Many proofs and results about manifolds use Hausdorffness in an essential way. For example, the classification of compact one-dimensional smooth manifolds is very simple, but if the Hausdorff condition were removed, we would have a very wild set.

So once armed with this definition, we can cheerfully exclude the line with doubled origin from civilized discussion, and we can (finally) define the notion of a *variety*, in a way that corresponds to the classical definition.

With our motivation from manifolds, we shouldn't be surprised that all of our affine and projective schemes are separated: certainly, in the land of smooth manifolds, the Hausdorff condition comes for free for "subsets" of manifolds. (More precisely, if Y is a manifold, and X is a subset that satisfies all the hypotheses of a manifold except possibly Hausdorffness, then Hausdorffness comes for free. Similarly, locally closed immersions in something separated are also separated: combine Exercise 11.1.B and Proposition 11.1.13(a).)

As an unexpected added bonus, a separated morphism to an affine scheme has the property that the intersection of two affine open sets in the source is affine (Proposition 11.1.8). This will make Čech cohomology work very easily on (quasicompact) schemes (Chapter 20). You might consider this an analogue of the fact that in $\mathbb{R}^n$, the intersection of two convex sets is also convex. As affine schemes are trivial from the point of view of quasicoherent cohomology, just as convex sets in $\mathbb{R}^n$ have no cohomology, this metaphor is apt.

A lesson arising from the construction is the importance of the *diagonal morphism*. More precisely, given a morphism $X \to Y$, good consequences can be leveraged from good behavior of the **diagonal morphism** $\delta : X \to X \times_Y X$ (the product of the identity morphism $X \to X$ with itself), usually through fun diagram chases. This lesson applies across many fields of geometry. (Another nice gift of the diagonal morphism: it will give us a good algebraic definition of differentials, in Chapter 23.)

Grothendieck taught us that one should try to define properties of morphisms, not of objects; then we can say that an object has that property if its morphism to the final object has that property. We discussed this briefly at the start of Chapter 8. In this spirit, separatedness will be a property of morphisms, not schemes.

11.1.2. Defining separatedness. Before we define separatedness, we make an observation about all diagonal morphisms.

11.1.3. Proposition. — *Let $\pi : X \to Y$ be a morphism of schemes. Then the diagonal morphism $\delta : X \to X \times_Y X$ is a locally closed immersion.*

We will often use δ to denote a diagonal morphism. This locally closed subscheme of $X \times_Y X$ (which we also call the **diagonal**) will be denoted Δ.

Proof. We will describe a union of open subsets of $X \times_Y X$ covering the image of X, such that the image of X is a closed immersion in this union.

Say Y is covered with affine open sets V_i and X is covered with affine open sets U_{ij}, with $\pi : U_{ij} \to V_i$. Note that $U_{ij} \times_{V_i} U_{ij}$ is an affine open subscheme of the product $X \times_Y X$ (basically this is how we constructed the product, by gluing together affine building blocks). Then the diagonal is covered by these affine open subsets $U_{ij} \times_{V_i} U_{ij}$. (Any point $p \in X$ lies in some U_{ij}; then $\delta(p) \in U_{ij} \times_{V_i} U_{ij}$. Figure 11.1 may be helpful.) Note that $\delta^{-1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij}$: clearly $U_{ij} \subset \delta^{-1}(U_{ij} \times_{V_i} U_{ij})$, and because $pr_1 \circ \delta = id_X$ (where pr_1 is the first projection), $\delta^{-1}(U_{ij} \times_{V_i} U_{ij}) \subset U_{ij}$. Finally, we check that $U_{ij} \to U_{ij} \times_{V_i} U_{ij}$ is a closed immersion. Say $V_i = \operatorname{Spec} B$ and $U_{ij} = \operatorname{Spec} A$. Then this corresponds to the natural ring map $A \otimes_B A \to A$ ($a_1 \otimes a_2 \mapsto a_1 a_2$), which is obviously surjective. $\square$

The open subsets we described may not cover $X \times_Y X$, so we have not shown that δ is a closed immersion.

11.1.4. *Definition.* A morphism $X \to Y$ is **separated** if the diagonal morphism $\delta : X \to X \times_Y X$ is a closed immersion. An A-scheme X is said to be **separated over** A if the structure morphism $X \to \operatorname{Spec} A$ is separated. When people say that a scheme (rather than a morphism) X is separated, they mean implicitly that some "structure morphism" is separated. For example, if they are talking about A-schemes, they mean that X is separated over A.

Thanks to Proposition 11.1.3, a morphism is separated if and only if the diagonal Δ is a closed subset — a purely topological condition on the diagonal. This is reminiscent of a definition of Hausdorff, as the next exercise shows.

11.1.A. UNIMPORTANT EXERCISE (FOR THOSE SEEKING TOPOLOGICAL MOTIVATION). Show that a topological space X is Hausdorff if and only if the diagonal is a closed subset of $X \times X$. (The reason separatedness of schemes doesn't give Hausdorffness — i.e. that for any two open points x and y there aren't necessarily disjoint open neighborhoods — is that in the category of schemes, the topological

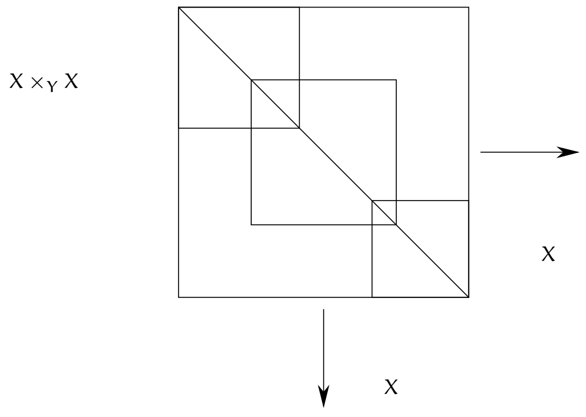


FIGURE 11.1. A neighborhood of the diagonal is covered by $U_{ij} \times_{V_j} U_{ij}$

space $X \times X$ is not in general the product of the topological space X with itself, see §10.1.2.)

11.1.B. IMPORTANT EASY EXERCISE. Show that open immersions, closed immersions, and hence locally closed immersions are separated. (Hint: Do this by hand. Alternatively, show that monomorphisms are separated. Open and closed immersions are monomorphisms, by Exercise 10.2.G.)

11.1.C. IMPORTANT EASY EXERCISE. Show that every morphism of affine schemes is separated. (Hint: this was essentially done in the proof of Proposition 11.1.3.)

11.1.D. EXERCISE. Show that the line with doubled origin X (Example 5.4.5) is not separated, by verifying that the image of the diagonal morphism is not closed. (Another argument is given below, in Exercise 11.1.M. A fancy argument is given in Exercise 13.5.C.)

We next come to our first example of something separated but not affine. The following single calculation will imply that all quasiprojective A-schemes are separated (once we know that the composition of separated morphisms are separated, Proposition 11.1.13).

11.1.5. Proposition. — $\mathbb{P}^n_A \to \operatorname{Spec} A$ *is separated.*

We give two proofs. The first is by direct calculation. The second requires no calculation, and just requires that you remember some classical constructions described earlier.

Proof 1: direct calculation. We cover $\mathbb{P}^n_A \times_A \mathbb{P}^n_A$ with open sets of the form $U_i \times_A U_j$, where $U_0, \dots, U_n$ form the "usual" affine open cover. The case $i = j$ was taken care of before, in the proof of Proposition 11.1.3. If $i \neq j$ then

$$U_i \times_A U_j \cong \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}, y_{0/j}, \dots, y_{n/j}]/(x_{i/i} - 1, y_{j/j} - 1).$$

Now the restriction of the diagonal Δ is contained in U_i (as the diagonal morphism composed with projection to the first factor is the identity), and similarly is contained in U_j. Thus the diagonal morphism over $U_i \times_A U_j$ is $U_i \cap U_j \to U_i \times_A U_j$. This is a closed immersion, as the corresponding map of rings

$$A[x_{0/i}, \dots, x_{n/i}, y_{0/j}, \dots, y_{n/j}] \to A[x_{0/i}, \dots, x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1)$$

(given by $x_{k/i} \mapsto x_{k/i}$, $y_{k/j} \mapsto x_{k/i}/x_{j/i}$) is clearly a surjection (as each generator of the ring on the right is clearly in the image — note that $x_{j/i}^{-1}$ is the image of $y_{i/j}$). $\square$

Proof 2: classical geometry. Note that the diagonal morphism $\delta : \mathbb{P}^n_A \to \mathbb{P}^n_A \times_A \mathbb{P}^n_A$ followed by the Segre embedding $S : \mathbb{P}^n_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{n^2+2n}$ (§10.5, a closed immersion) can also be factored as the second Veronese embedding $\nu_2 : \mathbb{P}^n_A \to \mathbb{P}^{\binom{n+2}{2}-1}$ (§9.2.6) followed by a linear map $L : \mathbb{P}^{\binom{n+2}{2}-1} \to \mathbb{P}^{n^2+2n}$ (another closed immersion, Exercise 9.2.D), both of which are closed immersions.

$$\begin{tikzcd}
& \mathbb{P}^n_A \times_A \mathbb{P}^n_A \arrow[dr, "S"] & \\
\mathbb{P}^n_A \arrow[ur, "\delta"] \arrow[dr, "\nu_2"'] & & \mathbb{P}^{n^2+2n} \\
& \mathbb{P}^{\binom{n+2}{2}-1} \arrow[ur, "L"'] &
\end{tikzcd}$$

Informally, in coordinates:

$$\begin{tikzcd}
& ([x_0; x_1; \cdots; x_n], [x_0; x_1; \cdots; x_n]) \arrow[dr, "S"] & \\
{[x_0; x_1; \cdots; x_n]} \arrow[ur, "\delta"] \arrow[dr, "\nu_2"'] & & \begin{bmatrix} x_0^2; & x_0 x_1; & \cdots & x_0 x_n; \\ x_1 x_0; & x_1^2; & \cdots & x_1 x_n; \\ \vdots & \vdots & \ddots & \vdots \\ x_n x_0; & x_n x_1; & \cdots & x_n^2 \end{bmatrix} \\
& {[x_0^2; x_0 x_1; \cdots; x_{n-1} x_n; x_n^2]} \arrow[ur, "L"'] &
\end{tikzcd}$$

The composed map $\mathbb{P}^n_A$ may be written as $[x_0; \cdots; x_n] \mapsto [x_0 x_0; x_0 x_1; \cdots; x_n x_n]$, where the subscripts on the right run over all ordered pairs (i, j) where $0 \leq i, j \leq n$.) This forces δ to send closed sets to closed sets (or else $S \circ \delta$ won't, but $L \circ \nu_2$ does). $\square$

We note for future reference a minor result proved in the course of Proof 1.

11.1.6. Small Proposition. — *If* U *and* V *are open subsets of an* A*-scheme* X*, then* $\Delta \cap (U \times_A V) \cong U \cap V$.

Figure 11.2 may help show why this is natural. You could also interpret this statement as

$$X \times_{(X \times_A X)} (U \times_A V) \cong U \times_X V$$

which follows from the magic diagram, Exercise 2.3.R.

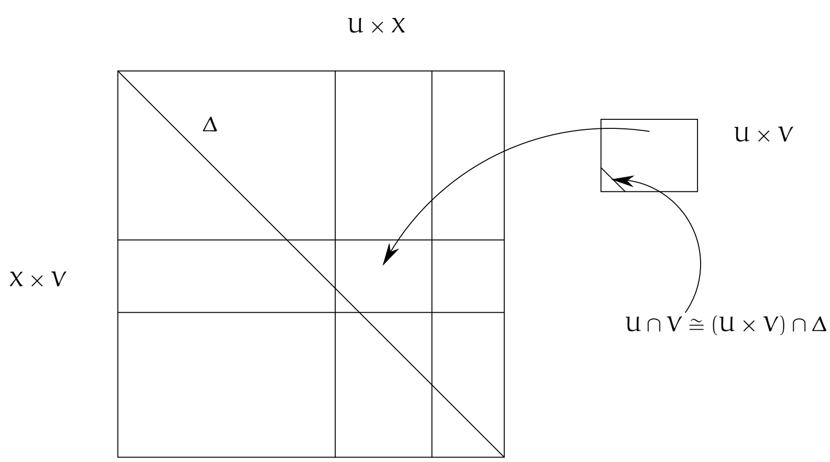


FIGURE 11.2. Small Proposition 11.1.6

We finally define *variety*!

11.1.7. *Definition.* A **variety** over a field k, or k**-variety**, is a reduced, separated scheme of finite type over k. For example, a reduced finite-type affine k-scheme is a variety. We will soon know that the composition of separated morphisms is separated (Exercise 11.1.13(a)), and then to check if $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ is a variety, you need only check reducedness. This generalizes our earlier notion of affine variety (§6.3.6) and projective variety (§6.3.6, see Proposition 11.1.14). (Notational caution: In some sources, the additional condition of irreducibility is imposed. Also, it is often assumed that k is algebraically closed.)

11.1.E. EXERCISE (PRODUCTS OF IRREDUCIBLE VARIETIES OVER $\overline{k}$ ARE IRREDUCIBLE VARIETIES). Use Exercise 10.4.E and properties of separatedness to show that the product of two irreducible $\overline{k}$-varieties is an irreducible $\overline{k}$-variety.

11.1.F. ⋆⋆ EXERCISE (COMPLEX ALGEBRAIC VARIETIES YIELD COMPLEX ANALYTIC VARIETIES; FOR THOSE WITH SUFFICIENT BACKGROUND). Show that the analytification (Exercises 6.3.7 and 7.3.J) of a complex algebraic variety is a complex analytic variety.

Here is a very handy consequence of separatedness.

11.1.8. Proposition. — *Suppose* $X \to \operatorname{Spec} A$ *is a separated morphism to an affine scheme, and* U *and* V *are affine open subsets of* X*. Then* $U \cap V$ *is an affine open subset of* X.

Before proving this, we state a consequence that is otherwise nonobvious. If $X = \operatorname{Spec} A$, then the intersection of any two affine open subsets is an affine open subset (just take $A = \mathbb{Z}$ in the above proposition). This is certainly not an obvious fact! We know the intersection of two distinguished affine open sets is affine (from $D(f) \cap D(g) = D(fg)$), but we have little handle on affine open sets in general.

Warning: this property does not characterize separatedness. For example, if $A = \operatorname{Spec} k$ and X is the line with doubled origin over k, then X also has this property.

Proof. By Proposition 11.1.6, $(U \times_A V) \cap \Delta \cong U \cap V$, where Δ is the diagonal. But $U \times_A V$ is affine (the fibered product of two affine schemes over an affine scheme is affine, Step 1 of our construction of fibered products, Theorem 10.1.1), and Δ is a closed subscheme of an affine scheme, and hence $U \cap V$ is affine. $\square$

11.1.9. Redefinition: Quasiseparated morphisms.

We say a morphism $f : X \to Y$ is **quasiseparated** if the diagonal morphism $\delta : X \to X \times_Y X$ is quasicompact.

11.1.G. EXERCISE. Show that this agrees with our earlier definition of quasiseparated (§8.3.1): show that $f : X \to Y$ is quasiseparated if and only if for any affine open $\operatorname{Spec} A$ of Y, and two affine open subsets U and V of X mapping to $\operatorname{Spec} A$, $U \cap V$ is a *finite* union of affine open sets. (Possible hint: compare this to Proposition 11.1.8. Another possible hint: the magic diagram, Exercise 2.3.R.)

Here are two large classes of morphisms that are quasiseparated.

11.1.H. EASY EXERCISE. Show that separated morphisms are quasiseparated. (Hint: closed immersions are affine, hence quasicompact.)

Second, if X is a Noetherian scheme, then any morphism to another scheme is quasicompact (easy, see Exercise 8.3.B(a)), so any $X \to Y$ is quasiseparated. Hence those working in the category of Noetherian schemes need never worry about this issue.

We now give four quick propositions showing that separatedness and quasiseparatedness behave well, just as many other classes of morphisms did.

11.1.10. Proposition. — *Both separatedness and quasiseparatedness are preserved by base change.*

Proof. Suppose

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

is a fiber diagram. We will show that if $Y \to Z$ is separated or quasiseparated, then so is $W \to X$. Then you can quickly verify that

$$\begin{tikzcd} W \arrow[r, "\delta_W"] \arrow[d] & W \times_X W \arrow[d] \\ Y \arrow[r, "\delta_Y"] & Y \times_Z Y \end{tikzcd}$$

is a fiber diagram. (This is true in any category with fibered products.) As the property of being a closed immersion is preserved by base change (§10.2 **(3)**), if δ_Y is a closed immersion, so is δ_X.

Quasiseparatedness follows in the identical manner, as quasicompactness is also preserved by base change (Exercise 10.4.B). $\square$

11.1.11. Proposition. — *The condition of being separated is local on the target. Precisely, a morphism $f : X \to Y$ is separated if and only if for any cover of Y by open subsets U_i, $f^{-1}(U_i) \to U_i$ is separated for each i.*

11.1.12. Hence affine morphisms are separated, as every morphism of affine schemes is separated (Exercise 11.1.C). In particular, finite morphisms are separated.

Proof. If $X \to Y$ is separated, then for any $U_i \hookrightarrow Y$, $f^{-1}(U_i) \to U_i$ is separated, as separatedness is preserved by base change (Theorem 11.1.10). Conversely, to check if $\Delta \hookrightarrow X \times_Y X$ is a closed subset, it suffices to check this on an open cover of $X \times_Y X$. Let $g : X \times_Y X \to Y$ be the natural map. We will use the open cover $g^{-1}(U_i)$, which by construction of the fiber product is $f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)$. As $f^{-1}(U_i) \to U_i$ is separated, $f^{-1}(U_i) \to f^{-1}(U_i) \times_{U_i} f(U_i)$ is a closed immersion by definition of separatedness. $\square$

11.1.I. EXERCISE. Prove that the condition of being quasiseparated is local on the target. (Hint: the condition of being quasicompact is local on the target by Exercise 8.3.C(a); use a similar argument as in Proposition 11.1.11.)

11.1.13. Proposition. — *(a) The condition of being separated is closed under composition. In other words, if $f : X \to Y$ is separated and $g : Y \to Z$ is separated, then $g \circ f : X \to Z$ is separated.*
(b) The condition of being quasiseparated is closed under composition.

Proof. (a) We are given that $\delta_f : X \hookrightarrow X \times_Y X$ and $\delta_g : Y \to Y \times_Z Y$ are closed immersions, and we wish to show that $\delta_h : X \to X \times_Z X$ is a closed immersion. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\delta_f"] & X \times_Y X \arrow[r, "c"] \arrow[d] & X \times_Z X \arrow[d] \\ & Y \arrow[r, "\delta_g"] & Y \times_Z Y. \end{tikzcd}$$

The square is the magic diagram (Exercise 2.3.R). As δ_g is a closed immersion, c is too (closed immersions are preserved by base change, §10.2 **(3)**). Thus $c \circ \delta_f$

is a closed immersion (the composition of two closed immersions is also a closed immersion, Exercise 9.1.B).

(b) The identical argument (with "closed immersion" replaced by "quasicompact") shows that the condition of being quasiseparated is closed under composition. □

11.1.14. Corollary. — *Any quasiprojective A-scheme is separated over A. In particular, any reduced quasiprojective k-scheme is a k-variety.*

Proof. Suppose $X \to \operatorname{Spec} A$ is a quasiprojective A-scheme. The structure morphism can be factored into an open immersion composed with a closed immersion followed by $\mathbb{P}^n_A \to A$. Open immersions and closed immersions are separated (Exercise 11.1.B), and $\mathbb{P}^n_A \to A$ is separated (Proposition 11.1.5). Compositions of separated morphisms are separated (Proposition 11.1.13), so we are done. □

11.1.15. Proposition. — *Suppose $f : X \to Y$ and $f' : X' \to Y'$ are separated (resp. quasiseparated) morphisms of S-schemes (where S is a scheme). Then the product morphism $f \times f' : X \times_S X' \to Y \times_S Y'$ is separated (resp. quasiseparated).*

Proof. Apply Exercise 10.4.F. □

11.1.16. Applications.

As a first application, we define the *graph morphism*.

11.1.17. *Definition.* Suppose $f : X \to Y$ is a morphism of Z-schemes. The morphism $\Gamma_f : X \to X \times_Z Y$ given by $\Gamma_f = (\mathrm{id}, f)$ is called the **graph morphism**. Then f factors as $\mathrm{pr}_2 \circ \Gamma_f$, where pr_2 is the second projection (see Figure 11.3).

11.1.18. Proposition. — *The graph morphism Γ is always a locally closed immersion. If Y is a separated Z-scheme (i.e. the structure morphism $Y \to Z$ is separated), then Γ is a closed immersion. If Y is a quasiseparated Z-scheme, then Γ is quasicompact.*

This will be generalized in Exercise 11.1.J.

Proof by Cartesian diagram. A special case of the magic diagram (Exercise 2.3.R) is:

(11.1.18.1)
$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_Z Y \arrow[d] \\ Y \arrow[r, "\delta"] & Y \times_Z Y. \end{tikzcd}$$

The notions of locally closed immersion and closed immersion are preserved by base change, so if the bottom arrow δ has one of these properties, so does the top. The same argument establishes the last sentence. □

We now come to a very useful, but bizarre-looking, result. Like the magic diagram, I find this result unexpected useful and ubiquitous.

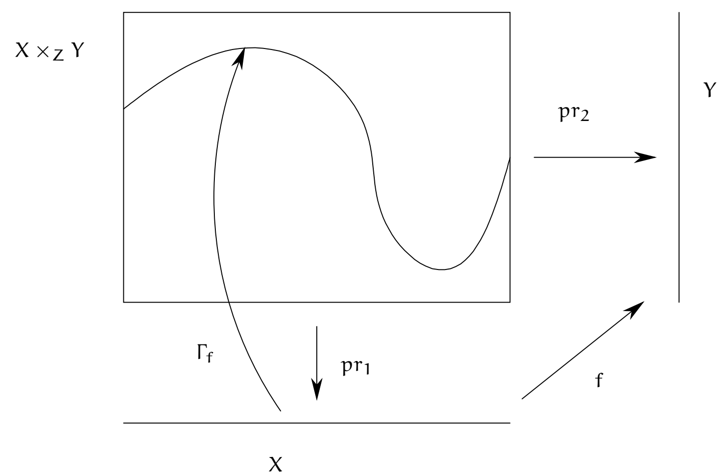


FIGURE 11.3. The graph morphism

11.1.19. Cancellation Theorem for a Property P of Morphisms. — *Let P be a class of morphisms that is preserved by base change and composition. Suppose*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

is a commuting diagram of schemes. Suppose that the diagonal morphism $\delta_g : Y \to Y \times_Z Y$ is in P and $h : X \to Z$ is in P. Then $f : X \to Y$ is in P. In particular:

(i) *Suppose that locally closed immersions are in P. If h is in P, then f is in P.*
(ii) *Suppose that closed immersions are in P (e.g. P could be finite morphisms, morphisms of finite type, closed immersions, affine morphisms). If h is in P and g is separated, then f is in P.*
(iii) *Suppose that quasicompact morphisms are in P. If h is in P and g is quasiseparated, then f is in P.*

The following diagram summarizes this important theorem:

$$\begin{tikzcd} X \arrow[rr, "\in P"] \arrow[dr, "\therefore \in P"'] & & Y \arrow[dl, "\delta \in P"] \\ & Z & \end{tikzcd}$$

When you plug in different P, you get very different-looking (and nonobvious) consequences. For example, if you factor a locally closed immersion $X \to Z$ into $X \to Y \to Z$, then $X \to Y$ *must* be a locally closed immersion.

Proof. By the graph Cartesian diagram (11.1.18.1)

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"] & X \times_Z Y \arrow[d] \\ Y \arrow[r, "\delta_g"] & Y \times_Z Y \end{tikzcd}$$

we see that the graph morphism $\Gamma_f : X \to X \times_Z Y$ is in P (Definition 11.1.17), as P is closed under base change. By the fibered square

$$\begin{tikzcd} X \times_Z Y \arrow[r, "h'"] \arrow[d] & Y \arrow[d, "g"'] \\ X \arrow[r, "h"] & Z \end{tikzcd}$$

the projection $h' : X \times_Z Y \to Y$ is in P as well. Thus $f = h' \circ \Gamma_f$ is in P $\square$

Here now are some fun and useful exercises.

11.1.J. EXERCISE. Suppose $\pi : Y \to X$ is a morphism, and $s : X \to Y$ is a *section* of a morphism, i.e. $\pi \circ s$ is the identity on X. Show that s is a locally closed immersion. Show that if π is separated, then s is a closed immersion. (This generalizes Proposition 11.1.18.) Give an example to show that s needn't be a closed immersion if π isn't separated.

11.1.K. LESS IMPORTANT EXERCISE. Show that an A-scheme is separated (over A) if and only if it is separated over $\mathbb{Z}$. In particular, a complex scheme is separated over $\mathbb{C}$ if and only if it is separated over $\mathbb{Z}$, so complex geometers and arithmetic geometers can communicate about separated schemes without confusion.

11.1.L. USEFUL EXERCISE: THE LOCUS WHERE TWO MORPHISMS AGREE. Suppose $f : X \to Y$ and $g : X \to Y$ are two morphisms over some scheme Z. We can now give meaning to the phrase 'the locus where f and g agree', and that in particular there is a largest locally closed subscheme where they agree — and even a closed immersion if Y is separated over Z. Suppose $h : W \to X$ is some morphism (perhaps a locally closed immersion). We say that f and g agree on h if $f \circ h = g \circ h$. Show that there is a locally closed subscheme $i : V \hookrightarrow X$ such that any morphism $h : W \to X$ on which f and g agree factors uniquely through i, i.e. there is a unique $j : W \to V$ such that $h = i \circ j$. Show further that if $Y \to Z$ is separated, then $i : V \hookrightarrow X$ is a closed immersion. Hint: define V to be the following fibered product:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d, "\delta"] \\ X \arrow[r, "(f,g)"] & Y \times_Z Y. \end{tikzcd}$$

As δ is a locally closed immersion, $V \to X$ is too. Then if $h : W \to X$ is any scheme such that $g \circ h = f \circ h$, then h factors through V.

Minor Remarks. 1) In the previous exercise, we are describing $V \hookrightarrow X$ by way of a universal property. Taking this as the definition, it is not a priori clear that V is a locally closed subscheme of X, or even that it exists.

2) Warning: consider two maps from $\operatorname{Spec}\mathbb{C}$ to itself $\operatorname{Spec}\mathbb{C}$ over $\operatorname{Spec}\mathbb{R}$, the identity and complex conjugation. These are both maps from a point to a point, yet they do not agree despite agreeing as maps of sets. (If you do not find this reasonable, this might help: after base change $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$, they do not agree as maps of sets.)

3) More generally, in the case of reduced finite type k-schemes, the locus where f and g agree can be interpreted as follows: f and g agree at x if $f(x) = g(x)$ and the two maps of residue fields are the same.

11.1.M. LESS IMPORTANT EXERCISE. Show that the line with doubled origin X (Example 5.4.5) is not separated, by finding two morphisms $f_1 : W \to X$, $f_2 : W \to X$ whose domain of agreement is not a closed subscheme (cf. Proposition 11.1.3). (Another argument was given above, in Exercise 11.1.D. A fancy argument will be given in Exercise 13.5.C.)

11.1.N. LESS IMPORTANT EXERCISE. Suppose P is a class of morphisms such that closed immersions are in P, and P is closed under fibered product and composition. Show that if $f : X \to Y$ is in P then $f^{red} : X^{red} \to Y^{red}$ is in P. (Two examples are the classes of separated morphisms and quasiseparated morphisms.) Hint:

$$\begin{tikzcd} X^{red} \arrow[r] \arrow[dr] & X \times_Y Y^{red} \arrow[r] \arrow[d] & Y^{red} \arrow[d] \\ & X \arrow[r] & Y \end{tikzcd}$$

11.2 Rational maps to separated schemes

When we introduced rational maps in §7.5, we promised that in good circumstances, a rational map has a "largest domain of definition". We are now ready to make precise what "good circumstances" means.

11.2.1. Reduced-to-separated Theorem (important!). — *Two S-morphisms $f_1 : U \to Z$, $f_2 : U \to Z$ from a reduced scheme to a separated S-scheme agreeing on a dense open subset of U are the same.*

Proof. Let V be the locus where f_1 and f_2 agree. It is a closed subscheme of U by Exercise 11.1.L, which contains a dense open set. But the only closed subscheme of a reduced scheme U whose underlying set is dense is all of U. $\square$

11.2.2. *Consequence 1.* Hence (as X is reduced and Y is separated) if we have two morphisms from open subsets of X to Y, say $f : U \to Y$ and $g : V \to Y$, and they agree on a dense open subset $Z \subset U \cap V$, then they necessarily agree on $U \cap V$.

Consequence 2. A rational map has a largest **domain of definition** on which $f : U \dashrightarrow Y$ is a morphism, which is the union of all the domains of definition. In particular, a rational function on a reduced scheme has a largest domain of definition. For example, the domain of definition of $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$

has domain of definition $\mathbb{A}^2_k \setminus \{(0,0)\}$ (cf. §7.5.3). This partially extends the definition of the domain of function of a rational function on a locally Noetherian scheme (Definition 6.5.2). The complement of the domain of definition is called the **locus of indeterminacy**, and its points are sometimes called **fundamental points** of the rational map, although we won't use these phrases. (We will see in Exercise 19.4.L that a rational map to a projective scheme can be upgraded to an honest morphism by "blowing up" a scheme-theoretic version of the locus of indeterminacy.)

11.2.A. EXERCISE. Show that the Reduced-to-separated Theorem 11.2.1 is false if we give up reducedness of the source or separatedness of the target. Here are some possibilities. For the first, consider the two maps from $\operatorname{Spec} k[x,y]/(y^2, xy)$ to $\operatorname{Spec} k[t]$, where we take f_1 given by $t \mapsto x$ and f_2 given by $t \mapsto x+y$; f_1 and f_2 agree on the distinguished open set $D(x)$, see Figure 11.4. For the second, consider the two maps from $\operatorname{Spec} k[t]$ to the line with the doubled origin, one of which maps to the "upper half", and one of which maps to the "lower half". These two morphisms agree on the dense open set $D(f)$, see Figure 11.5.

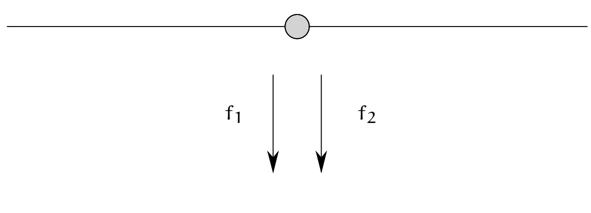


FIGURE 11.4. Two different maps from a nonreduced scheme agreeing on a dense open set

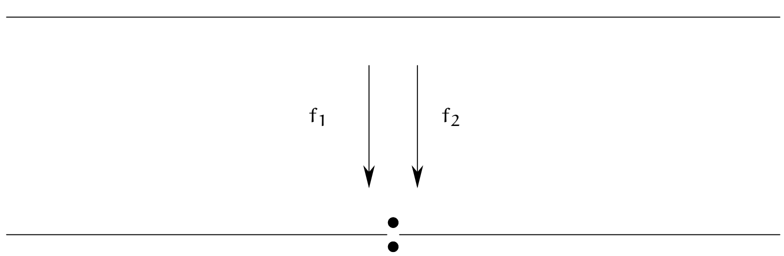


FIGURE 11.5. Two different maps to a nonseparated scheme agreeing on a dense open set

11.2.3. Proposition. — *Suppose Y and Z are integral separated schemes. Then Y and Z are birational if and only if there is a dense (=non-empty) open subscheme U of Y and a dense open subscheme V of Z such that $U \cong V$.*

This gives you a good idea of how to think of birational maps. For example, a variety is rational if it has a dense open subscheme isomorphic to an open subscheme of $\mathbb{A}^n$.

Proof. I find this proof surprising and unexpected.

Clearly if Y and Z have isomorphic open sets U and V respectively, then they are birational (with birational maps given by the isomorphisms $U \to V$ and $V \to U$ respectively).

For the other direction, assume that $f : Y \dashrightarrow Z$ is a birational map, with inverse birational map $g : Z \dashrightarrow Y$. Choose representatives for these rational maps $F : W \to Z$ (where W is an open subscheme of Y) and $G : X \to Y$ (where X is an open subscheme of Z). We will see that $F^{-1}(G^{-1}(W)) \subset Y$ and $G^{-1}(F^{-1}(X)) \subset Z$ are isomorphic open subschemes.

$$\begin{tikzcd}
F^{-1}(G^{-1}(W)) \arrow[d, hook] \arrow[dr, "F"] & G^{-1}(F^{-1}(X)) \arrow[d, hook] \arrow[dl, "G"'] \\
F^{-1}(X) \arrow[d, hook] \arrow[dr, "F"] & G^{-1}(W) \arrow[d, hook] \arrow[dl, "G"'] \\
W \arrow[d, hook] \arrow[dr, "F"] & X \arrow[d, hook] \arrow[dl, "G"'] \\
Y & Z
\end{tikzcd}$$

The key observation is that the two morphisms $G \circ F$ and the identity from $F^{-1}(G^{-1}(W)) \to W$ represent the same rational map, so by the Reduced-to-separated Theorem 11.2.1 they are the same morphism on $F^{-1}(G^{-1}(W))$. Thus $G \circ F$ gives the identity map from $F^{-1}(G^{-1}(W))$ to itself. Similarly $F \circ G$ gives the identity map on $G^{-1}(F^{-1}(X))$.

All that remains is to show that F maps $F^{-1}(G^{-1}(W))$ into $G^{-1}(F^{-1}(X))$, and that G maps $G^{-1}(F^{-1}(X))$ into $F^{-1}(G^{-1}(W))$, and by symmetry it suffices to show the former. Suppose $q \in F^{-1}(G^{-1}(W))$. Then $F(G(F(q)) = F(q) \in X$, from which $F(q) \in G^{-1}(F^{-1}(X))$. (Another approach is to note that each "parallelogram" in the diagram above is a fibered diagram, and to use the key observation of the previous paragraph to construct a morphism $G^{-1}(F^{-1}(X)) \to F^{-1}(G^{-1}(X))$ and vice versa, and showing that they are inverses.) $\square$

11.2.4. Variations.

Variations of the short proof of Theorem 11.2.1 yield other useful theorems.

11.2.B. EXERCISE: MAPS OF VARIETIES ARE DETERMINED BY THE MAPS ON CLOSED POINTS. Suppose $f_1 : X \to Y$ and $f_2 : X \to Y$ are two maps of varieties over $\overline{k}$, such that $f_1(p) = f_2(p)$ for all closed points. Show that $f_1 = f_2$. (This implies that the functor from the category of "classical varieties over $\overline{k}$", which we won't define here, to the category of $\overline{k}$-schemes, is fully faithful. Can you generalize this appropriately to non-algebraically closed fields?)

11.2.C. EXERCISE (MAPS TO A SEPARATED SCHEME CAN BE EXTENDED OVER AN EFFECTIVE CARTIER DIVISOR IN AT MOST ONE WAY). Suppose $\sigma : X \to Z$ and

$\tau : Y \to Z$ are two morphisms, and τ is separated. Suppose further that D is an effective Cartier divisor on X. Show that any Z-morphism $X \setminus D \to Y$ can be extended in at most one way to a Z-morphism $X \to Y$. (Hint: reduce to the case where $X = \operatorname{Spec} A$, and D is the vanishing scheme of $t \in A$. Reduce to showing that the scheme-theoretic $D(t)$ in X is all of X. Show this by showing that $R \to R_t$ is an inclusion.)

As noted in §7.5.2, rational maps can be defined from any X that has associated points to any Y. The Reduced-to-separated Theorem 11.2.1 can be extended to this setting, as follows.

11.2.D. EXERCISE (THE "ASSOCIATED-TO-SEPARATED THEOREM"). Prove that two S-morphisms $f_1 : U \to Z$ and $f_2 : U \to Z$ from a locally Noetherian scheme X to a separated S-scheme, agreeing on a dense open subset of U containing the associated points of X, are the same.

11.3 Proper morphisms

Recall that a map of topological spaces (also known as a continuous map!) is said to be *proper* if the preimage of any compact set is compact. *Properness* of morphisms is an analogous property. For example, a variety over $\mathbb{C}$ will be proper if it is compact in the classical topology. Alternatively, we will see that projective A-schemes are proper over A — this is the hardest thing we will prove — so you can see this as a nice property satisfied by projective schemes, and quite convenient to work with.

Recall (§8.3.6) that a (continuous) map of topological spaces $f : X \to Y$ is *closed* if for each closed subset $S \subset X$, $f(S)$ is also closed. A morphism of schemes is closed if the underlying continuous map is closed. We say that a morphism of schemes $f : X \to Y$ is **universally closed** if for every morphism $g : Z \to Y$, the induced morphism $Z \times_Y X \to Z$ is closed. In other words, a morphism is universally closed if it remains closed under any base change. (More generally, if P is some property of schemes, then a morphism of schemes is said to be **universally** P if it remains P under any base change.)

To motivate the definition of properness, we remark that a map $f : X \to Y$ of locally compact Hausdorff spaces which have countable bases for their topologies is universally closed if and only if it is proper in the usual topology. (You are welcome to prove this as an exercise.)

11.3.1. Definition. *A morphism $f : X \to Y$ is* **proper** *if it is separated, finite type, and universally closed.* A scheme X is often said to be proper if some implicit structure morphism is proper. For example, a k-scheme X is often described as proper if $X \to \operatorname{Spec} k$ is proper. (A k-scheme is often said to be **complete** if it is proper. We will not use this terminology.)

Let's try this idea out in practice. We expect that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper, because the complex manifold corresponding to $\mathbb{A}^1_{\mathbb{C}}$ is not compact. However, note that this map is separated (it is a map of affine schemes), finite type, and (trivially) closed. So the "universally" is what matters here.

11.3.A. EXERCISE. Show that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper, by finding a base change that turns this into a non-closed map. (Hint: Consider a well-chosen map $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ or $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.)

11.3.2. As a first example: closed immersions are proper. They are clearly separated, as affine morphisms are separated, §11.1.12. They are finite type. After base change, they remain closed immersions, and closed immersions are always closed. This easily extends further as follows.

11.3.3. Proposition. — *Finite morphisms are proper.*

Proof. Finite morphisms are separated (as they are affine by definition, and affine morphisms are separated, §11.1.12), and finite type (basically because finite modules over a ring are automatically finitely generated). To show that finite morphism are closed after any base change, we note that they remain finite after any base change (finiteness is preserved by base change, Exercise 10.4.B(d)), and finite morphisms are closed (Exercise 8.3.M). □

11.3.4. Proposition. —

(a) *The notion of "proper morphism" is stable under base change.*
(b) *The notion of "proper morphism" is local on the target (i.e.* $f : X \to Y$ *is proper if and only if for any affine open cover* $U_i \to Y$, $f^{-1}(U_i) \to U_i$ *is proper). Note that the "only if" direction follows from (a) — consider base change by* $U_i \hookrightarrow Y$.
(c) *The notion of "proper morphism" is closed under composition.*
(d) *The product of two proper morphisms is proper: if* $f : X \to Y$ *and* $g : X' \to Y'$ *are proper, where all morphisms are morphisms of* Z*-schemes, then* $f \times g : X \times_Z X' \to Y \times_Z Y'$ *is proper.*
(e) *Suppose*

(11.3.4.1)
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

is a commutative diagram, and g *is proper, and* h *is separated. Then* f *is proper.*

A sample application of (e): a morphism (over $\operatorname{Spec} k$) from a proper k-scheme to a separated k-scheme is always proper.

Proof. (a) The notions of separatedness, finite type, and universal closedness are all preserved by fibered product. (Notice that this is why universal closedness is better than closedness — it is automatically preserved by base change!)

(b) We have already shown that the notions of separatedness and finite type are local on the target. The notion of closedness is local on the target, and hence so is the notion of universal closedness.

(c) The notions of separatedness, finite type, and universal closedness are all preserved by composition.

(d) By (a) and (c), this follows from Exercise 10.4.F.

(e) Closed immersions are proper, so we invoke the Cancellation Theorem 11.1.19 for proper morphisms. □

We now come to the most important example of proper morphisms.

11.3.5. Theorem. — *Projective A-schemes are proper over A.*

(As finite morphisms to $\operatorname{Spec} A$ are projective A-schemes, Exercise 8.3.J, Theorem 11.3.5 can be used to give a second proof that finite morphisms are proper, Proposition 11.3.3.)

11.3.6. *Remark: "Reasonable" proper schemes are projective.* It is not easy to come up with an example of an A-scheme that is proper but not projective! Over a field, all proper curves are projective (we will see this in Exercise 20.6.E), and all smooth surfaces over a field are projective. (Smoothness of course is not yet defined.) We will meet a first example of a proper but not projective variety (a singular threefold) in §17.4.8. We will later see an example of a proper nonprojective surface in Exercise 22.2.G. Once we know about flatness, we will see Hironaka's example of a proper nonprojective irreducible nonsingular ("smooth") threefold over $\mathbb{C}$ (§25.6.8).

Proof. The structure morphism of a projective A-scheme $X \to \operatorname{Spec} A$ factors as a closed immersion followed by $\mathbb{P}^n_A \to \operatorname{Spec} A$. Closed immersions are proper, and compositions of proper morphisms are proper, so it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper. We have already seen that this morphism is finite type (Easy Exercise 6.3.I) and separated (Prop. 11.1.5), so it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is universally closed. As $\mathbb{P}^n_A = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} A$, it suffices to show that $\mathbb{P}^n_X := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} X \to X$ is closed for any scheme X. But the property of being closed is local on the target on X, so by covering X with affine open subsets, it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is closed. This is the Fundamental Theorem of Elimination Theory (Theorem 8.4.5). $\square$

11.3.7. Facts (not yet proved) that may help you correctly think about finiteness. The following facts may shed some light on the notion of finiteness. We will prove them later.

A morphism is finite if and only if it is proper and affine, if and only if it is proper and quasifinite. We have verified the "only if" parts of this statement; the "if" parts are harder (and involve Zariski's Main Theorem, cf. §8.3.12).

As an application: quasifinite morphisms from proper schemes to separated schemes are finite. Here is why: suppose $f : X \to Y$ is a quasifinite morphism over Z, where X is proper over Z. Then by the Cancellation Theorem 11.1.19 for proper morphisms, $X \to Y$ is proper. Hence as f is quasifinite and proper, f is finite.

As an explicit example, consider the map $\pi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $[x;y] \mapsto [f(x,y);g(x,y)]$, where f and g are homogeneous polynomials of the same degree with no common roots in $\mathbb{P}^1$. The fibers are finite, and π is proper (from the Cancellation Theorem 11.1.19 for proper morphisms, as discussed after the statement of Theorem 11.3.4), so π is finite. This could be checked directly as well, but now we can save ourselves the annoyance.

Part IV

Harder properties of schemes

CHAPTER 12

Dimension

12.1 Dimension and codimension

Everyone knows what a curve is, until he has studied enough mathematics to become confused ... – F. Klein

At this point, you know a fair bit about schemes, but there are some fundamental notions you cannot yet define. In particular, you cannot use the phrase "smooth surface", as it involves the notion of dimension and of smoothness. You may be surprised that we have gotten so far without using these ideas. You may also be disturbed to find that these notions can be subtle, but you should keep in mind that they are subtle in all parts of mathematics.

In this chapter, we will address the first notion, that of dimension of schemes. This should agree with, and generalize, our geometric intuition. Although we think of dimension as a basic notion in geometry, it is a slippery concept, as it is throughout mathematics. Even in linear algebra, the definition of dimension of a vector space is surprising the first time you see it, even though it quickly becomes second nature. The definition of dimension for manifolds is equally nontrivial. For example, how do we know that there isn't an isomorphism between some 2-dimensional manifold and some 3-dimensional manifold? Your answer will likely use topology, and hence you should not be surprised that the notion of dimension is often quite topological in nature.

A caution for those thinking over the complex numbers: our dimensions will be algebraic, and hence half that of the "real" picture. For example, we will see very shortly that $\mathbb{A}^1_{\mathbb{C}}$, which you may picture as the complex numbers (plus one generic point), has dimension 1.

12.1.1. *Definition(s): dimension.* Surprisingly, the right definition is purely topological — it just depends on the topological space, and not on the structure sheaf. We define the **dimension** of a topological space X (denoted $\dim X$) as the supremum of lengths of chains of closed irreducible sets, starting the indexing with 0. (The dimension may be infinite.) Scholars of the empty set can take the dimension of the empty set to be $-\infty$. Define the **dimension** of a ring as the Krull dimension of its spectrum — the supremum of the lengths of the chains of nested prime ideals (where indexing starts at zero). These two definitions of dimension are sometimes called **Krull dimension**. (You might think a Noetherian ring has finite dimension because all chains of prime ideals are finite, but this isn't necessarily true — see Exercise 12.1.H.)

12.1.A. EASY EXERCISE. Show that $\dim \operatorname{Spec} A = \dim A$. (Hint: Exercise 4.7.E gives a bijection between irreducible closed subsets of $\operatorname{Spec} A$ and prime ideals of A. It is "inclusion-reversing".)

The homeomorphism between $\operatorname{Spec} A$ and $\operatorname{Spec} A/\mathfrak{N}(A)$ (§4.4.5: the Zariski topology disregards nilpotents) implies that $\dim \operatorname{Spec} A = \dim \operatorname{Spec} A/\mathfrak{N}(A)$.

Examples. We have identified all the prime ideals of $k[t]$ (they are 0, and $(f(t))$ for irreducible polynomials $f(t)$), $\mathbb{Z}$ ((0) and (p)), k (only (0)), and $k[x]/(x^2)$ (only (x)), so we can quickly check that $\dim \mathbb{A}^1_k = \dim \operatorname{Spec} \mathbb{Z} = 1$, $\dim \operatorname{Spec} k = 0$, $\dim \operatorname{Spec} k[x]/(x^2) = 0$.

We must be careful with the notion of dimension for reducible spaces. If Z is the union of two closed subsets X and Y, then $\dim_Z = \max(\dim X, \dim Y)$. Thus dimension is not a "local" characteristic of a space. This sometimes bothers us, so we try to only talk about dimensions of irreducible topological spaces. If a topological space can be expressed as a finite union of irreducible subsets, then we say that it is **equidimensional** or **pure dimensional** (resp. equidimensional of dimension n or pure dimension n) if each of its components has the same dimension (resp. they are all of dimension n).

An equidimensional dimension 1 (resp. 2, n) topological space is said to be a **curve** (resp. **surface**, **n-fold**).

12.1.B. IMPORTANT EXERCISE. Show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to an integral *extension* of rings, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$. Hint: show that a chain of prime ideals downstairs gives a chain upstairs of the same length, by the Going-up Theorem (Exercise 8.2.F). Conversely, a chain upstairs gives a chain downstairs. We need to check that no two elements of the chain upstairs goes to the same element $[\mathfrak{q}] \in \operatorname{Spec} B$ of the chain downstairs. As integral extensions are preserved by localization and quotients of $\operatorname{Spec} B$ (Exercise 8.2.B), we can replace B by $B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ (and A by $A \otimes_B (B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}})$). **[This is wrong and needs fixing: integral extensions are not preserved by quotients.]** Thus we can assume B is a field. Hence we must show that if $\phi : k \to A$ is an integral extension, then $\dim A = 0$. Outline of proof: Suppose $\mathfrak{p} \subset \mathfrak{m}$ are two prime ideals of A. Mod out by $\mathfrak{p}$, so we can assume that A is a domain. I claim that any non-zero element is invertible: Say $x \in A$, and $x \neq 0$. Then the minimal monic polynomial for x has non-zero constant term. But then x is invertible — recall the coefficients are in a field.

12.1.C. EXERCISE. Show that if $\tilde{X} \to X$ is the normalization of a scheme (possibly in a finite field extension), then $\dim \tilde{X} = \dim X$.

12.1.D. EXERCISE. Suppose X is a k-scheme of *pure* dimension n, and $k \subset K$ is a field extension. Show that $X_K := X \times_k K$ also has pure dimension n.

12.1.E. EXERCISE. Show that $\dim \mathbb{Z}[x] = 2$. (Hint: The primes of $\mathbb{Z}[x]$ were implicitly determined in Exercise 4.2.N.)

12.1.2. Codimension. Because dimension behaves oddly for disjoint unions, we need some care when defining codimension, and in using the phrase. For example, if Y is a closed subset of X, we might define the codimension to be $\dim X - \dim Y$, but this behaves badly. For example, if X is the disjoint union of a point Y and a curve Z, then $\dim X - \dim Y = 1$, but this has nothing to do with the local behavior of X near Y.

A better definition is as follows. In order to avoid excessive pathology, we define the codimension of Y in X *only when* Y *is irreducible*. (Use extreme caution when using this word in any other setting.) Define the **codimension of an irreducible closed subset** $Y \subset X$ of a topological space as the supremum of lengths of *increasing* chains of irreducible closed subsets starting with Y (where indexing starts at 0). So the **codimension of a point** is the codimension of its closure.

We say that a prime ideal $\mathfrak{p}$ in a ring has **codimension** (denoted codim) equal to the supremum of lengths of the chains of decreasing prime ideals starting at $\mathfrak{p}$, with indexing starting at 0. Thus in an integral domain, the ideal (0) has codimension 0; and in $\mathbb{Z}$, the ideal (23) has codimension 1. Note that the codimension of the prime ideal $\mathfrak{p}$ in A is $\dim A_{\mathfrak{p}}$ (see §4.2.6). (This notion is often called **height**.) Thus the codimension of $\mathfrak{p}$ in A is the codimension of $[\mathfrak{p}]$ in Spec A.

12.1.F. EXERCISE. Show that if Y is an irreducible closed subset of a scheme X with generic point η, then the codimension of Y is the dimension of the local ring $\mathcal{O}_{X,\eta}$ (cf. §4.2.6).

Notice that Y is codimension 0 in X if it is an irreducible component of X. Similarly, Y is codimension 1 if it is strictly contained in an irreducible component Y', and there is no irreducible subset strictly between Y and Y'. (See Figure 12.1 for examples.) An closed subset all of whose irreducible components are codimension 1 in some ambient space X is said to be a **hypersurface** in X.

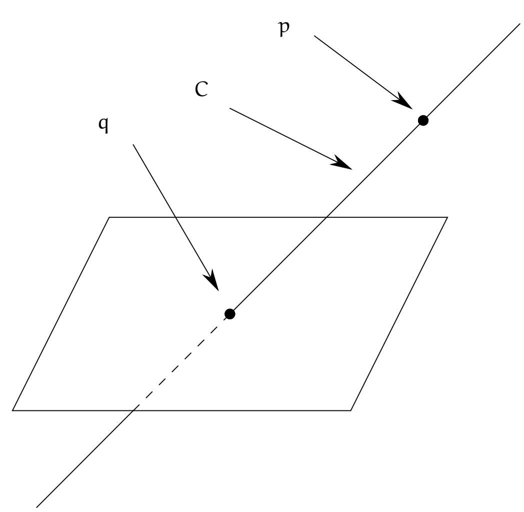


FIGURE 12.1. Behavior of codimension

12.1.G. EASY EXERCISE. Show that

$$\operatorname{codim}_X Y + \dim Y \leq \dim X. \tag{12.1.2.1}$$

We will soon see that equality always holds if X and Y are varieties (Exercise 12.2.E), but equality doesn't hold in general (§12.3.8).

Warning. The notion of codimension still can behave slightly oddly. For example, consider Figure 12.1. (You should think of this as an intuitive sketch.) Here the total space X has dimension 2, but point p is dimension 0, and codimension 1. We also have an example of a codimension 2 subset q contained in a codimension 0 subset C with no codimension 1 subset "in between".

Worse things can happen; we will soon see an example of a closed point in an *irreducible* surface that is nonetheless codimension 1, not 2, in §12.3.8. However, for irreducible *varieties* this can't happen, and inequality (12.1.2.1) must be an equality (Proposition 12.2.E).

12.1.3. *A fun but unimportant counterexample.* We end this introductory section with a fun pathology. As a Noetherian ring has no infinite chain of prime ideals, you may think that Noetherian rings must have finite dimension. Nagata, the master of counterexamples, shows you otherwise with the following example.

12.1.H. ⋆⋆ EXERCISE: AN INFINITE-DIMENSIONAL NOETHERIAN RING. Let $A = k[x_1, x_2, \dots]$. Choose an increasing sequence of positive integers $m_1, m_2, \dots$ whose *differences* are also increasing ($m_{i+1} - m_i > m_i - m_{i-1}$). Let $\mathfrak{p}_i = (x_{m_i+1}, \dots, x_{m_{i+1}})$ and $S = A - \cup_i \mathfrak{p}_i$. Show that S is a multiplicative set. Show that $S^{-1}A$ is Noetherian. Show that each $S^{-1}\mathfrak{p}$ is the smallest prime ideal in a chain of prime ideals of length $m_{i+1} - m_i$. Hence conclude that $\dim S^{-1}A = \infty$.

12.1.4. *Remark: local Noetherian rings have finite dimension.* However, we shall see in Exercise 12.3.G(a) that Noetherian *local* rings always have finite dimension. (This requires a surprisingly hard fact, a form of Krull's Ideal Theorem, Theorem 12.3.7.) Thus points of locally Noetherian schemes always have finite codimension.

12.2 Dimension, transcendence degree, and Noether normalization

We now prove a powerful alternative interpretation for dimension for irreducible varieties, in terms of transcendence degree. In case you haven't seen transcendence theory, here is a lightning introduction.

12.2.A. EXERCISE/DEFINITION. An element of a field extension E/F is *algebraic* over F if it is integral over F. A field extension is *algebraic* if it is integral. The composition of two algebraic extensions is algebraic, by Exercise 8.2.C. If E/F is a field extension, and F' and F'' are two intermediate field extensions, then we write $F' \sim F''$ if $F'F''$ is algebraic over both F' and F''. Here $F'F''$ is the *compositum* of F' and F'', the smallest field extension in E containing F' and F''. *(a) Show that $\sim$ is an equivalence relation on subextensions of* E/F. A **transcendence basis** of E/F is a set of elements $\{x_i\}$ that are algebraically independent over F (there is no nontrivial polynomial relation among the x_i with coefficients in F) such that $F(\{x_i\}) \sim E$. *(b) Show that the if* E/F *has two transcendence bases, and one has cardinality* n*, then both have cardinality* n. (Hint: show that you can substitute elements from the one basis into the other one at a time.) The size of any transcendence basis is called the

transcendence degree (which may be ∞), and is denoted tr. deg. Any finitely generated field extension necessarily has finite transcendence degree.

12.2.1. Theorem (dimension = transcendence degree). — *Suppose A is a finitely-generated integral domain over a field k. Then* $\dim \operatorname{Spec} A = \operatorname{tr.deg} K(A)/k$.

By "finitely generated domain over k", we mean "a finitely generated k-algebra that is an integral domain".

We will prove Theorem 12.2.1 shortly (§12.2.10). But we first show that it is useful by giving some immediate consequences. We seem to have immediately $\dim \mathbb{A}^n_k = n$. However, our proof of Theorem 12.2.1 will go *through* this fact, so it isn't really a Corollary. Instead, we begin with a proof of the Nullstellensatz, promised earlier.

12.2.B. EXERCISE: THE NULLSTELLENSATZ FROM DIMENSION THEORY. Prove Hilbert's Nullstellensatz 4.2.3: Suppose $A = k[x_1, \dots, x_n]/I$. Show that the residue field of any maximal ideal of A is a finite extension of k. (Hint: the maximal ideals correspond to dimension 0 points, which correspond to transcendence degree 0 extensions of k, i.e. finite extensions of k.)

12.2.C. EXERCISE. If X is an irreducible finite type k-scheme, and U is a nonempty open subset of X, show that $\dim U = \dim X$. (Warning: this is false without the finite type hypothesis, even in quite reasonable circumstances: let X be the two-point space $\operatorname{Spec} k[x]_{(x)}$, and U be the generic point, see Exercise 4.4.J.)

For further applications, we make a short observation.

12.2.2. Lemma. — *In a unique factorization domain A, all codimension 1 prime ideals are principal.*

We will see that the converse (when A is a Noetherian integral domain) holds as well (Proposition 12.3.5).

Proof. Suppose $\mathfrak{p}$ is a codimension 1 prime. Choose any $f \neq 0$ in $\mathfrak{p}$, and let g be any irreducible/prime factor of f that is in $\mathfrak{p}$ (there is at least one). Then (g) is a prime ideal contained in $\mathfrak{p}$, so $(0) \subset (g) \subset \mathfrak{p}$. As $\mathfrak{p}$ is codimension 1, we must have $\mathfrak{p} = (g)$, and thus $\mathfrak{p}$ is principal. $\square$

12.2.3. *Points of $\mathbb{A}^2_k$.* We can find a second proof that we have named all the primes of $k[x, y]$ where k is algebraically closed (promised in Exercise 4.2.D when $k = \mathbb{C}$). Recall that we have discovered the primes (0), $f(x, y)$ where f is irreducible, and $(x - a, y - b)$ where $a, b \in k$. As $\mathbb{A}^2_k$ is irreducible, there is only one irreducible subset of codimension 0. By Lemma 12.2.2, all codimension 1 primes are principal. By inequality (12.1.2.1), there are no primes of codimension greater than 2, and any prime of codimension 2 must be maximal. We have identified all the maximal ideals of $k[x, y]$ by the Nullstellensatz.

12.2.D. EXERCISE. Suppose X is an irreducible variety. Show that $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X. (This is not true in general, see §12.3.8.)

12.2.E. EXERCISE. Suppose $Y \subset X$ is an inclusion of irreducible k-varieties, and η is the generic point of Y. Show that $\dim Y + \dim \mathcal{O}_{X,\eta} = \dim X$. Hence by Exercise 12.1.F, $\dim Y + \operatorname{codim}_X Y = \dim X$. Thus for varieties, the inequality (12.1.2.1) is always an equality.

12.2.F. EXERCISE. Show that the equations $wz - xy = 0$, $wy - x^2 = 0$, $xz - y^2 = 0$ cut out an integral *surface* S in $\mathbb{A}^4_k$. (You may recognize these equations from Exercises 4.6.H and 9.2.A.) You might expect S to be a curve, because it is cut out by three equations in 4-space. One of many ways to proceed: cut S into pieces. For example, show that $D(w) \cong \operatorname{Spec} k[x, w]_w$. (You may recognize S as the affine cone over the twisted cubic. The twisted cubic was defined in Exercise 9.2.A.) It turns out that you need three equations to cut out this surface. The first equation cuts out a threefold in $\mathbb{A}^4_k$ (by Krull's Principal Ideal Theorem 12.3.3, which we will meet soon). The second equation cuts out a surface: our surface, along with another surface. The third equation cuts out our surface, and removes the "extraneous component". One last aside: notice once again that the cone over the quadric surface $k[w, x, y, z]/(wz - xy)$ makes an appearance.)

12.2.4. A first example of the utility of dimension theory. Although dimension theory is not central to the following statement, it is essential to the proof.

12.2.G. ENLIGHTENING STRENUOUS EXERCISE. For any $d > 3$, show that most degree d surfaces in $\mathbb{P}^3_{\overline{k}}$ contain no lines. Here, "most" means "all closed points of a Zariski-open subset of the parameter space for degree d homogeneous polynomials in 4 variables, up to scalars. As there are $\binom{d+3}{3}$ such monomials, the degree d hypersurfaces are parametrized by $\mathbb{P}^{\binom{d+3}{3}-1}_{\overline{k}}$. Hint: Construct an incidence correspondence

$$X = \{(\ell, H) : [\ell] \in \mathbb{G}(1,3), [H] \in \mathbb{P}^{\binom{d+3}{3}-1}, \ell \subset H\},$$

parametrizing lines in $\mathbb{P}^3$ contained in a hypersurface: define a closed subscheme X of $\mathbb{G}(1,3) \times \mathbb{P}^{\binom{d+3}{3}-1}$ that makes this notion precise. Show that X is a $\mathbb{P}^{\binom{d+3}{3}-1-(d+1)}$-bundle over $\mathbb{G}(1,3)$. (Possible hint for this: how many degree d hypersurfaces contain the line $x = y = 0$?) Show that $\dim \mathbb{G}(1,3) = 4$ (see §7.7: $\mathbb{G}(1,3)$ has an open cover by $\mathbb{A}^4$'s). Show that $\dim X = \binom{d+3}{3} - 1 - (d+1) + 4$. Show that the image of the projection $X \to \mathbb{P}^{d+33} - 1$ must lie in a proper closed subset. The following diagram may help.

$$\begin{tikzcd}
 & \dim \binom{d+3}{3} - 1 - (d+1) + 4 & & \\
 & X \arrow[dl] \arrow[dr, "\mathbb{P}^{\binom{d+3}{3}-1-(d+1)}"] & & \\
\mathbb{P}^{\binom{d+3}{3}-1} & & \mathbb{G}(1,3) & \dim 4
\end{tikzcd}$$

12.2.5. *Side Remark.* If you do the previous Exercise, your dimension count will suggest the true facts that degree 1 hypersurfaces — i.e. hyperplanes — have 2-dimensional families of lines, and that most degree 2 hypersurfaces have 1-dimensional families of lines, as shown in Exercise 9.2.M. They will also suggest that most degree 3 hypersurfaces contain a finite number of lines, which reflects the celebrated fact that nonsingular cubic surfaces over an algebraically closed field always contain 27 lines. The statement about quartics generalizes to the Noether-Lefschetz theorem implying that a very general surface of degree d at least 4 contains no curves that are not the intersection of the surface with a hypersurface. **"Very general"** means that in the parameter space (in this case, the projective space parametrizing surfaces of degree d), the statement is true away from a countable union of proper Zariski-closed subsets. It is a weaker version of the phrase "almost every" than "general".

12.2.6. Noether Normalization.

To set up the proof of Theorem 12.2.1 on dimension and transcendence degree, we introduce another important classical notion, Noether Normalization.

12.2.7. Noether Normalization Lemma. — *Suppose A is an integral domain, finitely generated over a field k. If $\operatorname{tr.deg}_k K(A) = n$, then there are elements $x_1, \dots, x_n \in A$, algebraically independent over k, such that A is a finite (hence integral by Corollary 8.2.2) extension of $k[x_1, \dots, x_n]$.*

The geometric content behind this result is that given any integral affine k-scheme X, we can find a surjective finite morphism $X \to \mathbb{A}_k^n$, where n is the transcendence degree of the function field of X (over k). Surjectivity follows from the Lying Over Theorem 8.2.5, in particular Exercise 12.1.B.

$\star$ *Nagata's proof of Noether normalization.* Suppose we can write $A = k[y_1, \dots, y_m]/\mathfrak{p}$, i.e. that A can be chosen to have m generators. Note that $m \geq n$. We show the result by induction on m. The base case $m = n$ is immediate.

Assume now that $m > n$, and that we have proved the result for smaller m. We will find $m-1$ elements $z_1, \dots, z_{m-1}$ of A such that A is finite over $A' := k[z_1, \dots, z_{m-1}]$ (i.e. the subring of A generated by $z_1, \dots, z_{m-1}$). Then by the inductive hypothesis, A' is finite over some $k[x_1, \dots, x_n]$, and A is finite over A', so by Exercise 8.3.I, A is finite over $k[x_1, \dots, x_n]$.

$$\begin{tikzcd} A \arrow[d, dash, "\text{finite}"] \\ A' = k[z_1, \dots, z_{m-1}]/\mathfrak{p} \arrow[d, dash, "\text{finite}"] \\ k[x_1, \dots, x_n] \end{tikzcd}$$

As $y_1, \dots, y_m$ are algebraically dependent, there is some non-zero algebraic relation $f(y_1, \dots, y_m) = 0$ among them (where f is a polynomial in m variables).

Let $z_1 = y_1 - y_m^{r_1}$, $z_2 = y_2 - y_m^{r_2}$, $\dots$, $z_{m-1} = y_{m-1} - y_m^{r_{m-1}}$, where $r_1, \dots, r_{m-1}$ are positive integers to be chosen shortly. Then

$$f(z_1 + y_m^{r_1}, z_2 + y_m^{r_2}, \dots, z_{m-1} + y_m^{r_{m-1}}, y_m) = 0.$$

Then upon expanding this out, each monomial in f (as a polynomial in m variables) will yield a single term in that is a constant times a power of y_m (with no z_i factors). By choosing the r_i so that $0 \ll r_1 \ll r_2 \ll \cdots \ll r_{m-1}$, we can ensure that the powers of y_m appearing are all distinct, and so that in particular there is a leading term y_m^N, and all other terms (including those with z_i-factors) are of smaller degree in y_m. Thus we have described an integral dependence of y_m on $z_1, \dots, z_{m-1}$ as desired. $\square$

12.2.8. Geometric interpretations and consequences.

12.2.9. *Aside: the geometry behind Nagata's proof.* Here is the geometric intuition behind Nagata's argument. Suppose we have an m-dimensional variety in $\mathbb{A}^n_k$ with $m < n$, for example $xy = 1$ in $\mathbb{A}^2$. One approach is to hope the projection to a hyperplane is a finite morphism. In the case of $xy = 1$, if we projected to the x-axis, it wouldn't be finite, roughly speaking because the asymptote $x = 0$ prevents the map from being closed (cf. Exercise 8.3.L). If we instead projected to a random line, we might hope that we would get rid of this problem, and indeed we usually can: this problem arises for only a finite number of directions. But we might have a problem if the field were finite: perhaps the finite number of directions in which to project each have a problem. (You can show that if k is an infinite field, then the substitution in the above proof $z_i = y_i - y_m^{r_i}$ can be replaced by the linear substitution $z_i = y_i - a_i y_m$ where $a_i \in k$, and that for a non-empty Zariski-open choice of a_i, we indeed obtain a finite morphism.) Nagata's trick in general is to "jiggle" the variables in a non-linear way, and this is jiggling kills the non-finiteness of the map.

12.2.H. EXERCISE (GEOMETRIC NOETHER NORMALIZATION). If X is an affine irreducible variety of dimension n over k, show that there is a dominant finite morphism $X \to \mathbb{A}^n_k$ (over k).

12.2.I. EXERCISE (DIMENSION IS ADDITIVE FOR FIBERED PRODUCTS OF FINITE TYPE k-SCHEMES). Suppose X and Y are finite type k-schemes. Show that $\dim X \times_k Y = \dim X + \dim Y$. (Hint: Use Noether normalization to find dominant finite morphisms $X \to \mathbb{A}_k^{\dim X}$ and $Y \to \mathbb{A}_k^{\dim Y}$, and use this to construct a dominant finite morphism $X \times_k Y \to \mathbb{A}_k^{\dim X + \dim Y}$.)

12.2.10. *Proof of Theorem 12.2.1 on dimension and transcendence degree.* Suppose X is an integral affine k-scheme. We show that $\dim X$ equals the transcendence degree n of its function field, by induction on n. (The idea is that we reduce from X to $\mathbb{A}^n$ to a hypersurface in $\mathbb{A}^n$ to $\mathbb{A}^{n-1}$.) Assume the result is known for all transcendence degrees less than n.

By Noether normalization, there exists a surjective finite morphism $X \to \mathbb{A}^n_k$. By Exercise 12.1.B, $\dim X = \dim \mathbb{A}^n_k$. If $n = 0$, we are done, as $\dim \mathbb{A}^0_k = 0$.

We now show that $\dim \mathbb{A}^n_k = n$ for $n > 0$, by induction. Clearly $\dim \mathbb{A}^n_k \geq n$, as we can describe a chain of irreducible subsets of length $n + 1$: if $x_1, \dots, x_n$ are coordinates on $\mathbb{A}^n$, consider the chain of ideals

$$(0) \subset (x_1) \subset \cdots \subset (x_1, \dots, x_n)$$

in $k[x_1, \dots, x_n]$. Suppose we have a chain of prime ideals of length at least n:

$$(0) = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m.$$

where $\mathfrak{p}_1$ is a codimension 1 prime ideal. Then $\mathfrak{p}_1$ is principal (as $k[x_1, \dots, x_n]$ is a unique factorization domain, Lemma 12.2.2) say $\mathfrak{p}_1 = (f(x_1, \dots, x_n))$, where f is an irreducible polynomial. Then $K(k[x_1, \dots, x_n]/(f(x_1, \dots, x_n)))$ has transcendence degree $n - 1$, so by induction,

$$\dim k[x_1, \dots, x_n]/(f) = n - 1.$$

□

12.3 Codimension one miracles: Krull and Hartogs

In this section, we will explore a number of results related to codimension one. We introduce two results that apply in more general situations, and link functions and the codimension one points where they vanish, Krull's Principal Ideal Theorem 12.3.3, and Algebraic Hartogs' Lemma 12.3.10. We will find these two theorems very useful. For example, Krull's Principal Ideal Theorem will help us compute codimensions, and will show us that codimension can behave oddly, and Algebraic Hartogs' Lemma will give us a useful characterization of unique factorization domains (Proposition 12.3.5). The results in this section will require (locally) Noetherian hypotheses.

12.3.1. Krull's Principal Ideal Theorem. The Principal Ideal Theorem generalizes the linear algebra fact that in a vector space, a single linear equation cuts out a subspace of codimension 0 or 1 (and codimension 0 occurs only when the equation is 0).

12.3.2. Krull's Principal Ideal Theorem (geometric version). — *Suppose X is a locally Noetherian scheme, and f is a function. The irreducible components of $V(f)$ are codimension 0 or 1.*

This is clearly a consequence of the following algebraic statement. You know enough to prove it for varieties (see Exercise 12.3.I), which is where we will use it most often. The full proof is technical, and included in §12.4 (see §12.4.2) only to show you that it isn't long.

12.3.3. Krull's Principal Ideal Theorem (algebraic version). — *Suppose A is a Noetherian ring, and $f \in A$. Then every prime $\mathfrak{p}$ minimal among those containing f has codimension at most 1. If furthermore f is not a zerodivisor, then every minimal prime $\mathfrak{p}$ containing f has codimension precisely 1.*

For example, the scheme $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ (the cone over the quadric surface) is cut out by one non-zero equation $wz - xy$ in $\mathbb{A}^4$, so it is a threefold. As another example, locally principal closed subschemes have "codimension 0 or 1", and effective Cartier divisors have "pure codimension 1".

12.3.A. EXERCISE. What is the dimension of $\operatorname{Spec} k[w, x, y, z]/(wz - xy, y^{17} + z^{17})$? (Check the hypotheses before invoking Krull!)

12.3.B. EXERCISE. Show that an irreducible homogeneous polynomial in $n + 1$ variables over a field k describes an integral scheme of dimension $n - 1$ in $\mathbb{P}^n_k$.

12.3.C. EXERCISE (VERY IMPORTANT FOR LATER). This is a pretty cool argument.
(a) *(Hypersurfaces meet everything of dimension at least* 1 *in projective space, unlike in affine space.)* Suppose X is a closed subset of $\mathbb{P}^n_k$ of dimension at least 1, and H is a nonempty hypersurface in $\mathbb{P}^n_k$. Show that H meets X. (Hint: note that the affine cone over H contains the origin in $\mathbb{A}^{n+1}_k$. Apply Krull's Principal Ideal Theorem 12.3.3 to the cone over X.)
(b) Suppose $X \hookrightarrow \mathbb{P}^n_k$ is a closed subset of dimension r. Show that any codimension r linear space meets X. Hint: Refine your argument in (a). (In fact any two things in projective space that you might expect to meet for dimensional reasons do in fact meet. We won't prove that here.)
(c) Show further that there is an intersection of $r+1$ nonempty hypersurfaces missing X. (The key step: show that there is a hypersurface of sufficiently high degree that doesn't contain every generic point of X. Show this by induction on the number of generic points. To get from n to $n+1$: take a hypersurface not vanishing on $p_1, \dots, p_n$. If it doesn't vanish on p_{n+1}, we are done. Otherwise, call this hypersurface f_{n+1}. Do something similar with $n+1$ replaced by i ($1 \leq i \leq n$). Then consider $\sum_i f_1 \cdots \hat{f_i} \cdots f_{n+1}$.) If k is infinite, show that there is a codimension $r+1$ linear subspace missing X. (The key step: show that there is a hyperplane not containing any generic point of a component of X.
(d) If k is an infinite field, show that there is an intersection of r hyperplanes meeting X in a finite number of points. (We will see in Exercise 26.5.C that if $k = \overline{k}$, the number of points for "most" choices of these r hyperplanes, the number of points is the degree of X. But first of course we must define "degree".)

12.3.D. EXERCISE (PRIME AVOIDANCE). As an aside, here is an exercise of a similar flavor to the previous one. Suppose $I \subseteq \cup_{i=1}^n \mathfrak{p}_i$. (The right side is not an ideal!) Show that $I \subset \mathfrak{p}_i$ for some i. (Can you give a geometric interpretation of this result?) Hint: by induction on n. Don't look in the literature — you might find a much longer argument! (See Exercise 12.3.C for a related problem.)

12.3.E. USEFUL EXERCISE. Suppose f is an element of a Noetherian ring A, contained in no codimension 1 primes. Show that f is a unit. (Hint: show that if a function vanishes nowhere, it is a unit.)

12.3.4. A useful characterization of unique factorization domains.

We can use Krull's Principal Ideal Theorem to prove one of the four useful criteria for unique factorization domains, promised in §6.4.5.

12.3.5. Proposition. — *Suppose that* A *is a Noetherian integral domain. Then* A *is a unique factorization domain if and only if all codimension* 1 *primes are principal.*

This contains Lemma 12.2.2 and (in some sense) its converse.

Proof. We have already shown in Lemma 12.2.2 that if A is a unique factorization domain, then all codimension 1 primes are principal. Assume conversely that all codimension 1 primes of A are principal. I claim that the generators of these ideals are irreducible, and that we can uniquely factor any element of A into these irreducibles, and a unit. First, suppose (f) is a codimension 1 prime ideal $\mathfrak{p}$. Then if $f = gh$, then either $g \in \mathfrak{p}$ or $h \in \mathfrak{p}$. As $\operatorname{codim} \mathfrak{p} > 0$, $\mathfrak{p} \neq (0)$, so by Nakayama's Lemma 8.2.H (as $\mathfrak{p}$ is finitely generated), $\mathfrak{p} \neq \mathfrak{p}^2$. Thus g and h cannot both be in $\mathfrak{p}$.

Say $g \notin \mathfrak{p}$. Then g is contained in no codimension 1 primes (as f was contained in only one, namely $\mathfrak{p}$), and hence is a unit by Exercise 12.3.E.

We next show that any non-zero element f of A can be factored into irreducibles. Now $V(f)$ is contained in a finite number of codimension 1 primes, as (f) has a finite number of associated primes (§6.5), and hence a finite number of minimal primes. We show that any nonzero f can be factored into irreducibles by induction on the number of codimension 1 primes containing f. In the base case where there are none, then f is a unit by Exercise 12.3.E. For the general case where there is at least one, say $f \in \mathfrak{p} = (g)$. Then $f = g^n h$ for some $h \notin (g)$. (Reason: otherwise, we have an ascending chain of ideals $(f) \subset (f/g) \subset (f/g^2) \subset \cdots$, contradicting Noetherianness.) Thus $f/g^n \in A$, and is contained in one fewer codimension 1 primes.

12.3.F. EXERCISE. Conclude the proof by showing that this factorization is unique. (Possible hint: the irreducible components of $V(f)$ give you the prime factors, but not the multiplicities.)

12.3.6. Generalizing Krull to more equations. The following generalization of Krull's Principal Ideal Theorem looks like it might follow by induction from Krull, but it is more subtle.

12.3.7. Krull's Principal Ideal Theorem, Strong Version. — *Suppose $X = \operatorname{Spec} A$ where A is Noetherian, and Z is an irreducible component of $V(r_1, \dots, r_n)$, where $r_1, \dots, r_n \in A$. Then the codimension of Z is at most n.*

A proof is given in §12.4.3.

12.3.G. EXERCISE. Suppose $(A, \mathfrak{m})$ is a Noetherian local ring.
(a) *(Noetherian local rings have finite dimension, promised in Remark 12.1.4)* Use Theorem 12.3.7 to prove that $(A, \mathfrak{m})$ has finite dimension. (Hint: if $\mathfrak{m} = (f_1, \dots, f_d)$, show that $\dim A \le d$.)
(b) Suppose that $g_1, \dots, g_k \in A$ with $V(g_1, \dots, g_k) = \{[\mathfrak{m}]\}$. Show that $k = d$ is possible. (Hint: in order to work by induction on d, you need to find a first equation that will knock the dimension down by 1, i.e. $\dim A/(g_d) = \dim A - 1$. Find g_d using prime avoidance, Exercise 12.3.D.) Show that $k \ge d$. (Geometric translation: given a d-dimensional "germ of a reasonable space" around a point p. Then p can be cut out set-theoretically by d equations, and you always need at least d equations. These d elements of A are called a **system of parameters** for the Noetherian local ring A, but we won't use this language beyond the next exercise.)

12.3.H. EXERCISE (CODIMENSION BEHAVES AS YOU MIGHT EXPECT FOR A MORPHISM). Suppose $\pi : X \to Y$ is a morphism of locally Noetherian schemes, and $p \in X$ and $q \in Y$ are points such that $q = \pi(p)$. Show that

$$\operatorname{codim}_X p \le \operatorname{codim}_Y q + \operatorname{codim}_{\pi^{-1} q} p.$$

(Does this agree with your geometric intuition? You should be able to come up with enlightening examples where equality holds, and where equality fails. We will see that equality always holds for sufficiently nice — flat — morphisms, see Proposition 25.5.5.) Hint: take a system of parameters for q "in Y", and a system of parameters for p "in $\pi^{-1} q$", and use them to find $\operatorname{codim}_Y q + \operatorname{codim}_{\pi^{-1} q} p$ elements of $\mathcal{O}_{X,p}$ cutting out $\{[\mathfrak{m}]\}$ in $\operatorname{Spec} \mathcal{O}_{X,p}$. Use the previous exercise.

12.3.I. EXERCISE. Prove Theorem 12.3.7 in the special case where X is an affine variety, i.e. if A is finitely generated over some field k. Show that $\dim Z \geq \dim X - n$. Hint: Exercise 12.2.E.

12.3.8. $\star$ **Pathologies of the notion of "codimension".** We can use Krull's Principal Ideal Theorem to produce the example of pathology in the notion of codimension promised earlier this chapter. Let $A = k[x]_{(x)}[t]$. In other words, elements of A are polynomials in t, whose coefficients are quotients of polynomials in x, where no factors of x appear in the denominator. (Warning: A is not $k[x,t]_{(x)}$.) Clearly, A is an integral domain, and $(xt-1)$ is not a zero divisor. You can verify that $A/(xt-1) \cong k[x]_{(x)}[1/x] \cong k(x)$ — "in $k[x]_{(x)}$, we may divide by everything but x, and now we are allowed to divide by x as well" — so $A/(xt-1)$ is a field. Thus $(xt-1)$ is not just prime but also maximal. By Krull's theorem, $(xt-1)$ is codimension 1. Thus $(0) \subset (xt-1)$ is a maximal chain. However, A has dimension at least 2: $(0) \subset (t) \subset (x,t)$ is a chain of primes of length 2. (In fact, A has dimension precisely 2, although we don't need this fact in order to observe the pathology.) Thus we have a codimension 1 prime in a dimension 2 ring that is dimension 0. Here is a picture of this poset of ideals.

$$\begin{tikzcd}[arrows=-]
(x,t) \arrow[d] & & \\
(t) \arrow[dr] & & (xt-1) \arrow[dl] \\
& (0) &
\end{tikzcd}$$

This example comes from geometry, and it is enlightening to draw a picture, see Figure 12.2. $\operatorname{Spec} k[x]_{(x)}$ corresponds to a "germ" of $\mathbb{A}^1_k$ near the origin, and $\operatorname{Spec} k[x]_{(x)}[t]$ corresponds to "this $\times$ the affine line". You may be able to see from the picture some motivation for this pathology — $V(xt-1)$ doesn't meet $V(x)$, so it can't have any specialization on $V(x)$, and there is nowhere else for $V(xt-1)$ to specialize. It is disturbing that this misbehavior turns up even in a relatively benign-looking ring.

12.3.9. Algebraic Hartogs' Lemma for Noetherian normal schemes.

Hartogs' Lemma in several complex variables states (informally) that a holomorphic function defined away from a codimension two set can be extended over that. We now describe an algebraic analog, for Noetherian normal schemes. We will use this repeatedly and relentlessly when connecting line bundles and divisors.

12.3.10. Algebraic Hartogs' Lemma. — *Suppose A is a Noetherian normal integral domain. Then*

$$A = \cap_{\mathfrak{p} \text{ codimension } 1} A_{\mathfrak{p}}.$$

The equality takes place inside $K(A)$; recall that any localization of an integral domain A is naturally a subset of $K(A)$ (Exercise 2.3.C). Warning: few people call this Algebraic Hartogs' Lemma. I call it this because it parallels the statement in

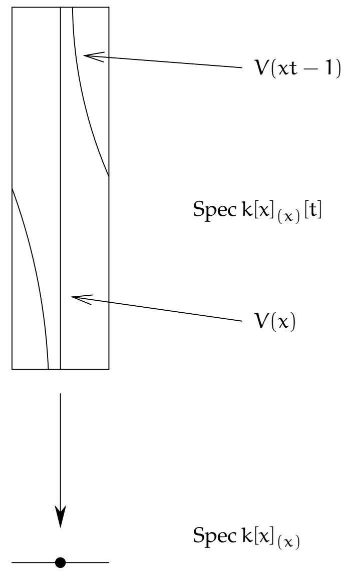


FIGURE 12.2. Dimension and codimension behave oddly on the surface $\operatorname{Spec} k[x]_{(x)}[t]$

complex geometry. The proof is technical and the details are less enlightening, so we postpone it to §12.3.11.

One might say that if $f \in K(A)$ does not lie in $A_{\mathfrak{p}}$ where $\mathfrak{p}$ has codimension 1, then f has a pole at $[\mathfrak{p}]$, and if $f \in FF(A)$ lies in $\mathfrak{p}A_{\mathfrak{p}}$ where $\mathfrak{p}$ has codimension 1, then f has a zero at $[\mathfrak{p}]$. It is worth interpreting Algebraic Hartogs' Lemma as saying that *a rational function on a normal scheme with no poles is in fact regular* (an element of A). Informally: *"Noetherian normal schemes have the Hartogs property."* (We will properly define zeros and poles in §13.4.7, see also Exercise 13.4.H.)

One can state Algebraic Hartogs' Lemma more generally in the case that $\operatorname{Spec} A$ is a Noetherian normal scheme, meaning that A is a product of Noetherian normal integral domains; the reader may wish to do so.

12.3.11. ⋆⋆ *Proof of Algebraic Hartogs' Lemma 12.3.10.* This proof sheds little light on the rest of this section, and thus should not be read. However, you should sleep soundly at night knowing that the proof is this short. The left side is obviously contained in the right. So assume we have some x in all $A_{\mathfrak{p}}$ but not in A. Let I be the "ideal of denominators" of x (cf. the proof of Proposition 6.4.2):

$$I := \{r \in A \ : \ rx \in A\}.$$

As $1 \notin I$, we have $I \neq A$, so choose a minimal prime $\mathfrak{q}$ containing I.

This construction behaves well with respect to localization — if $\mathfrak{p}$ is any prime, then the ideal of denominators x in $A_{\mathfrak{p}}$ is $I_{\mathfrak{p}}$, and it again measures "the failure of Algebraic Hartogs' Lemma for x," this time in $A_{\mathfrak{p}}$. But Algebraic Hartogs' Lemma is vacuously true for dimension 1 rings, so no codimension 1 prime contains I.

Thus $\mathfrak{q}$ has codimension at least 2. By localizing at $\mathfrak{q}$, we can assume that A is a local ring with maximal ideal $\mathfrak{q}$, and that $\mathfrak{q}$ is the *only* prime containing I. Thus $\sqrt{I} = \mathfrak{q}$ (Exercise 4.4.F), so as $\mathfrak{q}$ is finitely generated, there is some n with $I \supset \mathfrak{q}^n$ (do you see why?). Take the minimal such n, so $I \not\supset \mathfrak{q}^{n-1}$, and choose any $y \in \mathfrak{q}^{n-1} - I$. Let $z = yx$. Now $\mathfrak{q}y \subset \mathfrak{q}^n \subset I$, so $\mathfrak{q}z \subset Ix \subset A$, so $\mathfrak{q}z$ is an ideal of A.

I claim $\mathfrak{q}z$ is not contained in $\mathfrak{q}$. Otherwise, we would have a finitely-generated A-module (namely $\mathfrak{q}$) with a faithful $A[z]$-action, forcing z to be integral over A (and hence in A, as A is integrally closed) by Exercise 8.2.J.

Thus $\mathfrak{q}z$ is an ideal of A not contained in the unique maximal ideal $\mathfrak{q}$, so it must be A! Thus $\mathfrak{q}z = A$ from which $\mathfrak{q} = A(1/z)$, from which $\mathfrak{q}$ is principal. But then $\operatorname{codim} \mathfrak{q} = \dim A \leq \dim_{A/\mathfrak{q}} \mathfrak{q}/\mathfrak{q}^2 \leq 1$ by Nakayama's lemma 8.2.H, contradicting the fact that $\mathfrak{q}$ has codimension at least 2. $\square$

12.4 ⋆⋆ Proof of Krull's Principal Ideal Theorem 12.3.3

The details of this proof won't matter to us, so you should probably not read it. It is included so you can glance at it and believe that the proof is fairly short, and you could read it if you really wanted to.

If A is a ring, an **Artinian A-module** is an A-module satisfying the descending chain condition for submodules (any infinite descending sequence of submodules must stabilize, §4.6.3). A **ring** is Artinian ring if it is Artinian over itself as a module. The notion of Artinian rings is very important, but we will get away without discussing it much.

If $\mathfrak{m}$ is a maximal ideal of A, then any finite-dimensional $(A/\mathfrak{m})$-vector space (interpreted as an A-module) is clearly Artinian, as any descending chain

$$M_1 \supset M_2 \supset \cdots$$

must eventually stabilize (as $\dim_{A/\mathfrak{m}} M_i$ is a non-increasing sequence of non-negative integers).

12.4.A. EXERCISE. Suppose $\mathfrak{m}$ is finitely generated. Show that for any n, $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finite-dimensional $(A/\mathfrak{m})$-vector space. (Hint: show it for $n = 0$ and $n = 1$. Show surjectivity of $\operatorname{Sym}^n \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ to bound the dimension for general n.) Hence $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an Artinian A-module.

12.4.B. EXERCISE. Suppose A is a Noetherian ring with one prime ideal $\mathfrak{m}$. Suppose $\mathfrak{m}$ is finitely generated. Prove that $\mathfrak{m}^n = (0)$ for some n. (Hint: As $\sqrt{0}$ is prime, it must be $\mathfrak{m}$. Suppose $\mathfrak{m}$ can be generated by r elements, each of which has kth power 0, and show that $\mathfrak{m}^{r(k-1)+1} = 0$.)

12.4.C. EXERCISE. Show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of modules, then M is Artinian if and only if M' and M'' are Artinian. (Hint: given a descending chain in M, produce descending chains in M' and M''.)

12.4.1. Lemma. — *If A is a Noetherian ring with one prime ideal $\mathfrak{m}$, then A is Artinian, i.e., it satisfies the descending chain condition for ideals.*

Proof. As we have a finite filtration

$$A \supset \mathfrak{m} \supset \cdots \supset \mathfrak{m}^n = (0)$$

all of whose quotients are Artinian, A is Artinian as well. $\square$

12.4.2. *Proof of Krull's Principal Ideal Theorem 12.3.3.* Suppose we are given $x \in A$, with $\mathfrak{p}$ a minimal prime containing x. By localizing at $\mathfrak{p}$, we may assume that A is a local ring, with maximal ideal $\mathfrak{p}$. Suppose $\mathfrak{q}$ is another prime strictly contained in $\mathfrak{p}$.

$$\begin{tikzcd}
x \arrow[dr, hook] & & \\
& \mathfrak{p} \arrow[r, hook] & A \\
\mathfrak{q} \arrow[ur, hook] & &
\end{tikzcd}$$

For the first part of the theorem, we must show that $A_\mathfrak{q}$ has dimension 0. The second part follows from our earlier work: if any minimal primes are height 0, f is a zerodivisor, by Theorem 6.5.4(c) and (e).

Now $\mathfrak{p}$ is the only prime ideal containing (x), so $A/(x)$ has one prime ideal. By Lemma 12.4.1, $A/(x)$ is Artinian.

We invoke a useful construction, the **nth symbolic power of a prime ideal**: if A is a ring, and $\mathfrak{q}$ is a prime ideal, then define

$$\mathfrak{q}^{(n)} := \{r \in A \ : \ rs \in \mathfrak{q}^n \text{ for some } s \in A - \mathfrak{q}\}.$$

We have a descending chain of ideals in A

$$\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots,$$

so we have a descending chain of ideals in $A/(x)$

$$\mathfrak{q}^{(1)} + (x) \supset \mathfrak{q}^{(2)} + (x) \supset \cdots$$

which stabilizes, as $A/(x)$ is Artinian. Say $\mathfrak{q}^{(n)} + (x) = \mathfrak{q}^{(n+1)} + (x)$, so

$$\mathfrak{q}^{(n)} \subset \mathfrak{q}^{(n+1)} + (x).$$

Hence for any $f \in \mathfrak{q}^{(n)}$, we can write $f = ax + g$ with $g \in \mathfrak{q}^{(n+1)}$. Hence $ax \in \mathfrak{q}^{(n)}$. As $\mathfrak{p}$ is minimal over x, $x \notin \mathfrak{q}$, so $a \in \mathfrak{q}^{(n)}$. Thus

$$\mathfrak{q}^{(n)} = (x)\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)}.$$

As x is in the maximal ideal $\mathfrak{p}$, the second version of Nakayama's lemma 8.2.9 gives $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$.

We now shift attention to the local ring $A_\mathfrak{q}$, which we are hoping is dimension 0. We have $\mathfrak{q}^{(n)}A_\mathfrak{q} = \mathfrak{q}^{(n+1)}A_\mathfrak{q}$ (the symbolic power construction clearly construction commutes with localization). For any $r \in \mathfrak{q}^n A_\mathfrak{q} \subset \mathfrak{q}^{(n)}A_\mathfrak{q}$, there is some $s \in A_\mathfrak{q} - \mathfrak{q}A_\mathfrak{q}$ such that $rs \in \mathfrak{q}^{n+1}A_\mathfrak{q}$. As s is invertible, $r \in \mathfrak{q}^{n+1}A_\mathfrak{q}$ as well. Thus $\mathfrak{q}^n A_\mathfrak{q} \subset \mathfrak{q}^{n+1}A_\mathfrak{q}$, but as $\mathfrak{q}^{n+1}A_\mathfrak{q} \subset \mathfrak{q}^n A_\mathfrak{q}$, we have $\mathfrak{q}^n A_\mathfrak{q} = \mathfrak{q}^{n+1}A_\mathfrak{q}$. By Nakayama's Lemma version 4 (Exercise 8.2.H),

$$\mathfrak{q}^n A_\mathfrak{q} = 0.$$

Finally, any local ring $(R, \mathfrak{m})$ such that $\mathfrak{m}^n = 0$ has dimension 0, as $\operatorname{Spec} R$ consists of only one point: $[\mathfrak{m}] = V(\mathfrak{m}) = V(\mathfrak{m}^n) = V(0) = \operatorname{Spec} R$. $\square$

12.4.3. *Proof of Theorem 12.3.7, following* [**E**, Thm. 10.2]. We argue by induction on n. The case $n = 1$ is Krull's Principal Ideal Theorem 12.3.3. Assume $n > 1$. Suppose $\mathfrak{p}$ is a minimal prime containing $r_1, \dots, r_n \in A$. We wish to show that $\operatorname{codim} \mathfrak{p} \leq n$. By localizing at $\mathfrak{p}$, we may assume that $\mathfrak{p}$ is the unique maximal ideal of A. Let $\mathfrak{q} \neq \mathfrak{p}$ be a prime ideal of A with no prime between $\mathfrak{p}$ and $\mathfrak{q}$. We shall show that $\mathfrak{q}$ is minimal over an ideal generated by $c - 1$ elements. Then $\operatorname{codim} \mathfrak{q} \leq c - 1$ by the inductive hypothesis, so we will be done.

Now $\mathfrak{q}$ cannot contain every r_i (as $V(r_1, \dots, r_n) = \{[\mathfrak{p}]\}$), so say $r_1 \notin \mathfrak{q}$. Then $V(\mathfrak{q}, r_1) = \{[\mathfrak{p}]\}$. As each $r_i \in \mathfrak{p}$, there is some N such that $r_i^N \in (\mathfrak{q}, r_1)$ (Exercise 4.4.I), so write $r_i^N = q_i + a_i r_1$ where $q_i \in \mathfrak{q}$ ($2 \leq i \leq n$) and $a_i \in A$. Note that

$$V(r_1, q_2, \dots, q_n) = V(r_1, r_2^N, \dots, r_n^N) = V(r_1, r_2, \dots, r_n) = \{[\mathfrak{p}]\}. \tag{12.4.3.1}$$

We shall show that $\mathfrak{q}$ is minimal among primes containing $q_2, \dots, q_n$, completing the proof. In the ring $A/(q_2, \dots, q_n)$, $V(r_1) = \{[\mathfrak{p}]\}$ by (12.4.3.1). By Krull's principal ideal theorem 12.3.3, $[\mathfrak{p}]$ is codimension at most 1, so $[\mathfrak{q}]$ must be codimension 0 in $\operatorname{Spec} A/(q_2, \dots, q_n)$, as desired. $\square$

CHAPTER 13

Nonsingularity ("smoothness") of Noetherian schemes

One natural notion we expect to see for geometric spaces is the notion of when an object is "smooth". In algebraic geometry, this notion, called *nonsingularity* (or *regularity*, although we won't use this term) is easy to define but a bit subtle in practice. We will soon define what it means for a scheme to be *nonsingular* (or *regular*) at a point. The Jacobian criterion will show that this corresponds to smoothness in situations where you may have seen it before. A point that is not nonsingular is (not surprisingly) called *singular* ("not smooth"). A scheme is said to be *nonsingular* if all its points are nonsingular, and singular if one of its points is singular.

The notion of nonsingularity is less useful than you might think. Grothendieck taught us that the more important notions are properties of morphisms, not of objects, and there is indeed a "relative notion" that applies to a morphism of schemes $f : X \to Y$ that is much better-behaved (corresponding to the notion of submersion in differential geometry). For this reason, the word "smooth" is reserved for these morphisms. (This is why "smooth" has often been in quotes when mentioned until now.) We will discuss smooth morphisms (without quotes!) in Chapter 26. However, nonsingularity is still useful, especially in (co)dimension 1, and we shall discuss this case (of *discrete valuation rings*) in §13.4.

13.1 The Zariski tangent space

We first define the tangent space of a scheme at a point. It behaves like the tangent space you know and love at "smooth" points, but also makes sense at other points. In other words, geometric intuition at the "smooth" points guides the definition, and then the definition guides the algebra at all points, which in turn lets us refine our geometric intuition.

This definition is short but surprising. The main difficulty is convincing yourself that it deserves to be called the tangent space. This is tricky to explain, because we want to show that it agrees with our intuition, but our intuition is worse than we realize. So I will just define it for you, and later try to convince you that it is reasonable.

13.1.1. *Definition.* Suppose $\mathfrak{p}$ is a prime ideal of a ring A, so $[\mathfrak{p}]$ is a point of $\operatorname{Spec} A$. Then $[\mathfrak{p}A_{\mathfrak{p}}]$ is a point of the scheme $\operatorname{Spec} A_{\mathfrak{p}}$. For convenience, we let $\mathfrak{m} := \mathfrak{p}A_{\mathfrak{p}} \subset A_{\mathfrak{p}} =: B$. Let $\kappa = B/\mathfrak{m}$ be the residue field. Then $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over the residue field κ: it is a B-module, and elements of $\mathfrak{m}$ acts like 0. This is defined to be the **Zariski cotangent space**. The dual vector space is the **Zariski tangent**

space. Elements of the Zariski cotangent space are called **cotangent vectors** or **differentials**; elements of the tangent space are called **tangent vectors**.

Note that this definition is intrinsic. It does not depend on any specific description of the ring itself (such as the choice of generators over a field k, which is equivalent to the choice of embedding in affine space). Notice that the cotangent space is more algebraically natural than the tangent space (the definition is shorter). There is a moral reason for this: the cotangent space is more naturally determined in terms of functions on a space, and we are very much thinking about schemes in terms of "functions on them". This will come up later.

Here are two plausibility arguments that this is a reasonable definition. Hopefully one will catch your fancy.

In differential geometry, the tangent space at a point is sometimes defined as the vector space of derivations at that point. A derivation is a function that takes in functions near the point that vanish at the point, and gives elements of the field k, and satisfies the Leibniz rule

$$(fg)' = f'g + g'f.$$

(We will later define derivations in more general settings, §23.2.16) Translation: a derivation is a map $\mathfrak{m} \to k$. But $\mathfrak{m}^2$ maps to 0, as if $f(p) = g(p) = 0$, then

$$(fg)'(p) = f'(p)g(p) + g'(p)f(p) = 0.$$

Thus we have a map $\mathfrak{m}/\mathfrak{m}^2 \to k$, i.e. an element of $(\mathfrak{m}/\mathfrak{m}^2)^\vee$.

13.1.A. EXERCISE. Check that this is reversible, i.e. that any map $\mathfrak{m}/\mathfrak{m}^2 \to k$ gives a derivation. In other words, verify that the Leibniz rule holds. (Your proof will not use the fact that B is a local ring; this will be important at the end of the proof of Proposition 23.2.17.)

Here is a second vaguer motivation that this definition is plausible for the cotangent space of the origin of $\mathbb{A}^n$. Functions on $\mathbb{A}^n$ should restrict to a linear function on the tangent space. What (linear) function does $x^2 + xy + x + y$ restrict to "near the origin"? You will naturally answer: $x + y$. Thus we "pick off the linear terms". Hence $\mathfrak{m}/\mathfrak{m}^2$ are the linear functionals on the tangent space, so $\mathfrak{m}/\mathfrak{m}^2$ is the cotangent space. In particular, you should picture functions vanishing at a point (i.e. lying in $\mathfrak{m}$) as giving functions on the tangent space in this obvious a way.

13.1.2. *Old-fashioned example.* Computing the Zariski-tangent space is actually quite hands-on, because you can compute it just as you did when you learned multivariable calculus. In $\mathbb{A}^3$, we have a curve cut out by $x + y + z^2 + xyz = 0$ and $x - 2y + z + x^2y^2z^3 = 0$. (You can use Krull's Principal Ideal Theorem 12.3.3 to check that this is a curve, but it is not important to do so.) What is the tangent line near the origin? (Is it even smooth there?) Answer: the first surface looks like $x + y = 0$ and the second surface looks like $x - 2y + z = 0$. The curve has tangent line cut out by $x + y = 0$ and $x - 2y + z = 0$. It is smooth (in the traditional sense). In multivariable calculus, the students do a page of calculus to get the answer, because we aren't allowed to tell them to just pick out the linear terms.

Let's make explicit the fact that we are using. If A is a ring, $\mathfrak{m}$ is a maximal ideal, and $f \in \mathfrak{m}$ is a function vanishing at the point $[\mathfrak{m}] \in \operatorname{Spec} A$, then the Zariski tangent space of $\operatorname{Spec} A/(f)$ at $\mathfrak{m}$ is cut out in the Zariski tangent space of $\operatorname{Spec} A$

(at $\mathfrak{m}$) by the single linear equation $f \pmod{\mathfrak{m}^2}$. The next exercise will force you think this through.

13.1.B. IMPORTANT EXERCISE ("KRULL'S PRINCIPAL IDEAL THEOREM FOR TANGENT SPACES" — BUT MUCH EASIER THAN KRULL'S PRINCIPAL IDEAL THEOREM 12.3.3!). Suppose A is a ring, and $\mathfrak{m}$ a maximal ideal. If $f \in \mathfrak{m}$, show that the Zariski tangent space of A/f is cut out in the Zariski tangent space of A by $f \pmod{\mathfrak{m}^2}$. (Note: we can quotient by f and localize at $\mathfrak{m}$ in either order, as quotienting and localizing commute, (5.3.4.1).) Hence the dimension of the Zariski tangent space of $\operatorname{Spec} A$ at $[\mathfrak{m}]$ is the dimension of the Zariski tangent space of $\operatorname{Spec} A/(f)$ at $[\mathfrak{m}]$, or one less. (That last sentence should be suitably interpreted if the dimension is infinite, although it is less interesting in this case.)

Here is another example to see this principle in action: $x + y + z^2 = 0$ and $x + y + x^2 + y^4 + z^5 = 0$ cuts out a curve, which obviously passes through the origin. If I asked my multivariable calculus students to calculate the tangent line to the curve at the origin, they would do a reams of calculations which would boil down (without them realizing it) to picking off the linear terms. They would end up with the equations $x + y = 0$ and $x + y = 0$, which cuts out a plane, not a line. They would be disturbed, and I would explain that this is because the curve isn't smooth at a point, and their techniques don't work. We on the other hand bravely declare that the cotangent space is cut out by $x + y = 0$, and (will soon) *define* this as a singular point. (Intuitively, the curve near the origin is very close to lying in the plane $x + y = 0$.) Notice: the cotangent space jumped up in dimension from what it was "supposed to be", not down. We will see that this is not a coincidence soon, in Theorem 13.2.1.

Here is a nice consequence of the notion of Zariski tangent space.

13.1.3. Problem. Consider the ring $A = k[x, y, z]/(xy - z^2)$. Show that (x, z) is not a principal ideal.

As $\dim A = 2$ (by Krull's Principal Ideal Theorem 12.3.3), and $A/(x, z) \cong k[y]$ has dimension 1, we see that this ideal is codimension 1 (as codimension is the difference of dimensions for irreducible varieties, Exercise 12.2.E). Our geometric picture is that $\operatorname{Spec} A$ is a cone (we can diagonalize the quadric as $xy - z^2 = ((x + y)/2)^2 - ((x - y)/2)^2 - z^2$, at least if $\operatorname{char} k \neq 2$ — see Exercise 6.4.J), and that (x, z) is a ruling of the cone. (See Figure 13.1 for a sketch.) This suggests that we look at the cone point.

Solution. Let $\mathfrak{m} = (x, y, z)$ be the maximal ideal corresponding to the origin. Then $\operatorname{Spec} A$ has Zariski tangent space of dimension 3 at the origin, and $\operatorname{Spec} A/(x, z)$ has Zariski tangent space of dimension 1 at the origin. But $\operatorname{Spec} A/(f)$ must have Zariski tangent space of dimension at least 2 at the origin by Exercise 13.1.B.

13.1.C. EXERCISE. Show that $(x, z) \subset k[w, x, y, z]/(wz - xy)$ is a codimension 1 ideal that is not principal. (See Figure 13.2 for the projectivization of this situation.) This example was promised in Exercise 6.4.D. You might use it again in Exercise 13.1.D.

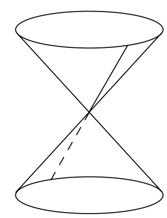


FIGURE 13.1. $V(x,z) \subset \operatorname{Spec} k[x,y,z]/(xy-z^2)$ is a ruling on a cone

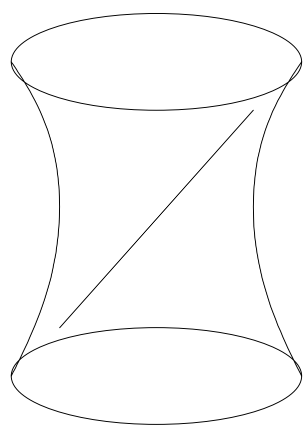


FIGURE 13.2. The ruling $V(x,z)$ on $V(wz-xy) \subset \mathbb{P}^3$.

13.1.D. EXERCISE. Let $A = k[w,x,y,z]/(wz-xy)$. Show that $\operatorname{Spec} A$ is not factorial. (Exercise 6.4.K shows that A is not a unique factorization domain, but this is not enough — why is the localization of A at the prime (w,x,y,z) not factorial? One possibility is to do this "directly", by trying to imitate the solution to Exercise 6.4.K, but this might be hard. Instead, use the intermediate result that in a unique factorization domain, any codimension 1 prime is principal, Lemma 12.2.2, and considering Exercise 13.1.C.) As A is integrally closed if $k = \overline{k}$ and $\operatorname{char} k \neq 2$ (Exercise 6.4.I(c)), this yields an example of a scheme that is normal but not factorial, as promised in Exercise 6.4.F. A slight generalization will be given in 19.4.N.

13.1.4. Morphisms and tangent spaces. Suppose $f : X \to Y$, and $f(p) = q$. Then if we were in the category of manifolds, we would expect a tangent map, from the tangent space of p to the tangent space at q. Indeed that is the case; we have a map of stalks $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, which sends the maximal ideal of the former $\mathfrak{n}$ to the maximal ideal of the latter $\mathfrak{m}$ (we have checked that this is a "local morphism" when we briefly discussed locally ringed spaces). Thus $\mathfrak{n}^2 \to \mathfrak{m}^2$, from which $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$. If $(\mathcal{O}_{X,p}, \mathfrak{m})$ and $(\mathcal{O}_{Y,q}, \mathfrak{n})$ have the same residue field κ, so $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is a linear map of κ-vector spaces, we have a natural map $(\mathfrak{m}/\mathfrak{m}^2)^\vee \to (\mathfrak{n}/\mathfrak{n}^2)^\vee$. This is the map from the tangent space of p to the tangent space at q that we sought. (Aside: note that the *cotangent* map *always* exists, without requiring p and q to have the same residue field — a sign that cotangent spaces are more natural than tangent spaces in algebraic geometry.)

Here are some exercises to give you practice with the Zariski tangent space. If you have some differential geometric background, the first will further convince you that this definition correctly captures the idea of (co)tangent space.

13.1.E. IMPORTANT EXERCISE (THE JACOBIAN COMPUTES THE ZARISKI TANGENT SPACE). Suppose X is a finite type k-scheme. Then locally it is of the form $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Show that the Zariski cotangent space at a k-valued point is given by the cokernel of the Jacobian map $k^r \to k^n$ given by the Jacobian matrix

(13.1.4.1)
$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}.$$

(This is makes precise our example of a curve in $\mathbb{A}^3$ cut out by a couple of equations, where we picked off the linear terms, see Example 13.1.2.) You might be alarmed: what does $\frac{\partial f}{\partial x_1}$ mean? Do you need deltas and epsilons? No! Just define derivatives formally, e.g.

$$\frac{\partial}{\partial x_1}(x_1^2 + x_1 x_2 + x_2^2) = 2x_1 + x_2.$$

Hint: Do this first when p is the origin, and consider linear terms, just as in Example 13.1.2 and Exercise 13.1.B. For the general case, "translate p to the origin".

13.1.5. *Warning.* It is more common in mathematics (but not universal) to define the Jacobian matrix as the dual of this. But for the way we use it, it will be more convenient to use this minority convention.

13.1.F. LESS IMPORTANT EXERCISE ("HIGHER-ORDER DATA"). In Exercise 4.7.B, you computed the equations cutting out the three coordinate axes of $\mathbb{A}^3_k$. (Call this scheme X.) Your ideal should have had three generators. Show that the ideal can't be generated by fewer than three elements. (Hint: working modulo $\mathfrak{m} = (x, y, z)$ won't give any useful information, so work modulo $\mathfrak{m}^2$.)

13.1.G. EXERCISE. Suppose X is a k-scheme. Describe a natural bijection from $\operatorname{Mor}_k(\operatorname{Spec} k[\epsilon]/(\epsilon^2), X)$ to the data of a point p with residue field k (necessarily a closed point) and a tangent vector at p. (This turns out to be very important, for example in deformation theory.)

13.1.H. EXERCISE. Find the dimension of the Zariski tangent space at the point $[(2, 2i)]$ of $\mathbb{Z}[2i] \cong \mathbb{Z}[x]/(x^2 + 4)$. Find the dimension of the Zariski tangent space at the point $[(2, x)]$ of $\mathbb{Z}[\sqrt{-2}] \cong \mathbb{Z}[x]/(x^2 + 2)$. (If you prefer geometric versions of the same examples, replace $\mathbb{Z}$ by or $\mathbb{C}$, and 2 by y: consider $\mathbb{C}[x, y]/(x^2 + y^2)$ and $\mathbb{C}[x, y]/(x^2 + y)$.)

13.2 Nonsingularity

The key idea in the definition of nonsingularity is contained in the following result, that "the dimension of the Zariski tangent space is at least the dimension of the local ring".

13.2.1. Theorem. — *Suppose* $(A, \mathfrak{m}, k)$ *is a Noetherian local ring. Then* $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.

If equality holds, we say that A is a **regular local ring**. (If a Noetherian ring A is regular at all of its primes, A is said to be a **regular ring**, but we won't use this terminology.) A locally Noetherian scheme X is **regular** or **nonsingular** at a point p if the local ring $\mathcal{O}_{X,p}$ is regular. It is **singular** at the point otherwise. A scheme is **regular** or **nonsingular** if it is regular at all points. It is **singular** otherwise (i.e. if it is singular at *at least one* point).

You will hopefully become convinced that this is the right notion of "smoothness" of schemes. Remarkably, Krull introduced the notion of a regular local ring for purely algebraic reasons, some time before Zariski realized that it was a fundamental notion in geometry in 1947.

13.2.2. *Proof of Theorem 13.2.1.* Note that $\mathfrak{m}$ is finitely generated (as A is Noetherian), so $\mathfrak{m}/\mathfrak{m}^2$ is a finitely generated $(A/\mathfrak{m} = k)$-module, hence finite-dimensional. Say $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$. Choose a basis of $\mathfrak{m}/\mathfrak{m}^2$, and lift them to elements $f_1, \dots, f_n$ of $\mathfrak{m}$. Then by Nakayama's lemma (version 4, Exercise 8.2.H), $(f_1, \dots, f_n) = \mathfrak{m}$.

Recall Krull's Theorem 12.3.7: any irreducible component of $V(f_1, \dots, f_n)$ has codimension at most n. In this case, $V((f_1, \dots, f_n)) = V(\mathfrak{m})$ is just the point $[\mathfrak{m}]$, so the codimension of $\mathfrak{m}$ is at most n. Thus the longest chain of prime ideals contained in $\mathfrak{m}$ is at most $n + 1$. But this is also the longest chain of prime ideals in A (as $\mathfrak{m}$ is the unique maximal ideal), so $n \geq \dim A$. □

13.2.A. EXERCISE. Show that Noetherian local rings have finite dimension. (Noetherian rings in general may have infinite dimension, see Exercise 12.1.H.)

13.2.B. EXERCISE (THE SLICING CRITERION FOR NONSINGULARITY). Suppose X is a Noetherian scheme, D is an effective Cartier divisor on X (Definition 9.1.2), and $p \in X$. Show that if p is a nonsingular point of D then p is a nonsingular point of X. (Hint: Krull's Principal Ideal Theorem for tangent spaces, Exercise 13.1.B.)

13.2.3. The Jacobian criterion for nonsingularity, and k-smoothness.

A finite type k-scheme is locally of the form $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. The Jacobian criterion for nonsingularity (Exercise 13.2.C) gives a hands-on method for checking for singularity at closed points, using the equations $f_1, \dots, f_r$, if $k = \overline{k}$.

13.2.C. IMPORTANT EXERCISE (THE JACOBIAN CRITERION — EASY, GIVEN EXERCISE 13.1.E). Suppose $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ has pure dimension d. Show that a k-valued point $p \in X$ is nonsingular if the **corank** of the Jacobian matrix (13.1.4.1) (the dimension of the cokernel) at p is d.

13.2.D. EASY EXERCISE. Suppose $k = \overline{k}$. Show that the singular *closed* points of the hypersurface $f(x_1, \dots, x_n) = 0$ in $\mathbb{A}^n_k$ are given by the equations

$$f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0.$$

(Translation: the singular points of $f = 0$ are where the gradient of f vanishes. This is not shocking.)

13.2.4. Smoothness over a field k, and the Jacobian criterion over non-algebraically closed fields. Before using the Jacobian criterion to get our hands dirty with some explicit varieties, I want to make some general philosophical comments. There seem to be two serious drawbacks with the Jacobian criterion. For finite type schemes over $\overline{k}$, the criterion gives a necessary condition for nonsingularity, but it is not obviously sufficient, as we need to check nonsingularity at non-closed points as well. We can prove sufficiency by working hard to show Fact 13.3.8, which shows that the non-closed points must be nonsingular as well. A second failing is that the criterion requires k to be algebraically closed. These problems suggest that old-fashioned ideas of using derivatives and Jacobians are ill-suited to the correct modern notion of nonsingularity. But in fact the fault is with nonsingularity. There is a better notion of *smoothness over a field*. Better yet, this idea generalizes to the notion of a smooth morphism of schemes, which behaves well in all possible ways (preserved by base change, composition, etc.). This is another sign that some properties we think of as of objects ("absolute notions") should really be thought of as properties of morphisms ("relative notions"). We know enough to imperfectly define what it means for a scheme to be k**-smooth**, or **smooth over** k: a k-scheme is smooth of dimension d if it is reduced and locally of finite type, pure dimension d, and there exist a cover by affine open sets $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ where the Jacobian matrix has corank d everywhere. You can check that any open subset of a smooth k-variety is also a smooth k-variety. We could check that this implies that this is equivalent to the Jacobian being corank d everywhere for *every* affine open cover (and by *any* choice of generators of the ring corresponding to such an open set), and also that it suffices to check at the closed points (rank of a matrix of functions is an upper semicontinuous function). But the cokernel of the Jacobian matrix is secretly the space of differentials (which might not be surprising if you have experience with differentials in differential geometry), so this will come for free when we give the right description of this definition in §26.2.1. The current imperfect definition will suffice for us to work out examples.

13.2.E. EXERCISE (PRACTICE WITH THE CONCEPT). Show that $\mathbb{A}^n_k$ is k-smooth for any n and k. For which characteristics is the curve $y^2 z = x^3 - xz^2$ in $\mathbb{P}^2_k$ smooth over k (cf. Exercise 13.2.I)?

13.2.5. *Nonsingularity vs.* k*-smoothness.* In Exercise 13.2.F, you will establish that a finite type $\overline{k}$-scheme is smooth if and only if it is nonsingular at its closed points (which we will soon see is the same as nonsingularity everywhere, Theorem 13.3.9). It is a nontrivial fact that (i) a smooth k-scheme is necessarily nonsingular, and (ii) a nonsingular finite type k-scheme is smooth *if* k *is perfect* (e.g. if $\operatorname{char} k = 0$ or k is a finite field). We will prove (ii) in §13.3.10. Perfection is necessary in (ii): Let $k = \mathbb{F}_p(u)$, and consider the hypersurface $X = \operatorname{Spec} k[x]/(x^p - u)$.

Now $k[x]/(x^p - u)$ is a field, hence nonsingular. But if $f(x) = x^p - u$, then $f(u^{1/p}) = \frac{df}{dx}(u^{1/p}) = 0$, so the Jacobian criterion fails.

13.2.F. EXERCISE. Show that X is a finite type scheme of pure dimension n over an algebraically closed field $k = \overline{k}$ is nonsingular at its closed points if and only if it is k-smooth. Hint to show nonsingularity implies k-smoothness: use the Jacobian criterion to show that the corank of the Jacobian is n at the closed points of X. Then use the fact that the rank of a matrix is upper semicontinuous.

13.2.6. Back to nonsingularity. We now return to nonsingularity, although many of the following statement are really about k-smoothness. In order to use the Jacobian criterion, we will usually work over an algebraically closed field.

13.2.G. EXERCISE. Suppose $k = \overline{k}$. Show that $\mathbb{A}^1_k$ and $\mathbb{A}^2_k$ are nonsingular. (Make sure to check nonsingularity at the non-closed points! Fortunately you know what all the points of $\mathbb{A}^2_k$ are; this is trickier for $\mathbb{A}^3_k$.) Show that $\mathbb{P}^1_k$ and $\mathbb{P}^2_k$ are nonsingular. (This holds even if k isn't algebraically closed, using the fact that smoothness implies nonsingularity, as discussed in §13.2.5, and in higher dimension, using Fact 13.3.8 below.)

13.2.H. EXERCISE (THE EULER TEST FOR PROJECTIVE HYPERSURFACES). There is an analogous Jacobian criterion for hypersurfaces $f = 0$ in $\mathbb{P}^n_k$. Suppose $k = \overline{k}$. Show that the singular *closed* points correspond to the locus

$$f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0.$$

If the degree of the hypersurface is not divisible by char k (e.g. if char $k = 0$), show that it suffices to check $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. Hint: show that $(\deg f) f = \sum_i x_i \frac{\partial f}{\partial x_1}$. (Fact: this will give the singular points in general, not just the closed points, cf. §13.2.4. I don't want to prove this, and I won't use it.)

13.2.I. EXERCISE. Suppose that $k = \overline{k}$ does not have characteristic 2. Show that $y^2 z = x^3 - xz^2$ in $\mathbb{P}^2_k$ is an irreducible nonsingular curve. (Eisenstein's criterion gives one way of showing irreducibility. Warning: we didn't specify char $k \neq 3$, so be careful when using the Euler test.)

13.2.J. EXERCISE. Suppose $k = \overline{k}$ has characteristic 0. Show that there exists a nonsingular plane curve of degree d. (Feel free to weaken the hypotheses.)

13.2.K. EXERCISE. Find all the singular closed points of the following plane curves. Here we work over $k = \overline{k}$ of characteristic 0 to avoid distractions.

(a) $y^2 = x^2 + x^3$. This is an example of a *node*.
(b) $y^2 = x^3$. This is called a *cusp*; we met it earlier in Exercise 10.6.F.
(c) $y^2 = x^4$. This is called a *tacnode*; we met it earlier in Exercise 10.6.G.

(A precise definition of a node etc. will be given in Definition 13.6.2.)

13.2.L. EXERCISE. Suppose $k = \overline{k}$. Use the Jacobian criterion to how that the twisted cubic $\operatorname{Proj} k[w, x, y, z]/(wz - xy, wy - x^2, xz - y^2)$ is nonsingular. (You can do this, without any hypotheses on k, using the fact that it is isomorphic to $\mathbb{P}^1$. But do this with the explicit equations, for the sake of practice. The twisted cubic was defined in Exercise 9.2.A.)

13.2.7. *Tangent planes and tangent lines.*

Suppose a scheme $X \subset \mathbb{A}^n$ is cut out by equations $f_1, \dots, f_r$, and X is nonsingular of dimension d at the k-valued point $a = (a_1, \dots, a_n)$. Then the **tangent d-plane to X at p** (sometimes denoted $T_p X$) is given by the r equations

$$\frac{\partial f_i}{\partial x_1}(x_1 - a_1) + \cdots + \frac{\partial f_i}{\partial x_n}(x_n - a_n) = 0.$$

13.2.M. EXERCISE. Why is this independent of the *choice* of defining equations f_1, $\dots$, f_r of X?

The Jacobian criterion (Exercise 13.2.C) ensures that these r equations indeed cut out a d-plane. If $d = 1$, this is called the **tangent line**. This is precisely the definition of tangent plane that we see in multivariable calculus, but note that here this is the *definition*, and thus don't have to worry about δ's and ϵ's. Instead we will have to just be careful that it behaves the way we want to.

13.2.N. EXERCISE. Compute the tangent line to the curve of Exercise 13.2.K(b) at $(1, 1)$.

13.2.O. EXERCISE. Suppose $X \subset \mathbb{P}^n_k$ (k as usual a field) is cut out by homogeneous equations $f_1, \dots, f_r$, and $p \in X$ is a k-valued point that is nonsingular of dimension d. Define the (projective) tangent d-plane to X at p. (Definition 9.2.3 gives the definition of a d-plane in $\mathbb{P}^n_k$, but you shouldn't need to refer there.)

13.2.8. *Side remark to help you think cleanly.* We would want the definition of tangent k-plane to be natural in the sense that for any automorphism q of $\mathbb{A}^n_k$ (or, in the case of the previous Exercise, $\mathbb{P}^n_k$), $q(T_p X) = T_{q(p)} q(X)$. You could verify this by hand, but you can also say this in a cleaner way, by interpreting the equations cutting out the tangent space in a coordinate free manner. Informally speaking, we are using the canonical identification of n-space with the tangent space to n-space at p, and using the fact that the Jacobian "linear transformation" cuts out $T_p X$ in $T_p \mathbb{A}^n$ in a way independent of choice of coordinates on $\mathbb{A}^n$ or defining equations of X. Your solution to Exercise 13.2.M will help you start to think in this way.

13.2.P. EXERCISE. Suppose $X \subset \mathbb{P}^n_k$ is a degree d hypersurface cut out by $f = 0$, and L is a line not contained in X. Exercise 9.2.E (a case of Bézout's theorem) showed that X and L meet at d points, counted "with multiplicity". multiplicity d. Suppose L meets X "with multiplicity at least 2" at a k-valued point $p \in L \cap X$, and that p is a nonsingular point of X. Show that L is contained in the tangent plane to X at p.

13.2.9. Arithmetic examples.

13.2.Q. EASY EXERCISE. Show that $\operatorname{Spec} \mathbb{Z}$ is a nonsingular curve.

13.2.R. EXERCISE. (This tricky exercise is for those who know about the primes of the Gaussian integers $\mathbb{Z}[i]$.) There are several ways of showing that $\mathbb{Z}[i]$ is dimension 1 (For example: (i) it is a principal ideal domain; (ii) it is the normalization of

$\mathbb{Z}$ in the field extension $\mathbb{Q}(i)/\mathbb{Q}$; (iii) using Krull's Principal Ideal Theorem 12.3.3 and the fact that $\dim \mathbb{Z}[x] = 2$ by Exercise 12.1.E). Show that $\operatorname{Spec} \mathbb{Z}[i]$ is a nonsingular curve. (There are several ways to proceed. You could use Exercise 13.1.B. For example, consider the prime $(2, 1+i)$, which is cut out by the equations 2 and $1 + x$ in $\operatorname{Spec} \mathbb{Z}[x]/(x^2+1)$.) We will later (§13.4.10) have a simpler approach once we discuss discrete valuation rings.

13.2.S. EXERCISE. Show that $[(5, 5i)]$ is the unique singular point of $\operatorname{Spec} \mathbb{Z}[5i]$. (Hint: $\mathbb{Z}[i]_5 \cong \mathbb{Z}[5i]_5$. Use the previous exercise.)

13.3 Two pleasant facts about regular local rings

Here are two pleasant facts. Because we won't prove them in full generality, we will be careful when using them. In this section only, you may assume these facts in doing exercises. In some sense, the first fact connects regular local rings to algebra, and the second connects them to geometry.

13.3.1. Pleasant Fact (Auslander-Buchsbaum, [E, Thm. 19.19]). — *Regular local rings are unique factorization domains.*

Thus regular schemes are factorial, and hence normal by Exercise 6.4.F.

In particular, as you might expect, a scheme is "locally irreducible" at a "smooth" point: a (Noetherian) regular local ring is an integral domain. This can be shown more directly, [**E**, Cor. 10.14]. (Of course, normality suffices to show that a Noetherian local ring is an integral domain — normal local rings are always integral domains.) Using "power series" ideas, we will prove the following case in §13.6, which will suffice for dealing with varieties.

13.3.2. Theorem. — *Suppose $(A, \mathfrak{m})$ is a regular local ring containing its residue field k (i.e. A is a k-algebra). Then A is an integral domain.*

13.3.A. EXERCISE. Suppose X is a variety over k, and p is a nonsingular k-valued point. Use Theorem 13.3.2 to show that only one irreducible component of X passes through p. (Your argument will apply without change to general Noetherian schemes using Fact 13.3.1.)

13.3.B. EASY EXERCISE. Show that a nonsingular Noetherian scheme is irreducible if and only if it is connected. (Hint: Exercise 6.2.I.)

13.3.3. *Remark: factoriality is weaker than nonsingularity.* There are local rings that are singular but still factorial, so the implication factorial implies nonsingular is strict. Here are is an example that we will verify later. Suppose k is an algebraically closed field of characteristic not 2. Let $A = k[x_1, \dots, x_n]/(x_1^2 + \cdots + x_n^2)$. Note that $\operatorname{Spec} A$ is clearly singular at the origin. In Exercise 15.2.S, we will show that A is a unique factorization domain when $n \geq 5$, so $\operatorname{Spec} A$ is factorial. Note that if $n = 4$, A is not a unique factorization domain, because of our friend the nonsingular quadric, see Exercise 13.1.D. (Aside: More generally, it is a consequence of Grothendieck's proof (of a conjecture of Samuel) that a local Noetherian ring that is a complete intersection — in particular a hypersurface — that is factorial in codimension at most 3 must be factorial, [**SGA2**, Exp. XI, Cor. 3.14].)

13.3.4. Local complete intersections.

(We discuss this now because we will invoke Theorem 13.3.2 in the proof of Theorem 13.3.5.) Suppose Y is a nonsingular (and hence implicitly locally Noetherian) scheme. A closed immersion $\pi : X \hookrightarrow Y$ is said to be a **local complete intersection** (of codimension m) if for each point $x \in X$, the ideal sheaf $\mathcal{I}_{X/Y,x}$ is generated by m elements, and each irreducible component of $\operatorname{Spec} \mathcal{O}_{X,x}$ has codimension m in $\operatorname{Spec} \mathcal{O}_{Y,x}$. (Note that by Theorem 12.3.7, an enhanced version of Krull's Principal Ideal Theorem 12.3.3, if $\mathcal{I}_{X/Y,x}$ is generated by m elements, then each irreducible component of $\operatorname{Spec} \mathcal{O}_{X,x}$ has codimension *at most* m in $\operatorname{Spec} \mathcal{O}_{Y,x}$.)

For example, the union of the three axes in $\mathbb{A}^3_k$ is not a complete intersection, by Exercise 13.1.F. Another example is the cone over the twisted cubic (Exercise 12.2.F), where a Zariski tangent space check will verify that you need three equations cut out this surface in $\mathbb{A}^4_k$.

13.3.C. EXERCISE. Suppose $i : X \hookrightarrow Y$ is a closed immersion into a nonsingular scheme of pure dimension n. Show that the locus of points $x \in X$ where i is a complete intersection is open in X. Hence show that if X is quasicompact, then to check that i is a local complete intersection it suffices to check at closed points of X.

13.3.5. Theorem: "$\overline{k}$-smooth in $\overline{k}$-smooth is always a local complete intersection". — *Suppose $\pi : X \to Y$ is a closed immersion of a pure dimension d $\overline{k}$-smooth variety into a pure dimension n $\overline{k}$-smooth variety. Then that π is a local complete intersection (of codimension $n-d$).*

(These hypotheses are more stringent than necessary, and we discuss how to weaken them in Remark 13.3.6.)

Proof. The final parenthetical comment follows from the rest of the statement, as for varieties, codimension is the difference of dimensions (Exercise 12.2.E).

By Exercise 13.3.C, it suffices to check that π is a local complete intersection at every closed point $x \in X$. Let $\phi : (B, \mathfrak{n}) \twoheadrightarrow (A, \mathfrak{m})$ be the corresponding surjection of local rings. Let I be the kernel of ϕ, and choose generators $f_1, \dots, f_r$ of I. By Exercise 13.1.B, these r equations induce a total of $n-d$ linearly independent equations on the Zariski tangent space $T_x Y$ to obtain the Zariski tangent space $T_x X$. Re-order the f_i so that the $n-d$ cut out the Zariski tangent space $T_x X$ in $T_x Y$. Let $X' = \operatorname{Spec} B/(f_1, \dots, f_{n-d})$. Then by Krull's Principal Ideal Theorem 12.3.3 applied $n-r$ times, $\dim X' \geq m$, while $\dim T_x X' = m$, so by Theorem 13.2.1, $\dim X' = m$, and X' is nonsingular at x. By Theorem 13.3.2, $B/(f_1, \dots, f_{n-d})$ is an integral domain. Thus we have a surjection $B/(f_1, \dots, f_{n-d}) \to B/I \cong A$ of integral domains of the same dimension, so we must have equality (any nonzero element in the kernel would be a non-zero divisor, so the quotient would have strictly smaller dimension by Krull's Principal Ideal Theorem 12.3.3). Thus $I = (f_1, \dots, f_{n-d})$ as desired. $\square$

13.3.6. *Remark: Relaxing hypotheses.* The main thing we needed to make this work is that codimension is the difference of dimension, which is true in reasonable circumstances, including varieties (Exercise 12.2.E), and more generally localizations of finite type algebras over the integers. Theorem 13.3.2 can be replaced by Fact 13.3.1, that regular local rings are always integral domains.

13.3.7. The second pleasant fact.

We come next to the second fact that will help us sleep well at night.

13.3.8. Pleasant Fact (due to Serre, [E, Cor. 19.14], [M-CRT, Thm. 19.3]). — *Suppose $(A, \mathfrak{m})$ is a Noetherian regular local ring. Any localization of A at a prime is also a regular local ring.*

Hence to check if $\operatorname{Spec} A$ is nonsingular (A Noetherian), it suffices to check at closed points (at maximal ideals). This major theorem was an open problem in commutative algebra for a long time until settled by homological methods by Serre. The special case of local rings that are localizations of finite type $\overline{k}$-algebras will be given in Exercise 25.9.E.

13.3.D. EXERCISE. Show (using Fact 13.3.8) that you can check nonsingularity of a Noetherian scheme by checking at closed points. (Caution: as mentioned in Exercise 6.1.E, a scheme in general needn't have any closed points!)

We will be able to prove two important cases of Exercise 13.3.D without invoking Fact 13.3.8. The first will be proved in §25.9.12.

13.3.9. Theorem. — *If X is a finite type $\overline{k}$-scheme that is nonsingular at all its closed points, then X is nonsingular.*

13.3.E. EXERCISE. Suppose X is a Noetherian dimension 1 scheme that is nonsingular at its closed points. Show that X is reduced. Hence show (without invoking Fact 13.3.8) that X is nonsingular.

13.3.F. EXERCISE (GENERALIZING EXERCISE 13.2.J). Suppose k is an algebraically closed field of characteristic 0. Show that there exists a nonsingular hypersurface of degree d in $\mathbb{P}^n$. (As in Exercise 13.2.J, feel free to weaken the hypotheses.)

Although we now know that $\mathbb{A}^n_{\overline{k}}$ is nonsingular (modulo our later proof of Theorem 13.3.9), you may be surprised to find that we never use this fact (although we might make use the fact that it is nonsingular in dimension 0 and codimension 1, which we knew beforehand). Perhaps surprisingly, it is more important to us that $\mathbb{A}^n_{\overline{k}}$ is factorial and hence normal, which we showed more simply. Similarly, geometers may be pleased to finally know that varieties are $\overline{k}$ are nonsingular if and only if they are nonsingular at closed points, but they likely cared only about the closed points anyway. In short, nonsingularity is less important than you might think, except in (co)dimension 1, which is the topic of the next section.

13.3.10. ⋆⋆ Checking nonsingularity of k-schemes at closed points by base changing to $\overline{k}$.

We conclude by fulfilling a promise made in §13.2.5. The Jacobian criterion is a great criterion for checking nonsingularity of finite type k-schemes at k-valued points. The following result extends its applicability to more general closed points.

Suppose X is a finite type k-scheme of pure dimension n, and $p \in X$ is a closed point with residue field k'. By the Nullstellensatz 4.2.3, $k \subset k'$ is a finite extension; suppose that it is separable. Define $\pi : X_{\overline{k}} := X \times_k \overline{k} \to X$ by base change from $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$.

13.3.G. EXERCISE. (a) Suppose $f(x) \in k[x]$ is a separable polynomial (i.e. f has distinct roots in $\overline{k}$), and irreducible, so $k'' := k[x]/(f(x))$ is a field extension of k. Show that $k'' \otimes_k \overline{k}$ is, as a ring, $\overline{k} \times \cdots \times \overline{k}$, where there are $\deg f = \deg k''/k$ factors.
(b) Show that $\pi^{-1}p$ consists of $\deg(\ell/k)$ *reduced* points.

13.3.H. EXERCISE. Suppose p is a closed point of X, with residue field k' that is separable over k of degree d. Show that $X_{\overline{k}}$ is nonsingular at all the preimages p_1, $\dots$, p_d of p if and only if X is nonsingular at p as follows.

(a) Reduce to the case $X = \operatorname{Spec} A$.

(b) Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to p. By tensoring the exact sequence $0 \to \mathfrak{m} \to A \to k' \to 0$ with $\overline{k}$ (field extensions preserve exactness of sequences of vector spaces), interpret

$$0 \to \overline{\mathfrak{m}} \to A \otimes_k \overline{k} \to k' \otimes_k \overline{k} \to 0$$

show that $\mathfrak{m} \otimes_k \overline{k} \subset A \otimes_k \overline{k}$ is the ideal corresponding to the pullback of p to $\operatorname{Spec} A \otimes_k \overline{k}$. Verify that $(\mathfrak{m} \otimes_k \overline{k})^2 = \mathfrak{m}^2 \otimes_k \overline{k}$.

(c) By tensoring the short exact sequence of k-vector spaces $0 \to \mathfrak{m}^2 \to \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \to 0$ with $\overline{k}$, show that

$$\sum_{i=1}^{d} \dim_{\overline{k}} T_{X_{\overline{k}}, p_i} = d \dim_k T_{X,p}.$$

(d) Use Exercise 12.1.D and the inequalities $\dim_{\overline{k}} T_{X_{\overline{k}}, p_i} \leq \dim X_{\overline{k}}$ and $\dim_k T_{X,p} \leq \dim X$ (Theorem 13.2.1) to conclude.

In fact, nonsingularity at a single p_i is enough to conclude nonsingularity at p. (The first idea in showing this: deal with the case when k'/k is Galois, and obtain some transitive group action of $\operatorname{Gal}(k'/k)$ on $\{p_1, \dots, p_d\}$.)

This can be used to extend most of the exercises earlier in this section, usually by replacing the statement that $k = \overline{k}$ with the statement that k is perfect. For example, if k is perfect, then the Jacobian criterion checks for nonsingularity at *all* closed points.

13.4 Discrete valuation rings: Dimension 1 Noetherian regular local rings

The case of (co)dimension 1 is important, because if you understand how primes behave that are separated by dimension 1, then you can use induction to prove facts in arbitrary dimension. This is one reason why Krull's Principal Ideal Theorem 12.3.3 is so useful.

A dimension 1 Noetherian regular local ring can be thought of as a "germ of a smooth curve" (see Figure 13.3). Two examples to keep in mind are $k[x]_{(x)} = \{f(x)/g(x) : x \not| g(x)\}$ and $\mathbb{Z}_{(5)} = \{a/b : 5 \not| b\}$. The first example is "geometric" and the second is "arithmetic", but hopefully it is clear that they are basically the same.

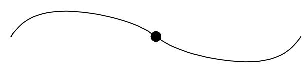


FIGURE 13.3. A germ of a curve

The purpose of this section is to give a long series of equivalent definitions of these rings. Before beginning, we quickly sketch these seven definitions. There are a number of ways a Noetherian local ring can be "nice". It can be regular, or a principal domain, or a unique factorization domain, or normal. In dimension 1, these are the same. Also equivalent are nice properties of ideals: if $\mathfrak{m}$ is principal; or if *all* ideals are either powers of the maximal ideal, or 0. Finally, the ring can have a *discrete valuation*, a measure of "size" of elements that behaves particularly well.

13.4.1. Theorem. — *Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring of dimension* 1*. Then the following are equivalent.*

(a) $(A, \mathfrak{m})$ *is regular.*
(b) $\mathfrak{m}$ *is principal.*

Here is why (a) implies (b). If A is regular, then $\mathfrak{m}/\mathfrak{m}^2$ is one-dimensional. Choose any element $t \in \mathfrak{m} - \mathfrak{m}^2$. Then t generates $\mathfrak{m}/\mathfrak{m}^2$, so generates $\mathfrak{m}$ by Nakayama's lemma 8.2.H. We call such an element a **uniformizer**.

Conversely, if $\mathfrak{m}$ is generated by one element t over A, then $\mathfrak{m}/\mathfrak{m}^2$ is generated by one element t over $A/\mathfrak{m} = k$. Since $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq 1$ by Theorem 13.2.1, we have $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$, and $(A, \mathfrak{m})$ is regular.

We will soon use a useful fact, and we may as well prove it in much more generality than we need, because the proof is so short.

13.4.2. Proposition. — *If* $(A, \mathfrak{m})$ *is a Noetherian local ring, then* $\cap_i \mathfrak{m}^i = 0$*.*

The geometric intuition for this is that any function that is analytically zero at a point (vanishes to all orders) actually vanishes in a neighborhood of that point. The geometric intuition also suggests an example showing that Noetherianness is necessary: consider the function e^{-1/x^2} in the germs of C^∞-functions on $\mathbb{R}$ at the origin.

It is tempting to argue that $\mathfrak{m}(\cap_i \mathfrak{m}^i) = \cap_i \mathfrak{m}^i$, and then to use Nakayama's lemma 8.2.H to argue that $\cap_i \mathfrak{m}^i = 0$. Unfortunately, it is not obvious that this first equality is true: product does not commute with infinite intersections in general. But we will still make this work.

Proof. (A better proof, putting the result into a larger context, is via the Artin-Rees lemma [**E**, Lem. 5.1], see [**E**, Cor. 5.4].) Let $I = \cap_i \mathfrak{m}^i$. We wish to show that $I \subset \mathfrak{m}I$;

then as $\mathfrak{m}I \subset I$, we have $I = \mathfrak{m}I$, and hence by Nakayama's Lemma 8.2.H, $I = 0$. Fix a primary decomposition of $\mathfrak{m}I$. It suffices to show that $\mathfrak{q}$ contains I for any $\mathfrak{q}$ in this primary decomposition, as then I is contained in all the primary ideals in the decomposition of $\mathfrak{m}I$, and hence $\mathfrak{m}I$. Let $\mathfrak{p} = \sqrt{\mathfrak{q}}$.

If $\mathfrak{p} \neq \mathfrak{m}$, then choose $x \in \mathfrak{m} \setminus \mathfrak{p}$. Now x is not nilpotent in $A/\mathfrak{q}$, and hence is not a zerodivisor. (Recall that $\mathfrak{q}$ is primary if and only if in $A/\mathfrak{q}$, each zerodivisor is nilpotent.) But $xI \subset \mathfrak{m}I \subset \mathfrak{q}$, so $I \subset \mathfrak{q}$.

On the other hand, if $\mathfrak{p} = \mathfrak{m}$, then as $\mathfrak{m}$ is finitely generated, and each generator is in $\sqrt{\mathfrak{q}} = \mathfrak{m}$, there is some a such that $\mathfrak{m}^a \subset \mathfrak{q}$. But $I \subset \mathfrak{m}^a$, so we are done. □

13.4.3. Proposition. — *Suppose* $(A, \mathfrak{m})$ *is a Noetherian regular local ring of dimension* 1 *(i.e. satisfying (a) above). Then* A *is an integral domain.*

Proof. Suppose $xy = 0$, and $x, y \neq 0$. Then by Proposition 13.4.2, $x \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$ for some $i \geq 0$, so $x = at^i$ for some $a \notin \mathfrak{m}$. Similarly, $y = bt^j$ for some $j \geq 0$ and $b \notin \mathfrak{m}$. As $a, b \notin \mathfrak{m}$, a and b are invertible. Hence $xy = 0$ implies $t^{i+j} = 0$. But as nilpotents don't affect dimension,

$$\dim A = \dim A/(t) = \dim A/\mathfrak{m} = \dim k = 0, \tag{13.4.3.1}$$

contradicting $\dim A = 1$. □

13.4.4. Theorem. — *Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring of dimension* 1*. Then (a) and (b) are equivalent to:*

(c) *all ideals are of the form* $\mathfrak{m}^n$ *or* (0)*.*

Proof. Assume (a): suppose $(A, \mathfrak{m}, k)$ is a Noetherian regular local ring of dimension 1. Then I claim that $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for any n. Otherwise, by Nakayama's lemma, $\mathfrak{m}^n = 0$, from which $t^n = 0$. But A is an integral domain, so $t = 0$, from which $A = A/\mathfrak{m}$ is a field, which can't have dimension 1, contradiction.

I next claim that $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is dimension 1. Reason: $\mathfrak{m}^n = (t^n)$. So $\mathfrak{m}^n$ is generated as as a A-module by one element, and $\mathfrak{m}^n/(\mathfrak{m}\mathfrak{m}^n)$ is generated as a $(A/\mathfrak{m} = k)$-module by 1 element (non-zero by the previous paragraph), so it is a one-dimensional vector space.

So we have a chain of ideals $A \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ with $\cap \mathfrak{m}^i = (0)$ (Proposition 13.4.2). We want to say that there is no room for any ideal besides these, because "each pair is "separated by dimension 1", and there is "no room at the end". Proof: suppose $I \subset A$ is an ideal. If $I \neq (0)$, then there is some n such that $I \subset \mathfrak{m}^n$ but $I \not\subset \mathfrak{m}^{n+1}$. Choose some $u \in I - \mathfrak{m}^{n+1}$. Then $(u) \subset I$. But u generates $\mathfrak{m}^n/\mathfrak{m}^{n+1}$, hence by Nakayama it generates $\mathfrak{m}^n$, so we have $\mathfrak{m}^n \subset I \subset \mathfrak{m}^n$, so we are done. Conclusion: in a Noetherian local ring of dimension 1, regularity implies all ideals are of the form $\mathfrak{m}^n$ or (0).

We now show that (c) implies (a). Assume (a) is false: suppose we have a dimension 1 Noetherian local integral domain that is not regular, so $\mathfrak{m}/\mathfrak{m}^2$ has dimension at least 2. Choose any $u \in \mathfrak{m} - \mathfrak{m}^2$. Then $(u, \mathfrak{m}^2)$ is an ideal, but $\mathfrak{m} \subsetneq (u, \mathfrak{m}^2) \subsetneq \mathfrak{m}^2$. □

13.4.A. EASY EXERCISE. Suppose $(A, \mathfrak{m})$ is a Noetherian dimension 1 local ring. Show that (a)–(c) above are equivalent to:

(d) A *is a principal ideal domain.*

13.4.5. Discrete valuation rings. We next define the notion of a discrete valuation ring. Suppose K is a field. A **discrete valuation** on K is a **surjective homomorphism** $\nu : K^\times \to \mathbb{Z}$ (in particular, $\nu(xy) = \nu(x) + \nu(y)$) satisfying

$$\nu(x + y) \geq \min(\nu(x), \nu(y))$$

except if $x + y = 0$ (in which case the left side is undefined). (Such a valuation is called *non-archimedean*, although we will not use that term.) It is often convenient to say $\nu(0) = \infty$. More generally, a **valuation** is a surjective homomorphism $\nu : K^\times \to G$ to a totally ordered group G, although this isn't so important to us.

Examples.

(i) *(the 5-adic valuation)* $K = \mathbb{Q}$, $\nu(r)$ is the "power of 5 appearing in r", e.g. $\nu(35/2) = 1$, $\nu(27/125) = -3$.

(ii) $K = k(x)$, $\nu(f)$ is the "power of x appearing in f."

(iii) $K = k(x)$, $\nu(f)$ is the negative of the degree. This is really the same as (ii), with x replaced by $1/x$.

Then $0 \cup \{x \in K^\times : \nu(x) \geq 0\}$ is a ring, which we denote $\mathcal{O}_\nu$. It is called the **valuation ring** of ν. (Not every valuation is discrete. Consider the ring of *Puisseux series* over a field k, $K = \cup_{n \geq 1} k((x^{1/n}))$, with $\nu : K^\times \to \mathbb{Q}$ given by $\nu(x^q) = q$.)

13.4.B. EXERCISE. Describe the valuation rings in the three examples above. (You will notice that they are familiar-looking dimension 1 Noetherian local rings. What a coincidence!)

13.4.C. EXERCISE. Show that $\{0\} \cup \{x \in K^\times : \nu(x) \geq 1\}$ is the unique maximal ideal of the valuation ring. (Hint: show that everything in the complement is invertible.) Thus the valuation ring is a local ring.

An integral domain A is called a **discrete valuation ring** (or **DVR**) if there exists a discrete valuation ν on its fraction field $K = K(A)$ for which $\mathcal{O}_\nu = A$. Similarly, A is a **valuation ring** if there exists a valuation ν on K for which $\mathcal{O}_\nu = A$.

Now if A is a Noetherian regular local ring of dimension 1, and t is a uniformizer (a generator of $\mathfrak{m}$ as an ideal, or equivalently of $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space) then any non-zero element r of A lies in some $\mathfrak{m}^n - \mathfrak{m}^{n+1}$, so $r = t^n u$ where u is a unit (as t^n generates $\mathfrak{m}^n$ by Nakayama, and so does r), so $K(A) = A_t = A[1/t]$. So any element of $K(A)$ can be written uniquely as ut^n where u is a unit and $n \in \mathbb{Z}$. Thus we can define a valuation $\nu(ut^n) = n$.

13.4.D. EXERCISE. Show that ν is a discrete valuation.

13.4.E. EXERCISE. Conversely, suppose $(A, \mathfrak{m})$ is a discrete valuation ring. Show that $(A, \mathfrak{m})$ is a Noetherian regular local ring of dimension 1. (Hint: Show that the ideals are all of the form (0) or $I_n = \{r \in A : \nu(r) \geq n\}$, and (0) and I_1 are the only primes. Thus we have Noetherianness, and dimension 1. Show that I_1/I_2 is generated by the image of any element of $I_1 - I_2$.)

Hence we have proved:

13.4.6. Theorem. — *An integral domain* A *is a Noetherian local ring of dimension* 1 *satisfying (a)–(d) if and only if*

(e) A *is a discrete valuation ring.*

13.4.F. EXERCISE. Show that there is only one discrete valuation on a discrete valuation ring.

13.4.7. *Definition.* Thus any Noetherian regular local ring of dimension 1 comes with a unique valuation on its fraction field. If the valuation of an element is $n > 0$, we say that the element has a **zero of order** n. If the valuation is $-n < 0$, we say that the element has a **pole of order** n. We will come back to this shortly, after dealing with (f) and (g).

13.4.8. Theorem. — *Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring of dimension* 1*. Then (a)–(e) are equivalent to:*

(f) A *is a unique factorization domain,*
(g) A *is integrally closed in its fraction field* $K = K(A)$*.*

Proof. (a)–(e) clearly imply (f), because we have the following stupid unique factorization: each non-zero element of r can be written uniquely as ut^n where $n \in \mathbb{Z}^{\geq 0}$ and u is a unit.

Now (f) implies (g), because unique factorization domains are integrally closed in their fraction fields (Exercise 6.4.F).

It remains to check that (g) implies (a)–(e). We will show that (g) implies (b).

Suppose $(A, \mathfrak{m})$ is a Noetherian local integral domain of dimension 1, integrally closed in its fraction field $K = K(A)$. Choose any nonzero $r \in \mathfrak{m}$. Then $S = A/(r)$ is a Noetherian local ring of dimension 0 — its only prime is the image of $\mathfrak{m}$, which we denote $\mathfrak{n}$ to avoid confusion. Then $\mathfrak{n}$ is finitely generated, and each generator is nilpotent (the intersection of all the prime ideals in any ring are the nilpotents, Theorem 4.2.10). Then $\mathfrak{n}^N = 0$, where N is sufficiently large. Hence there is some n such that $\mathfrak{n}^n = 0$ but $\mathfrak{n}^{n-1} \neq 0$.

Now comes the crux of the argument. Thus in A, $\mathfrak{m}^n \subseteq (r)$ but $\mathfrak{m}^{n-1} \not\subseteq (r)$. Choose $s \in \mathfrak{m}^{n-1} - (r)$. Consider $s/r \in K(A)$. As $s \notin (r)$, $s/r \notin A$, so as A is integrally closed, s/r is not integral over A.

Now $\frac{s}{r}\mathfrak{m} \not\subset \mathfrak{m}$ (or else $\frac{s}{r}\mathfrak{m} \subset \mathfrak{m}$ would imply that $\mathfrak{m}$ is a faithful $A[\frac{s}{r}]$-module, contradicting Exercise 8.2.J). But $s\mathfrak{m} \subset \mathfrak{m}^n \subset rA$, so $\frac{s}{r}\mathfrak{m} \subset A$. Thus $\frac{s}{r}\mathfrak{m} = A$, from which $\mathfrak{m} = \frac{r}{s}A$, so $\mathfrak{m}$ is principal. $\square$

13.4.9. Geometry of normal Noetherian schemes. We can finally make precise (and generalize) the fact that the function $(x-2)^2x/(x-3)^4$ on $\mathbb{A}^1_{\mathbb{C}}$ has a double zero at $x = 2$ and a quadruple pole at $x = 3$. Furthermore, we can say that 75/34 has a double zero at 5, and a single pole at 2. (What are the zeros and poles of $x^3(x+y)/(x^2+xy)^3$ on $\mathbb{A}^2$?) Suppose X is a locally Noetherian scheme. Then for any regular codimension 1 points (i.e. any point p where $\mathcal{O}_{X,p}$ is a regular local ring of dimension 1), we have a discrete valuation ν. If f is any non-zero element of the fraction field of $\mathcal{O}_{X,p}$ (e.g. if X is integral, and f is a non-zero element of the function field of X), then if $\nu(f) > 0$, we say that the element has a **zero of order** $\nu(f)$, and if $\nu(f) < 0$, we say that the element has a **pole of order** $-\nu(f)$. (We aren't yet allowed to discuss order of vanishing at a point that is not regular or codimension 1. One can make a definition, but it doesn't behave as well as it does when have you have a discrete valuation.)

13.4.G. EXERCISE (FINITENESS OF ZEROS AND POLES ON NOETHERIAN SCHEMES). Suppose X is an integral Noetherian scheme, and $f \in K(X)^\times$ is a non-zero element of its function field. Show that f has a finite number of zeros and poles. (Hint: reduce to $X = \operatorname{Spec} A$. If $f = f_1/f_2$, where $f_i \in A$, prove the result for f_i.)

Suppose A is a Noetherian integrally closed domain. Then it is **regular in codimension** 1 (translation: its points of codimension at most 1 are regular). If A is dimension 1, then obviously A is nonsingular.

13.4.H. EXERCISE. If f is a rational function on a locally Noetherian normal scheme with no poles, show that f is regular. (Hint: Algebraic Hartogs' Lemma 12.3.10.)

13.4.10. For example (cf. Exercise 13.2.R), $\operatorname{Spec} \mathbb{Z}[i]$ is nonsingular, because it is dimension 1, and $\mathbb{Z}[i]$ is a unique factorization domain. Hence $\mathbb{Z}[i]$ is normal, so all its closed (codimension 1) points are nonsingular. Its generic point is also nonsingular, as $\mathbb{Z}[i]$ is an integral domain.

13.4.11. *Remark.* A (Noetherian) scheme can be singular in codimension 2 and still be normal. For example, you have shown that the cone $x^2 + y^2 = z^2$ in $\mathbb{A}^3$ is normal (Exercise 6.4.I(b)), but it is singular at the origin (the Zariski tangent space is visibly three-dimensional).

But singularities of normal schemes are not so bad. For example, we have already seen Hartogs' Theorem 12.3.10 for Noetherian normal schemes, which states that you could extend functions over codimension 2 sets.

13.4.12. *Remark.* We know that for Noetherian rings we have implications

$$\text{unique factorization domain} \Longrightarrow \text{integrally closed} \Longrightarrow \text{regular in codimension 1.}$$

Hence for locally Noetherian schemes, we have similar implications:

$$\text{factorial} \Longrightarrow \text{normal} \Longrightarrow \text{regular in codimension 1.}$$

Here are two examples to show you that these inclusions are strict.

13.4.I. EXERCISE (THE KNOTTED PLANE). Let A be the subring $k[x^3, x^2, xy, y] \subset k[x, y]$. (Informally, we allow all polynomials that don't include a non-zero multiple of the monomial x.) Show that $\operatorname{Spec} k[x, y] \to \operatorname{Spec} A$ is a normalization. Show that A is not integrally closed. Show that $\operatorname{Spec} A$ is regular in codimension 1 (hint: show it is dimension 2, and when you throw out the origin you get something nonsingular, by inverting x^2 and y respectively, and considering A_{x^2} and A_y).

13.4.13. *Example.* Suppose k is algebraically closed of characteristic not 2. Then $k[w, x, y, z]/(wz - xy)$ is integrally closed, but not a unique factorization domain, see Exercise 6.4.K (and Exercise 13.1.D).

13.4.14. Dedekind domains. A **Dedekind domain** is a Noetherian integral domain of dimension at most one that is normal (integrally closed in its fraction field). The localization of a Dedekind domain at any prime but (0) (i.e. a codimension one prime) is hence a discrete valuation ring. This is an important notion, but we won't use it much. Rings of integers of number fields are examples, see §10.6.1. In particular, if n is a square free integer congruent to $3 \pmod 4$, then $\mathbb{Z}[\sqrt{n}]$ is a Dedekind domain, by Exercise 6.4.I(a).

13.4.15. *Remark: Serre's criterion that "normal = R1+S2".* Suppose A is a reduced Noetherian integral domain. *Serre's criterion* for normality states that A is normal if and only if A is regular in codimension 1, and every associated prime of a principal ideal generated by a non-zerodivisor is of codimension 1 (i.e. if b is a non-zerodivisor, then Spec $A/(b)$ has no embedded points). The first hypothesis is sometimes called "R1", and the second is called "Serre's S2 criterion". The S2 criterion says rather precisely what is needed for normality in addition to regularity in codimension 1. We won't use this, so we won't prove it here. (See [**E**, §11.2] for a proof.) Note that the necessity of R1 follows from the equivalence of (a) and (g) in Theorem 13.4.8.) An example of a variety satisfying R1 but not S2 is the knotted plane, Exercise 13.4.I.

13.4.J. EXERCISE. Consider two planes in $\mathbb{A}^4_k$ meeting at a point, $V(x, y)$ and $V(z, w)$. Their union $V(xz, xw, yz, yw)$ is not normal, but it is regular in codimension 1. Show that it fails the S2 condition by considering the function $x+z$. (This is a useful example: it is a simple example of a variety that is not Cohen-Macaulay.)

13.4.16. *Remark: Finitely generated modules over a discrete valuation ring.* We record a useful fact for future reference. Recall that finitely generated modules over a principal ideal domain are finite direct sums of cyclic modules (see for example [**DF**, §12.1, Thm. 5]). Hence any finitely generated module over a discrete valuation ring A with uniformizer t is a finite direct sum of terms A and $A/(t^r)$ (for various r). See Proposition 14.7.3 for an immediate consequence.

13.5 Valuative criteria for separatedness and properness

In reasonable circumstances, it is possible to verify separatedness by checking only maps from spectra of discrete valuations rings. There are three reasons you might like this (even if you never use it). First, it gives useful intuition for what separated morphisms look like. Second, given that we understand schemes by maps to them (the Yoneda philosophy), we might expect to understand morphisms by mapping certain maps of schemes to them, and this is how you can interpret the diagram appearing in the valuative criterion. And the third concrete reason is that one of the two directions in the statement is much easier (a special case of the Reduced-to-separated Theorem 11.2.1, see Exercise 13.5.A), and this is the direction we will repeatedly use.

We begin with a valuative criterion that applies in a case that will suffice for the interests of most people, that of finite type morphisms of Noetherian schemes. We will then give a more general version for more general readers.

13.5.1. Theorem (Valuative criterion for separatedness for morphisms of finite type of Noetherian schemes). — *Suppose $f : X \to Y$ is a morphism of finite type of Noetherian schemes. Then f is separated if and only if the following condition holds. For* *any discrete valuation ring* A*, and any diagram of the form*

(13.5.1.1)
$$\begin{tikzcd} \operatorname{Spec} K(A) \arrow[r] \arrow[d, hook, "\textit{open imm.}"'] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

(where the vertical morphism on the left corresponds to the inclusion $A \hookrightarrow K(A)$*), there is at most one morphism* $\operatorname{Spec} A \to X$ *such that the diagram*

(13.5.1.2)
$$\begin{tikzcd} \operatorname{Spec} K(A) \arrow[r] \arrow[d, hook, "\textit{open imm.}"'] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dotted, "\leq 1"] & Y \end{tikzcd}$$

commutes.

13.5.A. EXERCISE (THE EASY DIRECTION). Use the Reduced-to-separated Theorem 11.2.1 to prove one direction of the theorem: that if f is separated, then the valuative criterion holds.

13.5.B. EXERCISE. Suppose X is an irreducible Noetherian separated curve. If $p \in X$ is a nonsingular point, then $\mathcal{O}_{X,p}$ is a discrete valuation ring, so each nonsingular point yields a discrete valuation on $K(X)$. Use the previous exercise to show that distinct points yield distinct valuations.

Here is the intuition behind the valuative criterion (see Figure 13.4). We think of Spec of a discrete valuation ring A as a "germ of a curve", and $\operatorname{Spec} K(A)$ as the "germ minus the origin" (even though it is just a point!). Then the valuative criterion says that if we have a map from a germ of a curve to Y, and have a lift of the map away from the origin to X, then there is at most one way to lift the map from the entire germ. In the case where Y is a field, you can think of this as saying that limits of one-parameter families are unique (if they exist).

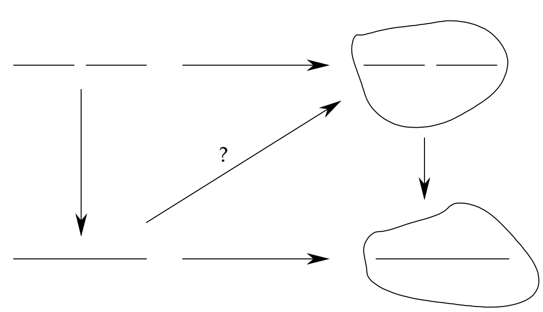


FIGURE 13.4. The line with the doubled origin fails the valuative criterion for separatedness

For example, this captures the idea of what is wrong with the map of the line with the doubled origin over k (Figure 13.5): we take $\operatorname{Spec} A$ to be the germ of the affine line at the origin, and consider the map of the germ minus the origin to the line with doubled origin. Then we have two choices for how the map can extend over the origin.

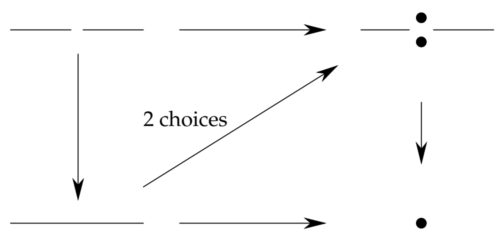


FIGURE 13.5. The valuative criterion for separatedness

13.5.C. EXERCISE. Make this precise: show that map of the line with doubled origin over k to $\operatorname{Spec} k$ fails the valuative criterion for separatedness. (Earlier arguments were given in Exercises 11.1.D and 11.1.M.)

13.5.2. *Remark for experts: moduli spaces and the valuative criterion of separatedness.* If $Y = \operatorname{Spec} k$, and X is a (fine) moduli space (a term I won't define here) of some type of object, then the question of the separatedness of X (over $\operatorname{Spec} k$) has a natural interpretation: given a family of your objects parametrized by a "punctured discrete valuation ring", is there always at most one way of extending it over the closed point?

13.5.3. *Idea behind the proof.* (One direction was done in Exercise 13.5.A.) If f is *not* separated, our goal is to produce a diagram (13.5.1.1) that cannot be completed to (13.5.1.2). If f is not separated, then $\delta : X \to X \times_Y X$ is a locally closed immersion that is not a closed immersion.

13.5.D. EXERCISE. Show that you can find points $p \notin X \times_Y X$ and $q \in X \times_Y X$ such that $p \in \overline{q}$, and there are no points "between p and q" (no points r distinct from p and q with $p \in \overline{r}$ and $r \in \overline{q}$).

Let Q be the scheme obtained by giving the induced reduced subscheme structure to $\overline{q}$. Let $B = \mathscr{O}_{Q,p}$ be the local ring of Q at p.

13.5.E. EXERCISE. Show that B is a Noetherian local integral domain of dimension 1.

If B were regular, then we would be done: composing the inclusion morphism $Q \to X \times_Y X$ with the two projections induces the same morphism $q \to X$ but different extensions to Q precisely because p is not in the diagonal. To complete the proof, one shows that the normalization of B is Noetherian; then localizing at any prime above p (there is one by the Lying Over Theorem 8.2.5) yields the desired discrete valuation ring A.

With a more powerful invocation of commutative algebra, we can prove a valuative criterion with much less restrictive hypotheses.

13.5.4. Theorem (Valuative criterion of separatedness). — *Suppose* $f : X \to Y$ *is a quasiseparated morphism. Then* f *is separated if and only if the following condition holds. For any valuation ring* A *with function field* K*, and any diagram of the form* (13.5.1.1)*, there is at most one morphism* $\operatorname{Spec} A \to X$ *such that the diagram* (13.5.1.2) *commutes.*

Because I have already proved something useful that we will never use, I feel no urge to prove this harder fact. The proof of one direction, that separated implies that the criterion holds, follows from the identical argument as in Exercise 13.5.A.

13.5.5. Valuative criteria of properness.

There is a valuative criterion for properness too. It is philosophically useful, and sometimes directly useful, although we won't need it.

13.5.6. Theorem (Valuative criterion for properness for morphisms of finite type of Noetherian schemes). — *Suppose* $f : X \to Y$ *is a morphism of finite type of locally Noetherian schemes. Then* f *is proper if and only if for any discrete valuation ring* A *and any diagram* (13.5.1.1)*, there is* exactly *one morphism* $\operatorname{Spec} A \to X$ *such that the diagram* (13.5.1.2) *commutes.*

Recall that the valuative criterion for separatedness was the same, except that *exact* was replaced by *at most*.

In the case where Y is a field, you can think of this as saying that limits of one-parameter families always exist, and are unique. This is a useful intuition for the notion of properness.

13.5.F. EXERCISE. Use the valuative criterion of properness to prove that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper if A is Noetherian. (This is a difficult way to prove a fact that we already showed in Theorem 11.3.5.)

13.5.7. *Remarks for experts.* There is a moduli-theoretic interpretation similar to that for separatedness (Remark 13.5.2): X is proper if and only if there is always precisely one way of filling in a family over a punctured discrete valuation ring.

Finally, here is a fancier version of the valuative criterion for properness.

13.5.8. Theorem (Valuative criterion of properness). — *Suppose* $f : X \to Y$ *is a quasiseparated, finite type (hence quasicompact) morphism. Then* f *is proper if and only if the following condition holds. For any valuation ring* A *and any diagram of the form* (13.5.1.1)*, there is* exactly *one morphism* $\operatorname{Spec} A \to X$ *such that the diagram* (13.5.1.2) *commutes.*

13.6 ⋆ Completions

This section will briefly introduce the notion of completions of rings, which generalizes the notion of power series. Our short-term goal is to show that regular local rings appearing on $\overline{k}$-varieties are integral domains (Theorem 13.3.2), and a key fact (§13.6.4) that will be used in the proof that nonsingularity for $\overline{k}$-varieties can be checked at closed points (Theorem 13.3.9). But we will also define some types of singularities such as nodes of curves.

13.6.1. *Definition.* Suppose that I is an ideal of a ring A. Define $\hat{A}$ to be $\varprojlim A/I^i$, the **completion** of A at I (or along I).

13.6.A. EXERCISE. Suppose that I is a maximal ideal $\mathfrak{m}$. Show that the completion construction factors through localization at $\mathfrak{m}$. More precisely, make sense of the following diagram, and show that it commutes.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \hat{A} \arrow[d, "\sim"] \\ A_{\mathfrak{m}} \arrow[r] & \widehat{A_{\mathfrak{m}}} \end{tikzcd}$$

For this reason, one informally thinks of the information in the completion as coming from an even smaller shred of a scheme than the localization.

13.6.B. EXERCISE. If $J \subset A$ is an ideal, figure out how to define the completion $\hat{J} \subset \hat{A}$ (an ideal of $\hat{A}$) using $(J + I^m)/I^m \subset A/I^m$. With your definition, you will observe an isomorphism $\widehat{A/J} \cong \hat{A}/\hat{J}$, which is helpful for computing completions in practice.

13.6.2. *Definition (cf. Exercise 13.2.K).* If X is a $\overline{k}$-variety of pure dimension 1, and p is a closed point, where $\operatorname{char} k \neq 2, 3$. We say that X has a **node** (resp. **cusp, tacnode, triple point**) at p if $\hat{\mathcal{O}}_{X,p}$ is isomorphic to the completion of the curve $\operatorname{Spec} \overline{k}[x, y]/(y^2 - x^2)$ (resp. $\operatorname{Spec} \overline{k}[x, y]/(y^2 - x^3)$, $\operatorname{Spec} \overline{k}[x, y]/(y^2 - x^4)$, $\operatorname{Spec} \overline{k}[x, y]/(y^3 - x^3)$). One can define other singularities similarly (see for example Definition 19.4.4, Exercise 19.4.F, and Remark 19.4.5). You may wish to extend these definitions to more general fields.

Suppose for the rest of this section that $(A, \mathfrak{m})$ is Noetherian local ring containing its residue field k (i.e. it is a k-algebra), of dimension n. Let $x_1, \dots, x_n$ be elements of A whose images are a basis for $\mathfrak{m}/\mathfrak{m}^2$.

13.6.C. EXERCISE. Show that the natural map $A \to \hat{A}$ is an injection. (Hint: Proposition 13.4.2.)

13.6.D. EXERCISE. Show that the map of k-algebras $k[[t_1, \dots, t_n]] \to \hat{A}$ defined by $t_i \mapsto x_i$ is a surjection. (First be clear why there *is* such a map!)

13.6.E. EXERCISE. Show that $\hat{A}$ is a Noetherian local ring. (Hint: By Exercise 4.6.K, $k[[t_1 \dots, t_n]]$ is Noetherian.)

13.6.F. EXERCISE. Show that $k[[t_1, \dots, t_n]]$ is an integral domain. (Possible hint: if $f \in k[[t_1, \dots, t_n]]$ is nonzero, make sense of its "degree", and its "leading term".)

13.6.G. EXERCISE. Show that $k[[t_1, \dots, t_n]]$ is dimension n. (Hint: find a chain of $n+1$ prime ideals to show that the dimension is at least n. For the other inequality, use the multi-equation generalization of Krull, Theorem 12.3.7.)

13.6.H. EXERCISE. If $\mathfrak{p} \subset A$, show that $\hat{\mathfrak{p}}$ is a prime ideal of $\hat{A}$. (Hint: if $f, g \notin \mathfrak{p}$, then let m_f, m_g be the first "level" where they are not in $\mathfrak{p}$ (i.e. the smallest m such that $f \notin \mathfrak{p}/\mathfrak{m}^{m+1}$). Show that $fg \notin \mathfrak{p}/\mathfrak{m}^{m_f+m_g+1}$.)

13.6.I. EXERCISE. Show that if $I \subsetneq J \subset A$ are nested ideals, then $\hat{I} \subsetneq \hat{J}$. Hence (applying this to prime ideals) show that $\dim \hat{A} \geq \dim A$.

Suppose for the rest of this section that $(A, \mathfrak{m})$ is a *regular* local ring.

13.6.J. EXERCISE. Show that $\dim \hat{A} = \dim A$. (Hint: argue $\dim \hat{A} \leq \dim \mathfrak{m}/\mathfrak{m}^2 = \dim A$.)

13.6.3. Theorem. — *Suppose* $(A, \mathfrak{m})$ *is a Noetherian regular local ring containing its residue field* k. *Then* $k[[t_1, \dots, t_n]] \to \hat{A}$ *is an isomorphism.*

(This is basically the Cohen Structure Theorem.) Thus you should think of the map $A \to \hat{A} = k[[x_1, \dots x_n]]$ as sending an element of A to its power series expansion in the variables x_i.

Proof. We wish to show that $k[[t_1, \dots, t_n]] \to \hat{A}$ is injective; we already know it is surjective (Exercise 13.6.D). Suppose $f \in k[[t_1, \dots, t_n]]$ maps to 0, so we get a surjection map $k[[t_1, \dots, t_n]/f \to \hat{A}$. Now f is not a zerodivisor, so by Krull's Principal Ideal Theorem 12.3.3, the left side has dimension $n-1$. But then any quotient of it has dimension at most $n-1$, yielding a contradiction. $\square$

13.6.K. EXERCISE. Prove Theorem 13.3.2, that regular local rings containing their residue field are integral domains.

13.6.4. *Fact for later.* We conclude by mentioning a fact we will use later. Suppose $(A, \mathfrak{m})$ is a regular local ring of dimension n, containing its residue field. Suppose $x_1, \dots, x_m$ are elements of $\mathfrak{m}$ such that their images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent (over k). Let $I = (x_1, \dots, x_m)$. Note that $(A/I, \mathfrak{m})$ is a regular local ring: by Krull's Principal Ideal Theorem 12.3.3, $\dim A/I \geq n - m$, and in A/I, $\mathfrak{m}/\mathfrak{m}^2$ is dimension $n - m$. Thus I is a prime ideal, and I/I^2 is an (A/I)-module.

13.6.L. EXERCISE. Show that $\dim_k (I/I^2) \otimes_{A/I} k = n - m$. (Hint: reduce this to a calculation in the completion. It will be convenient to choose coordinates by extending $x_1, \dots, x_m$ to $x_1, \dots, x_n$.)

Part V

Quasicoherent sheaves

CHAPTER 14

Quasicoherent and coherent sheaves

Quasicoherent and coherent sheaves generalize the notion of a vector bundle. To motivate them, we first discuss vector bundles, and their interpretation as locally free shaves.

A **free sheaf** on X is an $\mathcal{O}_X$-module isomorphic to $\mathcal{O}_X^{\oplus I}$ where the sum is over some index set I. A **locally free sheaf** on a ringed space X is an $\mathcal{O}_X$-module locally isomorphic to a free sheaf. This corresponds to the notion of a vector bundle (§14.1). Quasicoherent sheaves form a convenient abelian category containing the locally free sheaves that is much smaller than the full category of $\mathcal{O}$-modules. Quasicoherent sheaves generalize free sheaves in much the way that modules generalize free modules. Coherent sheaves are roughly speaking a finite rank version of quasicoherent sheaves, which form a well-behaved abelian category containing finite rank locally free sheaves (or equivalently, finite rank vector bundles).

14.1 Vector bundles and locally free sheaves

We recall the notion of vector bundles on smooth manifolds. Nontrivial examples to keep in mind are the tangent bundle to a manifold, and the Möbius strip over a circle (interpreted as a line bundle). Arithmetically-minded readers shouldn't tune out: for example, fractional ideals of the ring of integers in a number field (defined in §10.6.1) turn out to be an example of a "line bundle on a smooth curve" (Exercise 14.1.K).

A **rank n vector bundle on a manifold** M is a fibration $\pi : V \to M$ with the structure of an n-dimensional real vector space on $\pi^{-1}(x)$ for each point $x \in M$, such that for every $x \in M$, there is an open neighborhood U and a homeomorphism

$$\phi : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

over U (so that the diagram

(14.1.0.1)
$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, leftrightarrow, "\cong"] \arrow[dr, "\pi|_{\pi^{-1}(U)}"'] & & U \times \mathbb{R}^n \arrow[dl, "\text{projection to first factor}"] \\ & U & \end{tikzcd}$$

commutes) that is an isomorphism of vector spaces over each $y \in U$. An isomorphism (14.1.0.1) is called a **trivialization over** U.

We call n the **rank** of the vector bundle. A rank 1 vector bundle is called a **line bundle**. (It can also be convenient to be agnostic about the rank of the vector bundle, so it can have different ranks on different connected components. It is also sometimes convenient to consider infinite-rank vector bundles.)

14.1.1. Transition functions. Given trivializations over U_1 and U_2, over their intersection, the two trivializations must be related by an element T_{12} of $GL(n)$ with entries consisting of functions on $U_1 \cap U_2$. If $\{U_i\}$ is a cover of M, and we are given trivializations over each U_i, then the $\{T_{ij}\}$ must satisfy the **cocycle condition**:

$$T_{ij}|_{U_i \cap U_j \cap U_k} \circ T_{jk}|_{U_i \cap U_j \cap U_k} = T_{ik}|_{U_i \cap U_j \cap U_K}. \tag{14.1.1.1}$$

(This implies $T_{ij} = T_{ji}^{-1}$.) The data of the T_{ij} are called **transition functions** (or *transition matrices* for the trivialization.

Conversely, given the data of a cover $\{U_i\}$ and transition functions T_{ij}, we can recover the vector bundle (up to unique isomorphism) by "gluing together the various $U_i \times \mathbb{R}^n$ along $U_i \cap U_j$ using T_{ij}".

14.1.2. The sheaf of sections. Fix a rank n vector bundle $V \to M$. The sheaf of sections $\mathcal{F}$ of V (Exercise 3.2.G) is an $\mathcal{O}_M$-module — given any open set U, we can multiply a section over U by a function on U and get another section.

Moreover, given a trivialization over U, the sections over U are naturally identified with n-tuples of functions of U.

$$\begin{tikzcd} U \times \mathbb{R}^n \arrow[d, shift right, "\pi"'] \\ U \arrow[u, shift right, bend right, "n\text{-tuple of functions}"'] \end{tikzcd}$$

Thus given a trivialization, over each open set U_i, we have an isomorphism $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$. We say that such an $\mathcal{F}$ is a **locally free sheaf of rank** n. (A sheaf $\mathcal{F}$ is **free of rank** n if $\mathcal{F} \cong \mathcal{O}^{\oplus n}$.)

14.1.3. Transition functions for the sheaf of sections. Suppose we have a vector bundle on M, along with a trivialization over an open cover U_i. Suppose we have a section of the vector bundle over M. (This discussion will apply with M replaced by any open subset.) Then over each U_i, the section corresponds to an n-tuple functions over U_i, say $\vec{s}^i$.

14.1.A. EXERCISE. Show that over $U_i \cap U_j$, the vector-valued function $\vec{s}^i$ is related to $\vec{s}^j$ by the transition functions: $T_{ij}\vec{s}^i = \vec{s}^j$. (Don't do this too quickly — make sure your i's and j's are on the correct side.)

Given a locally free sheaf $\mathcal{F}$ with rank n, and a trivializing neighborhood of $\mathcal{F}$ (an open cover $\{U_i\}$ such that over each U_i, $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$ as $\mathcal{O}$-modules), we have transition functions $T_{ij} \in GL(n, \mathcal{O}(U_i \cap U_j))$ satisfying the cocycle condition (14.1.1.1). Thus in conclusion the data of a locally free sheaf of rank n is equivalent to the data of a vector bundle of rank n. This change of perspective is useful, and is similar to an earlier change of perspective when we introduced ringed spaces: understanding spaces is the same as understanding (sheaves of) functions on the spaces, and understanding vector bundles (a type of "space over M") is the same as understanding functions.

14.1.4. *Definition.* A rank 1 locally free sheaf is called an **invertible sheaf**. (Unimportant aside: "invertible sheaf" is a heinous term for something that is essentially a line bundle. The motivation is that if X is a locally ringed space, and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules with $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{O}_X$, then $\mathcal{F}$ and $\mathcal{G}$ are invertible sheaves [**MO**]. Thus in the monoid of $\mathcal{O}_X$-modules under tensor product, invertible sheaves are the invertible elements. We will never use this fact.)

14.1.5. Locally free sheaves on schemes.

We can generalize the notion of locally free sheaves to schemes without change. A **locally free sheaf of rank** n **on a scheme** X is defined as an $\mathcal{O}_X$-module $\mathcal{F}$ that is locally a free sheaf of rank n. Precisely, there is an open cover $\{U_i\}$ of X such that for each U_i, $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$. This open cover determines transition functions — the data of a cover $\{U_i\}$ of X, and functions $T_{ij} \in GL(n, \mathcal{O}(U_i \cap U_j))$ satisfying the cocycle condition (14.1.1.1) — which in turn determine the locally free sheaf. As before, given this data, we can find the sections over any open set U. Informally, they are sections of the free sheaves over each $U \cap U_i$ that agree on overlaps. More formally, for each i, they are $\vec{s}^i = \begin{pmatrix} s_1^i \\ \vdots \\ s_n^i \end{pmatrix} \in \Gamma(U \cap U_i, \mathcal{O}_X)^n$, satisfying $T_{ij}\vec{s}^i = \vec{s}^j$ on $U \cap U_i \cap U_j$.

You should think of these as vector bundles, but just keep in mind that they are not the "same", just equivalent notions. We will later (Definition 18.1.4) define the "total space" of the vector bundle $V \to X$ (a scheme over X) in terms of the sheaf version of Spec (precisely, $\mathcal{S}\mathrm{pec}\,\mathrm{Sym}\, V^\bullet$). But the locally free sheaf perspective will prove to be more useful. As one example: the definition of a locally free sheaf is much shorter than that of a vector bundle.

As in our motivating discussion, it is sometimes convenient to let the rank vary among connected components, or to consider infinite rank locally free sheaves.

14.1.6. Useful constructions, in the form of a series of important exercises.

We now give some useful constructions in the form of a series of exercises. Two hints: Exercises 14.1.B–14.1.G will apply for ringed spaces in general, so you shouldn't use special properties of schemes. Furthermore, they are all local on X, so you can reduce to the case where the locally free sheaves in question are actually free.

14.1.B. EXERCISE. Suppose $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves on X of rank m and n respectively. Show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a locally free sheaf of rank mn.

14.1.C. EXERCISE. If $\mathcal{E}$ is a (finite rank) locally free sheaf on X of rank n, Exercise 14.1.B implies that $\mathcal{E}^\vee := \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ is also a locally free sheaf of rank n. This is called the **dual** of $\mathcal{E}$. Given transition functions for $\mathcal{E}$, describe transition functions for $\mathcal{E}^\vee$. (Note that if $\mathcal{E}$ is rank 1, i.e. invertible, the transition functions of the dual are the inverse of the transition functions of the original.) Show that $\mathcal{E} \cong \mathcal{E}^{\vee\vee}$. (Caution: your argument showing that there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$ better not also show that there is an isomorphism $\mathcal{F}^\vee \cong \mathcal{F}$! We will see an example in §15.1 of a locally free $\mathcal{F}$ that is not isomorphic to its dual: the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n$.)

14.1.D. EXERCISE. If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is a locally free sheaf. (Here $\otimes$ is tensor product as $\mathcal{O}_X$-modules, defined in Exercise 3.5.I.) If $\mathcal{F}$ is an invertible sheaf, show that $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$.

14.1.E. EXERCISE. Recall that tensor products tend to be only right-exact in general. Show that tensoring by a locally free sheaf is exact. More precisely, if $\mathcal{F}$ is a locally free sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ is an exact sequence of $\mathcal{O}_X$-modules, then then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F}$. (Possible hint: it may help to check exactness by checking exactness at stalks. Recall that the tensor product of stalks can be identified with the stalk of the tensor product, so for example there is a "natural" isomorphism $(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F})_x \cong \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x$, Exercise 3.5.I(b).)

14.1.F. EXERCISE. If $\mathcal{E}$ is a locally free sheaf of finite rank, and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, show that $\mathcal{H}om(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}) \cong \mathcal{H}om(\mathcal{F} \otimes \mathcal{E}^\vee, \mathcal{G})$. (Possible hint: first consider the case where $\mathcal{E}$ is free.)

14.1.G. EXERCISE AND IMPORTANT DEFINITION. Show that the invertible sheaves on X, up to isomorphism, form an abelian group under tensor product. This is called the **Picard group** of X, and is denoted $\operatorname{Pic} X$.

Unlike the previous exercises, the next one is specific to schemes.

14.1.H. EXERCISE. Suppose s is a section of a locally free sheaf $\mathcal{F}$ on a scheme X. Define the notion of the **subscheme cut out by** $s = 0$. (Hint: given a trivialization over an open set U, s corresponds to a number of functions $f_1, \dots$ on U; on U, take the scheme cut out by these functions.)

14.1.7. Random concluding remarks.
We define **rational (and regular) sections of a locally free sheaf** on a scheme X just as we did rational (and regular) functions (see for example §6.5 and §7.5).

14.1.I. EXERCISE. Show that locally free sheaves on Noetherian normal schemes satisfy "Hartogs' lemma": sections defined away from a set of codimension at least 2 extend over that set. (Hartogs' lemma for Noetherian normal schemes is Theorem 12.3.10.)

14.1.8. *Remark.* Based on your intuition for line bundles on manifolds, you might hope that every point has a "small" open neighborhood on which all invertible sheaves (or locally free sheaves) are trivial. Sadly, this is not the case. We will eventually see (§21.9.1) that for the curve $y^2 - x^3 - x = 0$ in $\mathbb{A}^2_{\mathbb{C}}$, every nonempty open set has nontrivial invertible sheaves. (This will use the fact that it is an open subset of an *elliptic curve*.)

14.1.J. $\star$ EXERCISE (FOR THOSE WITH SUFFICIENT COMPLEX-ANALYTIC BACKGROUND). Recall the analytification functor (Exercises 7.3.J and 11.1.F), that takes a complex finite type reduced scheme and produces a complex analytic space.
(a) If $\mathcal{L}$ is an invertible sheaf on a complex (algebraic) variety X, define (up to unique isomorphism) the corresponding invertible sheaf on the complex variety X_{an}.
(b) Show that the induced map $\operatorname{Pic} X \to \operatorname{Pic} X_{an}$ is a group homomorphism.
(c) Show that this construction is functorial: if $\pi : X \to Y$ is a morphism of complex

varieties, the following diagram commutes:

$$\begin{tikzcd} \operatorname{Pic} Y \arrow[r, "\pi^*"] \arrow[d] & \operatorname{Pic} X \arrow[d] \\ \operatorname{Pic} Y_{an} \arrow[r, "\pi^*_{an}"] & \operatorname{Pic} X_{an} \end{tikzcd}$$

where the vertical maps are the ones you have defined.

14.1.K. $\star$ EXERCISE (FOR THOSE WITH SUFFICIENT ARITHMETIC BACKGROUND; SEE ALSO PROPOSITION 15.2.7 AND §15.2.10). Recall the definition of the ring of integers $\mathcal{O}_K$ in a number field K, Remark 10.6.1. A **fractional ideal** $\mathfrak{a}$ of $\mathcal{O}_K$ is an $\mathcal{O}_K$-submodule of K such that there is a nonzero $a \in \mathcal{O}_K$ such that $a\mathfrak{a} \subset \mathcal{O}_K$. Products of fractional ideals are defined analogously to products of ideals in a ring (defined in Exercise 4.4.C): $\mathfrak{a}\mathfrak{b}$ consists of (finite) $\mathcal{O}_K$-linear combinations of products of elements of $\mathfrak{a}$ and elements of $\mathfrak{b}$. Thus fractional ideals form a semigroup under multiplication, with $\mathcal{O}_K$ as the identity. In fact fractional ideals of $\mathcal{O}_K$ form a group.

(a) Explain how a fractional ideal on a ring of integers in a number field yields an invertible sheaf.
(b) A fractional ideal is **principal** if it is of the form $r\mathcal{O}_K$ for some $r \in K$. Show that any two that differ by a principal ideal yield the same invertible sheaf.
(c) Show that two fractional ideals that yield the same invertible sheaf differ by a principal ideal.

The *class group* is defined to be the group of fractional ideals modulo the principal ideals (i.e. modulo $K^\times$). This exercise shows that the class group is (isomorphic to) the Picard group of $\mathcal{O}_K$. (This discussion applies to the ring of integers in any global field.)

14.1.9. The problem with locally free sheaves.

Recall that $\mathcal{O}_X$-modules form an abelian category: we can talk about kernels, cokernels, and so forth, and we can do homological algebra. Similarly, vector spaces form an abelian category. But locally free sheaves (i.e. vector bundles), along with reasonably natural maps between them (those that arise as maps of $\mathcal{O}_X$-modules), don't form an abelian category. As a motivating example in the category of differentiable manifolds, consider the map of the trivial line bundle on $\mathbb{R}$ (with coordinate t) to itself, corresponding to multiplying by the coordinate t. Then this map jumps rank, and if you try to define a kernel or cokernel you will get confused.

This problem is resolved by enlarging our notion of nice $\mathcal{O}_X$-modules in a natural way, to quasicoherent sheaves. (You can turn this into two *definitions* of quasicoherent sheaves, equivalent to those we will give. We want a notion that is local on X of course. So we ask for the smallest abelian subcategory of $Mod_{\mathcal{O}_X}$ that is "local" and includes vector bundles. It turns out that the main obstruction to vector bundles to be an abelian category is the failure of cokernels of maps of locally free sheaves — as $\mathcal{O}_X$-modules — to be locally free; we could define quasicoherent sheaves to be those $\mathcal{O}_X$-modules that are locally cokernels, yielding a description that works more generally on ringed spaces, as described in Exercise 14.4.B. You may wish to later check that our future definitions are equivalent to these.)

$\mathcal{O}_X$-modules	$\supset$	quasicoherent sheaves	$\supset$	locally free sheaves
(abelian category)		(abelian category)		(not an abelian category)

Similarly, finite rank locally free sheaves will sit in a nice smaller abelian category, that of *coherent sheaves*.

quasicoherent sheaves	$\supset$	coherent sheaves	$\supset$	finite rank locally free sheaves
(abelian category)		(abelian category)		(not an abelian category)

14.1.10. *Remark: Quasicoherent and coherent sheaves on ringed spaces in general.* We will discuss quasicoherent and coherent sheaves on schemes, but they can be defined more generally on ringed spaces. Many of the results we state will hold in this greater generality, but because the proofs look slightly different, we restrict ourselves to schemes to avoid distraction.

14.2 Quasicoherent sheaves

We now define the notion of *quasicoherent sheaf*. In the same way that a scheme is defined by "gluing together rings", a quasicoherent sheaf over that scheme is obtained by "gluing together modules over those rings". Given an A-module M, we defined an $\mathcal{O}$-module $\tilde{M}$ on $\operatorname{Spec} A$ long ago (Exercise 5.1.D) — the sections over $D(f)$ were M_f.

14.2.1. Theorem. — *Let X be a scheme, and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then let P be the property of affine open sets that $\mathcal{F}|_{\operatorname{Spec} A} \cong \tilde{M}$ for an A-module M. Then P satisfies the two hypotheses of the Affine Communication Lemma 6.3.2.*

We prove this in a moment.

14.2.2. *Definition.* If X is a scheme, then an $\mathcal{O}_X$-module $\mathcal{F}$ is **quasicoherent** if for every affine open subset $\operatorname{Spec} A \subset X$, $\mathcal{F}|_{\operatorname{Spec} A} \cong \tilde{M}$ for some A-module M. By Theorem 14.2.1, it suffices to check this for a collection of affine open sets covering X. For example, $\tilde{M}$ is a quasicoherent sheaf on X, and all locally free sheaves on X are quasicoherent.

14.2.A. UNIMPORTANT EXERCISE (NOT EVERY $\mathcal{O}_X$-MODULE IS A QUASICOHERENT SHEAF). (a) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the origin $[(t)]$, with group $k(t)$ and the usual $k[t]$-module structure. Show that this is an $\mathcal{O}_X$-module that is not a quasicoherent sheaf. (More generally, if X is an integral scheme, and $p \in X$ that is not the generic point, we could take the skyscraper sheaf at p with group the function field of X. Except in a silly circumstances, this sheaf won't be quasicoherent.) See Exercises 9.1.D and 14.3.F for more (pathological) examples of $\mathcal{O}_X$-modules that are not quasicoherent.
(b) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the generic point $[(0)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this *is* a quasicoherent sheaf. Describe the restriction maps in the distinguished topology of X. (Remark: your argument will apply more generally, for example

when X is an integral scheme with generic point η, and $\mathcal{F}$ is the skyscraper sheaf $i_{\eta,*}K(X)$.)

14.2.B. UNIMPORTANT EXERCISE (NOT EVERY QUASICOHERENT SHEAF IS LOCALLY FREE). Use the example of Exercise 14.2.A(b) to show that not every quasicoherent sheaf is locally free.

Proof of Theorem 14.2.1. Clearly if $\operatorname{Spec} A$ has property P, then so does the distinguished open $\operatorname{Spec} A_f$: if M is an A-module, then $\tilde{M}|_{\operatorname{Spec} A_f} \cong \tilde{M}_f$ as sheaves of $\mathcal{O}_{\operatorname{Spec} A_f}$-modules (both sides agree on the level of distinguished open sets and their restriction maps).

We next show the second hypothesis of the Affine Communication Lemma 6.3.2. Suppose we have modules $M_1, \dots, M_n$, where M_i is an A_{f_i}-module, along with isomorphisms $\phi_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ of $A_{f_i f_j}$-modules, satisfying the cocycle condition (14.1.1.1). We want to construct an M such that $\tilde{M}$ gives us $\tilde{M}_i$ on $D(f_i) = \operatorname{Spec} A_{f_i}$, or equivalently, isomorphisms $\rho_i : \Gamma(D(f_i), \tilde{M}) \to M_i$, so that the bottom triangle of

(14.2.2.1)
$$\begin{tikzcd}
& & M \arrow[dl, "\otimes A_{f_i}"'] \arrow[dr, "\otimes A_{f_j}"] & & \\
& M_{f_i} \arrow[dl, "\rho_i"', "\sim"] \arrow[dr, "\otimes A_{f_j}"] & & M_{f_j} \arrow[dl, "\otimes A_{f_j}"'] \arrow[dr, "\rho_j", "\sim"'] & \\
M_i \arrow[dr, "\otimes A_{f_i}"'] & & M_{f_i f_j} & & M_j \arrow[dl, "\otimes A_{f_j}"] \\
& (M_i)_{f_j} \arrow[ur, leftrightarrow, "\sim"'] \arrow[rr, "\phi_{ij}", "\sim"'] & & (M_j)_{f_i} \arrow[ul, leftrightarrow, "\sim"] &
\end{tikzcd}$$

commutes.

14.2.C. EXERCISE. Why does this suffice to prove the result? In other words, why does this imply that $\mathcal{F}|_{\operatorname{Spec} A} \cong \tilde{M}$?

We already know that M should be $\Gamma(\mathcal{F}, \operatorname{Spec} A)$, as $\mathcal{F}$ is a sheaf. Consider elements of $M_1 \times \cdots \times M_n$ that "agree on overlaps"; let this set be M. In other words,

(14.2.2.2)
$$0 \longrightarrow M \longrightarrow M_1 \times \cdots \times M_n \xrightarrow{\gamma} M_{12} \times M_{13} \times \cdots \times M_{(n-1)n}$$

is an exact sequence (where $M_{ij} = (M_i)_{f_j} \cong (M_j)_{f_i}$, and the map γ is the "difference" map. So M is a kernel of a morphism of A-modules, hence an A-module. We are left to show that $M_i \cong M_{f_i}$ (and that this isomorphism satisfies (14.2.2.1)). (At this point, we may proceed in a number of ways, and the reader may wish to find their own route rather than reading on.)

For convenience assume $i = 1$. Localization is exact (Exercise 2.6.F(a)), so tensoring (14.2.2.2) by A_{f_1} yields

(14.2.2.3)
$$0 \longrightarrow M_{f_1} \longrightarrow (M_1)_{f_1} \times (M_2)_{f_1} \times \cdots \times (M_n)_{f_1}$$
$$\longrightarrow M_{12} \times \cdots \times M_{1n} \times (M_{23})_{f_1} \times \cdots \times (M_{(n-1)n})_{f_1}$$

is an exact sequence of A_{f_1}-modules.

We now identify many of the modules appearing in (14.2.2.3) in terms of M_1. First of all, f_1 is invertible in A_{f_1}, so $(M_1)_{f_1}$ is canonically M_1. Also, $(M_j)_{f_1} \cong (M_1)_{f_j}$ via ϕ_{ij}. Hence if $i, j \neq 1$, $(M_{ij})_{f_1} \cong (M_1)_{f_i f_j}$ via ϕ_{1i} and ϕ_{1j} (here the cocycle condition is implicitly used). Furthermore, $(M_{1i})_{f_1} \cong (M_1)_{f_i}$ via ϕ_{1i}. Thus we can write (14.2.2.3) as

$$0 \longrightarrow M_{f_1} \longrightarrow M_1 \times (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \tag{14.2.2.4}$$

$$\xrightarrow{\alpha} (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \times (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n}$$

By assumption, $\mathcal{F}|_{\operatorname{Spec} A_{f_1}} \cong \widetilde{M_1}$ for some M_1, so by considering the cover

$$\operatorname{Spec} A_{f_1} = \operatorname{Spec} A_{f_1} \cup \operatorname{Spec} A_{f_1 f_2} \cup \operatorname{Spec} A_{f_1 f_3} \cup \cdots \cup \operatorname{Spec} A_{f_1 f_n}$$

(notice the "redundant" first term), and identifying sections of $\mathcal{F}$ over $\operatorname{Spec} A_{f_1}$ in terms of sections over the open sets in the cover and their pairwise overlaps, we have an exact sequence of A_{f_1}-modules

$$0 \longrightarrow M_1 \longrightarrow M_1 \times (M_1)_{f_2} \times \cdots \times (M_1)_{f_n}$$

$$\xrightarrow{\beta} (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \times (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n}$$

which is very similar to (14.2.2.4). Indeed, the final map β of the above sequence is the same as the map α of (14.2.2.4), so $\ker \alpha = \ker \beta$, i.e. we have an isomorphism $M_1 \cong M_{f_1}$.

Finally, the triangle of (14.2.2.1) is commutative, as each vertex of the triangle can be identified as the sections of $\mathcal{F}$ over $\operatorname{Spec} A_{f_1 f_2}$. $\square$

14.3 Characterizing quasicoherence using the distinguished affine base

Because quasicoherent sheaves are locally of a very special form, in order to "know" a quasicoherent sheaf, we need only know what the sections are over every affine open set, and how to restrict sections from an affine open set U to a *distinguished* affine open subset of U. We make this precise by defining what I will call the *distinguished affine base* of the Zariski topology — not a base in the usual sense. The point of this discussion is to give a useful characterization of quasicoherence, but you may wish to just jump to §14.3.3.

The open sets of the distinguished affine base are the affine open subsets of X. We have already observed that this forms a base. But forget that fact. We like distinguished open sets $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, and we don't really understand open immersions of one random affine open subset in another. So we just remember the "nice" inclusions.

14.3.1. Definition. The **distinguished affine base** of a scheme X is the data of the affine open sets and the distinguished inclusions.

In other words, we remember only some of the open sets (the affine open sets), and *only some of the morphisms between them* (the distinguished morphisms). For experts: if you think of a topology as a category (the category of open sets), we have described a subcategory.

We can define a sheaf on the distinguished affine base in the obvious way: we have a set (or abelian group, or ring) for each affine open set, and we know how to restrict to distinguished open sets.

Given a sheaf $\mathcal{F}$ on X, we get a sheaf on the distinguished affine base. You can guess where we are going: we will show that all the information of the sheaf is contained in the information of the sheaf on the distinguished affine base.

As a warm-up, we can recover stalks as follows. (We will be implicitly using only the following fact. We have a collection of open subsets, and *some* subsets, such that if we have any $x \in U, V$ where U and V are in our collection of open sets, there is some W containing x, and contained in U and V such that $W \hookrightarrow U$ and $W \hookrightarrow V$ are both in our collection of inclusions. In the case we are considering here, this is the key Proposition 6.3.1 that given any two affine open sets $\operatorname{Spec} A$, $\operatorname{Spec} B$ in X, $\operatorname{Spec} A \cap \operatorname{Spec} B$ could be covered by affine open sets that were simultaneously distinguished in $\operatorname{Spec} A$ and $\operatorname{Spec} B$. In fancy language: the category of affine open sets, and distinguished inclusions, forms a filtered set.)

The stalk $\mathcal{F}_x$ is the colimit $\varinjlim(f \in \mathcal{F}(U))$ where the limit is over all open sets contained in X. We compare this to $\varinjlim(f \in \mathcal{F}(U))$ where the limit is over all affine open sets, and all distinguished inclusions. You can check that the elements of one correspond to elements of the other. (Think carefully about this!)

14.3.A. EXERCISE. Show that a section of a sheaf on the distinguished affine base is determined by the section's germs.

14.3.2. Theorem. —

(a) *A sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)*

(b) *A morphism of sheaves on a distinguished affine base uniquely determines a morphism of sheaves.*

(c) *An $\mathcal{O}_X$-module "on the distinguished affine base" yields an $\mathcal{O}_X$-module.*

This proof is identical to our argument of §3.7 showing that sheaves are (essentially) the same as sheaves on a base, using the "sheaf of compatible germs" construction. The main reason for repeating it is to let you see that all that is needed is for the open sets to form a filtered set (or in the current case, that the category of open sets and distinguished inclusions is filtered).

For experts: (a) and (b) are describing an equivalence of categories between sheaves on the Zariski topology of X and sheaves on the distinguished affine base of X.

Proof. (a) Suppose $\mathcal{F}^b$ is a sheaf on the distinguished affine base. Then we can define stalks.

For *any* open set U of X, define the sheaf of compatible germs

$$\begin{aligned}\mathcal{F}(U) \quad := \quad & \{(f_x \in \mathcal{F}^b_x)_{x \in U} : \text{ for all } x \in U, \\ & \text{there exists } U_x \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}^b(U_x) \\ & \text{such that } F^x_y = f_y \text{ for all } y \in U_x\}\end{aligned}$$

where each U_x is in our base, and F^x_y means "the germ of F^x at y". (As usual, those who want to worry about the empty set are welcome to.)

This is a sheaf: convince yourself that we have restriction maps, identity, and gluability, really quite easily.

I next claim that if U is in our base, that $\mathcal{F}(U) = \mathcal{F}^b(U)$. We clearly have a map $\mathcal{F}^b(U) \to \mathcal{F}(U)$. This is an isomorphism on stalks, and hence an isomorphism by Exercise 3.4.E.

14.3.B. EXERCISE. Prove (b) (cf. Exercise 3.7.C).

14.3.C. EXERCISE. Prove (c) (cf. Remark 3.7.3)

□

14.3.3. A characterization of quasicoherent sheaves in terms of distinguished inclusions. We use this perspective to give a useful characterization of quasicoherent sheaves. Suppose $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A \subset X$ is a distinguished open set. Let $\phi : \Gamma(\operatorname{Spec} A, \mathcal{F}) \to \Gamma(\operatorname{Spec} A_f, \mathcal{F})$ be the restriction map. The source of ϕ is an A-module, and the target is an A_f-module, so by the universal property of localization, ϕ naturally factors as:

$$\begin{tikzcd} \Gamma(\operatorname{Spec} A, \mathcal{F}) \arrow[rr, "\phi"] \arrow[dr] & & \Gamma(\operatorname{Spec} A_f, \mathcal{F}) \\ & \Gamma(\operatorname{Spec} A, \mathcal{F})_f \arrow[ur, "\alpha"'] & \end{tikzcd}$$

14.3.D. VERY IMPORTANT EXERCISE. Show that an $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for each such distinguished $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, α is an isomorphism.

Thus a quasicoherent sheaf is (equivalent to) the data of one module for each affine open subset (a module over the corresponding ring), such that the module over a distinguished open set $\operatorname{Spec} A_f$ is given by localizing the module over $\operatorname{Spec} A$. The next exercise shows that this will be an easy criterion to check.

14.3.E. IMPORTANT EXERCISE (CF. THE QCQS LEMMA 8.3.4). Suppose X is a quasicompact and quasiseparated scheme (i.e. covered by a finite number of affine open sets, the pairwise intersection of which is also covered by a finite number of affine open sets). Suppose $\mathcal{F}$ is a quasicoherent sheaf on X, and let $f \in \Gamma(X, \mathcal{O}_X)$ be a function on X. Show that the restriction map $\operatorname{res}_{X_f \subset X} : \Gamma(X, \mathcal{F}) \to \Gamma(X_f, \mathcal{F})$ (here X_f is the open subset of X where f doesn't vanish) is precisely localization. In other words show that there is an isomorphism $\Gamma(X, \mathcal{F})_f \to \Gamma(X_f, \mathcal{F})$ making the following diagram commute.

$$\begin{tikzcd} \Gamma(X, \mathcal{F}) \arrow[rr, "\operatorname{res}_{X_f \subset X}"] \arrow[dr, "\otimes_A A_f"'] & & \Gamma(X_f, \mathcal{F}) \\ & \Gamma(X, \mathcal{F})_f \arrow[ur, "\sim"'] & \end{tikzcd}$$

All that you should need in your argument is that X admits a cover by a finite number of open sets, and that their pairwise intersections are each quasicompact. (Hint: Apply the exact functor $\otimes_A A_f$ to the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \oplus_i \Gamma(U_i, \mathcal{F}) \to \oplus \Gamma(U_{ijk}, \mathcal{F})$$

where the U_i form a finite affine cover of X and U_{ijk} form a finite affine cover of $U_i \cap U_j$.)

14.3.F. LESS IMPORTANT EXERCISE. Give a counterexample to show that the above statement need not hold if X is not quasicompact. (Possible hint: take an infinite disjoint union of affine schemes. The key idea is that infinite direct products do not commute with localization.)

14.3.G. EXERCISE (GOOD PRACTICE: THE SHEAF OF NILPOTENTS). If A is a ring, and $f \in A$, show that $\mathfrak{N}(A_f) \cong \mathfrak{N}(A)_f$. Use this to show construct the quasicoherent **sheaf of nilpotents** on any scheme X. This is an example of an ideal sheaf (of $\mathcal{O}_X$).

14.3.H. EXERCISE (TO BE USED REPEATEDLY IN §16.3). Generalize Exercise 14.3.E as follows. Suppose X is a quasicompact quasiseparated scheme, $\mathcal{L}$ is an invertible sheaf on X with section s, and $\mathcal{F}$ is a quasicoherent sheaf on X. As in Exercise 14.3.E, let X_s be the open subset of X where s doesn't vanish. Show that any section of $\mathcal{F}$ over X_s can be interpreted as a the quotient of a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ by s^n. More precisely: note that $\oplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ is a graded ring, and we interpret s as a degree 1 element of it. Note also that $\oplus_{n \geq 0} \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is a graded module over this ring. Describe a natural map

$$\left(\left(\oplus_{n \geq 0} \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \right)_s \right)_0 \to \Gamma(X_s, \mathcal{F})$$

and show that it is an isomorphism. (Hint: after showing the existence of the natural map, show the result in the affine case.)

14.3.I. IMPORTANT EXERCISE (COROLLARY TO EXERCISE 14.3.E: PUSHFORWARDS OF QUASICOHERENT SHEAVES ARE QUASICOHERENT IN NON-PATHOLOGICAL CIRCUMSTANCES). Suppose $\pi : X \to Y$ is a quasicompact quasiseparated morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X. Show that $\pi_* \mathcal{F}$ is a quasicoherent sheaf on Y.

14.3.4. $\star\star$ **Grothendieck topologies.** The distinguished affine base isn't a topology in the usual sense — the union of two affine sets isn't necessarily affine, for example. It is however a first new example of a generalization of a topology — the notion of a **site** or a **Grothendieck topology**. We give the definition to satisfy the curious, but we certainly won't use this notion. (For a clean statement, see [**Stacks**, 00VH]; this is intended only as motivation.) The idea is that we should abstract away only those notions we need to define sheaves. We need the notion of open set, but it turns out that we won't even need an underling set, i.e. we won't even need the notion of points! Let's think through how little we need. For our discussion of sheaves to work, we needed to know what the open sets were, and what the (allowed) inclusions were, and these should "behave well", and in particular the data of the open sets and inclusions should form a category. (For example, the composition of an allowed inclusion with another allowed inclusion should be an allowed inclusion — in the distinguished affine base, a distinguished open set of a distinguished open set is a distinguished open set.) So we just require the data of *this category*. At this point, we can already define presheaf (as just a contravariant functor from this category of "open sets"). We saw this idea earlier in Exercise 3.2.A.

In order to extend this definition to that of a sheaf, we need to know more information. We want two open subsets of an open set to intersect in an open set, so *we want the category to be closed under fiber products* (cf. Exercise 2.3.N). For the identity and gluability axioms, we need to know *when some open sets cover another*, so we also remember this as part of the data of a Grothendieck topology. This data of the coverings satisfy some obvious properties. Every open set covers itself (i.e. *the identity map in the category of open sets is a covering*). Coverings pull back: *if we have a map* $Y \to X$*, then any cover of* X *pulls back to a cover of* Y. Finally, *a cover of a cover should be a cover*. Such data (satisfying these axioms) is called a *Grothendieck topology* or a *site*. We can define the notion of a sheaf on a Grothendieck topology in the usual way, with no change. A **topos** is a scary name for a category of sheaves on a Grothendieck topology.

Grothendieck topologies are used in a wide variety of contexts in and near algebraic geometry. Etale cohomology (using the etale topology), a generalization of Galois cohomology, is a central tool, as are more general flat topologies, such as the smooth topology. The definition of a Deligne-Mumford or Artin stack uses the etale and smooth topologies, respectively. Tate developed a good theory of non-archimedean analytic geometry over totally disconnected ground fields such as $\mathbb{Q}_p$ using a suitable Grothendieck topology. Work in K-theory (related for example to Voevodsky's work) uses exotic topologies.

14.4 Quasicoherent sheaves form an abelian category

The category of A-modules is an abelian category. Indeed, this is our motivating example for the notion of abelian category. Similarly, quasicoherent sheaves on a scheme X form an abelian category, which we call $QCoh_X$. Here is how.

When you show that something is an abelian category, you have to check many things, because the definition has many parts. However, if the objects you are considering lie in some ambient abelian category, then it is much easier. You have seen this idea before: there are several things you have to do to check that something is a group. But if you have a subset of group elements, it is much easier to check that it forms a subgroup.

You can look at back at the definition of an abelian category, and you will see that in order to check that a subcategory is an abelian subcategory, you need to check only the following things:

(i) 0 is in the subcategory
(ii) the subcategory is closed under finite sums
(iii) the subcategory is closed under kernels and cokernels

In our case of $QCoh_X \subset Mod_{\mathcal{O}_X}$, the first two are cheap: 0 is certainly quasicoherent, and the subcategory is closed under finite sums: if $\mathcal{F}$ and $\mathcal{G}$ are sheaves on

X, and over $\operatorname{Spec} A$, $\mathcal{F} \cong \tilde{M}$ and $\mathcal{G} \cong \tilde{N}$, then $\mathcal{F} \oplus \mathcal{G} = \widetilde{M \oplus N}$ (do you see why?), so $\mathcal{F} \oplus \mathcal{G}$ is a quasicoherent sheaf.

We now check (iii), using the characterization of Important Exercise 14.3.3. Suppose $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves. Then on any affine open set U, where the morphism is given by $\beta : M \to N$, define $(\ker \alpha)(U) = \ker \beta$ and $(\operatorname{coker} \alpha)(U) = \operatorname{coker} \beta$. Then these behave well under inversion of a single element: if

$$0 \to K \to M \to N \to P \to 0$$

is exact, then so is

$$0 \to K_f \to M_f \to N_f \to P_f \to 0,$$

from which $(\ker \beta)_f \cong \ker(\beta_f)$ and $(\operatorname{coker} \beta)_f \cong \operatorname{coker}(\beta_f)$. Thus both of these define quasicoherent sheaves. Moreover, by checking stalks, they are indeed the kernel and cokernel of α (exactness can be checked stalk-locally). Thus the quasicoherent sheaves indeed form an abelian category.

14.4.A. EXERCISE. Show that a sequence of quasicoherent sheaves $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ on X is exact if and only if it is exact on each open set in any given affine cover of X. (In particular, taking sections over an affine open $\operatorname{Spec} A$ is an exact functor from the category of quasicoherent sheaves on X to the category of A-modules. Recall that taking sections is only left-exact in general, see §3.5.E.) In particular, we may check injectivity or surjectivity of a morphism of quasicoherent sheaves by checking on an affine cover of our choice.

Warning: If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of quasicoherent sheaves, then for any open set

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact, and exactness on the right is guaranteed to hold only if U is affine. (To set you up for cohomology: whenever you see left-exactness, you expect to eventually interpret this as a start of a long exact sequence. So we are expecting H^1's on the right, and now we expect that $H^1(\operatorname{Spec} A, \mathcal{F}) = 0$. This will indeed be the case.)

14.4.B. LESS IMPORTANT EXERCISE (CONNECTION TO ANOTHER DEFINITION, AND QUASICOHERENT SHEAVES ON RINGED SPACES IN GENERAL). Show that an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is quasicoherent if and only if there exists an open cover by U_i such that on each U_i, $\mathcal{F}|_{U_i}$ is isomorphic to the cokernel of a map of two free sheaves:

$$\mathcal{O}_{U_i}^{\oplus I} \to \mathcal{O}_{U_i}^{\oplus J} \to \mathcal{F}|_{U_i} \to 0$$

is exact. We have thus connected our definitions to the definition given at the very start of the chapter. This is the definition of a quasicoherent sheaf on a ringed space in general. It is useful in many circumstances, for example in complex analytic geometry.

14.5 Module-like constructions

In a similar way, basically any nice construction involving modules extends to quasicoherent sheaves. (One exception: the Hom of two A-modules is an A-module, but the $\mathcal{H}om$ of two quasicoherent sheaves is quasicoherent only in "reasonable" circumstances, see Exercise 14.7.A.)

14.5.1. *Locally free sheaves from free modules.*

14.5.A. EXERCISE (POSSIBLE HELP FOR LATER PROBLEMS). (a) Suppose

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \tag{14.5.1.1}$$

is a short exact sequence of locally free sheaves on X. Suppose $U = \operatorname{Spec} A$ is an affine open set where $\mathcal{F}'$, $\mathcal{F}''$ are free, say $\mathcal{F}'|_{\operatorname{Spec} A} = \tilde{A}^{\oplus a}$, $\mathcal{F}''|_{\operatorname{Spec} A} = \tilde{A}^{\oplus b}$. (Here a and b are assumed to be finite for convenience, but this is not necessary, so feel free to generalize to the infinite rank case.) Show that $\mathcal{F}$ is also free, and that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ can be interpreted as coming from the tautological exact sequence $0 \to A^{\oplus a} \to A^{\oplus(a+b)} \to A^{\oplus b} \to 0$. (As a consequence, given an exact sequence of quasicoherent sheaves (14.5.1.1) where $\mathcal{F}'$ and $\mathcal{F}''$ are locally free, $\mathcal{F}$ must also be locally free.)
(b) In the finite rank case, show that given such an open cover, the transition functions (really, matrices) of $\mathcal{F}$ may be interpreted as block upper-diagonal matrices, where the top $a \times a$ block are transition functions for $\mathcal{F}'$, and the bottom $b \times b$ blocks are transition functions for $\mathcal{F}''$.

14.5.B. EXERCISE. Suppose (14.5.1.1) is an exact sequence of quasicoherent sheaves on X. (a) If $\mathcal{F}'$ and $\mathcal{F}''$ are locally free, show that $\mathcal{F}$ is locally free. (Hint: Use the previous exercise.)
(b) If $\mathcal{F}$ and $\mathcal{F}''$ are locally free of finite rank, show that $\mathcal{F}'$ is too. (Hint: Reduce to the case $X = \operatorname{Spec} A$ and $\mathcal{F}$ and $\mathcal{F}''$ free. Interpret the map $\phi : \mathcal{F} \to \mathcal{F}''$ as an $n \times m$ matrix M with values in A, with m the rank of $\mathcal{F}$ and n the rank of $\mathcal{F}''$. For each point p of X, show that there exist n columns $\{c_1, \dots, c_n\}$ of M that are linearly independent at p and hence near p (as linear independence is given by nonvanishing of the appropriate $n \times n$ determinant). Thus X can be covered by distinguished open subsets in bijection with the choices of n columns of M. Restricting to one subset and renaming columns, reduce to the case where the determinant of the first n columns of M is invertible. Then change coordinates on $A^{\oplus m} = \mathcal{F}(\operatorname{Spec} A)$ so that M with respect to the new coordinates is the identity matrix in the first n columns, and 0 thereafter. Finally, in this case interpret $\mathcal{F}'$ as $\widetilde{A^{\oplus(m-n)}}$.
(c) If $\mathcal{F}'$ and $\mathcal{F}$ are both locally free, show that $\mathcal{F}''$ need not be. (Hint: over $k[t]$, consider $0 \to tk[t] \to k[t] \to k[t]/(t) \to 0$. We will soon interpret this as the closed subscheme exact sequence (14.5.5.1) for a point on $\mathbb{A}^1$.)

14.5.2. *Tensor products.* Another important example is tensor products.

14.5.C. EXERCISE. If $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is a quasicoherent sheaf described by the following information: If $\operatorname{Spec} A$ is an affine open, and $\Gamma(\operatorname{Spec} A, \mathcal{F}) = M$ and $\Gamma(\operatorname{Spec} A, \mathcal{G}) = N$, then $\Gamma(\operatorname{Spec} A, \mathcal{F} \otimes \mathcal{G}) = M \otimes N$, and the restriction map $\Gamma(\operatorname{Spec} A, \mathcal{F} \otimes \mathcal{G}) \to \Gamma(\operatorname{Spec} A_f, \mathcal{F} \otimes \mathcal{G})$ is precisely the localization map $M \otimes_A N \to (M \otimes_A N)_f \cong M_f \otimes_{A_f} N_f$. (We are using the algebraic fact that $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. You can prove this by universal property if you want, or by using the explicit construction.)

Note that thanks to the machinery behind the distinguished affine base, sheafification is taken care of. This is a feature we will use often: constructions involving quasicoherent sheaves that involve sheafification for general sheaves don't require sheafification when considered on the distinguished affine base. Along with the fact that injectivity, surjectivity, kernels and so on may be computed on affine opens, this is the reason that it is particularly convenient to think about quasicoherent sheaves in terms of affine open sets.

Given a section s of $\mathcal{F}$ and a section t of $\mathcal{G}$, we have a section $s \otimes t$ of $\mathcal{F} \otimes \mathcal{G}$. If $\mathcal{F}$ is an invertible sheaf, this section is often denoted st.

14.5.3. *Tensor algebra constructions.*

For the next exercises, recall the following. If M is an A-module, then the **tensor algebra** $T^\bullet(M)$ is a non-commutative algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as follows. $T^0(M) = A$, $T^n(M) = M \otimes_A \cdots \otimes_A M$ (where n terms appear in the product), and multiplication is what you expect.

The **symmetric algebra** $\operatorname{Sym}^\bullet M$ is a symmetric algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as the quotient of $T^\bullet(M)$ by the (two-sided) ideal generated by all elements of the form $x \otimes y - y \otimes x$ for all $x, y \in M$. Thus $\operatorname{Sym}^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - m'_1 \otimes \cdots \otimes m'_n$ where $(m'_1, \ldots, m'_n)$ is a rearrangement of $(m_1, \ldots, m_n)$.

The **exterior algebra** $\wedge^\bullet M$ is defined to be the quotient of $T^\bullet M$ by the (two-sided) ideal generated by all elements of the form $x \otimes x$ for all $x \in M$. Expanding $(a+b) \otimes (a+b)$, we see that $a \otimes b = -b \otimes a$ in $\wedge^2 M$. This implies that if 2 is invertible in A (e.g. if A is a field of characteristic not 2), $\wedge^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - (-1)^{\operatorname{sgn}(\sigma)} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}$ where σ is a permutation of $\{1, \ldots, n\}$. The exterior algebra is a "skew-commutative" A-algebra.

It is most correct to write $T^\bullet_A(M)$, $\operatorname{Sym}^\bullet_A(M)$, and $\wedge^\bullet_A(M)$, but the "base ring" A is usually omitted for convenience. (Better: both Sym and $\wedge$ can defined by universal properties. For example, $\operatorname{Sym}^n_A(M)$ is universal among modules such that any map of A-modules $M^{\otimes n} \to N$ that is symmetric in the n entries factors uniquely through $\operatorname{Sym}^n_A(M)$.)

14.5.D. EXERCISE. Suppose $\mathcal{F}$ is a quasicoherent sheaf. Define the quasicoherent sheaves $\operatorname{Sym}^n \mathcal{F}$ and $\wedge^n \mathcal{F}$. (One possibility: describe them on each affine open set, and use the characterization of Important Exercise 14.3.3.) If $\mathcal{F}$ is locally free of rank m, show that $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$ are locally free, and find their ranks.

You can also define the sheaf of non-commutative algebras $T^\bullet \mathcal{F}$, the sheaf of commutative algebras $\operatorname{Sym}^\bullet \mathcal{F}$, and the sheaf of skew-commutative algebras $\wedge^\bullet \mathcal{F}$.

14.5.E. EXERCISE. Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves. Show that for any r, there is a filtration of $\operatorname{Sym}^r \mathcal{F}$

$$\operatorname{Sym}^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = 0$$

with subquotients

$$F^p / F^{p+1} \cong (\operatorname{Sym}^p \mathcal{F}') \otimes (\operatorname{Sym}^{r-p} \mathcal{F}'').$$

(Here are two different possible hints for this and Exercise 14.5.G: (1) Interpret the transition matrices for $\mathcal{F}$ as block upper-diagonal, with two blocks, where one diagonal block gives the transition matrices for $\mathcal{F}'$, and the other gives the transition matrices for $\mathcal{F}''$ (cf. Exercise 14.5.1.1(b)). Then appropriately interpret the transition matrices for $\operatorname{Sym}^r \mathcal{F}$ as block upper-diagonal as well, with $r+1$ blocks. (2) It suffices to consider a small enough affine open set $\operatorname{Spec} A$, where $\mathcal{F}'$, $\mathcal{F}$, $\mathcal{F}''$ are free, and to show that your construction behaves well with respect to localization at an element $f \in A$. In such an open set, the sequence is $0 \to A^{\oplus p} \to A^{\oplus(p+q)} \to A^{\oplus q} \to 0$ by the Exercise 14.5.A. Let $e_1, \dots, e_n$ be the standard basis of A^n, and $f_1, \dots, f_q$ be the the standard basis of $A^{\oplus q}$. Let $e'_1, \dots, e'_p$ be denote the images of $e_1, \dots, e_p$ in A^{p+q}. Let $f'_1, \dots, f'_q$ be any lifts of $f_1, \dots, f_q$ to $A^{\oplus(p+q)}$. Note that f'_i is well-defined modulo $e'_1, \dots, e'_p$. Note that

$$\operatorname{Sym}^s \mathcal{F}|_{\operatorname{Spec} A} \cong \oplus_{i=0}^s \operatorname{Sym}^i \mathcal{F}'|_{\operatorname{Spec} A} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \operatorname{Sym}^{s-i} \mathcal{F}''|_{\operatorname{Spec} A}.$$

Show that $\mathcal{F}^p := \oplus_{i=p}^s \operatorname{Sym}^i \mathcal{F}'|_{\operatorname{Spec} A} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \operatorname{Sym}^{s-i} \mathcal{F}''|_{\operatorname{Spec} A}$ gives a well-defined (locally free) subsheaf that is independent of the choices made, e.g. of the basis $e_1, \dots, e_p$, $f_1, \dots, f_q$, and the lifts $f'_1, \dots, f'_q$.)

14.5.F. EXERCISE. Suppose $\mathcal{F}$ is locally free of rank n. Then $\wedge^n \mathcal{F}$ is called the **determinant (line) bundle** or (both better and worse) the **determinant locally free sheaf**. Describe a map $\wedge^r \mathcal{F} \times \wedge^{n-r} \mathcal{F} \to \wedge^n \mathcal{F}$ that induces an isomorphism of $\wedge^r \mathcal{F} \to (\wedge^{n-r} \mathcal{F}) \otimes \wedge^n \mathcal{F}$. This is called a **perfect pairing of vector bundles**. (If you know about perfect pairings of vector spaces, do you see why this is a generalization?) You might use this later showing duality of Hodge numbers of nonsingular varieties over algebraically closed fields, Exercise 23.4.K.

14.5.G. USEFUL EXERCISE. Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves. Show that for any r, there is a filtration of $\wedge^r \mathcal{F}$:

$$\wedge^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with subquotients

$$F^p / F^{p+1} \cong (\wedge^p \mathcal{F}') \otimes (\wedge^{r-p} \mathcal{F}'')$$

for each p. In particular, $\det \mathcal{F} = (\det \mathcal{F}') \otimes (\det \mathcal{F}'')$. In fact we only need that $\mathcal{F}''$ is locally free.

14.5.H. EXERCISE (DETERMINANT LINE BUNDLES BEHAVE WELL IN EXACT SEQUENCES). Suppose $0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$ is an exact sequence of finite rank locally free sheaves on X. Show that "the alternating product of determinant bundles is trivial":

$$\det(\mathcal{F}_1) \otimes \det(\mathcal{F}_2)^\vee \otimes \det(\mathcal{F}_3) \otimes \det(\mathcal{F}_4)^\vee \otimes \cdots \otimes \det(\mathcal{F}_n)^{(-1)^n} \cong \mathcal{O}_X.$$

(Hint: break the exact sequence into short exact sequences. Use Exercise 14.5.B(b) to show that they are short exact exact sequences of *finite rank locally free sheaves*. Then use the previous Exercise 14.5.G.

14.5.4. Torsion-free sheaves (a stalk-local condition). An A-module M is said to be **torsion-free** if $rm = 0$ implies $r = 0$ or $m = 0$. An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be **torsion-free** if $\mathcal{F}_p$ is a torsion-free $\mathcal{O}_{X,p}$-module for all p.

14.5.I. EXERCISE. Show that if M is a torsion-free A-module, then so is any localization of M. Hence show that $\tilde{M}$ is a torsion free sheaf on $\operatorname{Spec} A$.

14.5.J. UNIMPORTANT EXERCISE (TORSION-FREENESS IS NOT AN AFFINE LOCAL CONDITION FOR STUPID REASONS). Find an example on a two-point space showing that $M := A$ might not be torsion-free on $\operatorname{Spec} A$ even though $\mathcal{O}_{\operatorname{Spec} A} = \tilde{M}$ is torsion-free.

14.5.5. Quasicoherent sheaves of ideals correspond to closed subschemes. Recall that if $i : X \hookrightarrow Y$ is a closed immersion, then we have a surjection of sheaves on Y: $\mathcal{O}_Y \twoheadrightarrow i_* \mathcal{O}_X$ (§9.1). (The i_* is often omitted, as we are considering the sheaf on X as being a sheaf on Y.) The kernel $\mathcal{I}_{X/Y}$ is a "sheaf of ideals" in Y: for each open subset U of Y, the sections form an ideal in the ring of functions on U.

Compare (hard) Exercise 9.1.F and the characterization of quasicoherent sheaves given in (hard) Exercise 14.3.D. You will see that a sheaf is ideas is quasicoherent if and only if it comes from a closed subscheme. (An example of a non-quasicoherent sheaf of ideals was given in Exercise 9.1.D.) We call

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to i_* \mathcal{O}_X \to 0 \tag{14.5.5.1}$$

the **closed subscheme exact sequence** corresponding to $X \hookrightarrow Y$.

14.6 Finite type and coherent sheaves

There are some natural finiteness conditions on an A-module M. I will tell you three. In the case when A is a Noetherian ring, which is the case that almost all of you will ever care about, they are all the same.

The first is the most naive: a module could be **finitely generated**. In other words, there is a surjection $A^p \to M \to 0$.

The second is reasonable too. It could be finitely presented — it could have a finite number of generators with a finite number of relations: there exists a **finite presentation**

$$A^q \to A^p \to M \to 0.$$

14.6.A. EXERCISE ("FINITELY PRESENTED IMPLIES ALWAYS FINITELY PRESENTED"). Suppose M is a finitely presented A-module, and $\phi : A^{p'} \to M$ is *any surjection*. Show that $\ker \phi$ is finitely generated. Hint: Write M as the kernel of A^p by a finitely generated module K. Figure out how to map the short exact sequence $0 \to K \to A^p \to M \to 0$ to the exact sequence $0 \to \ker \phi \to A^{p'} \to M \to 0$, and use the Snake Lemma.

The third notion is frankly a bit surprising, and I will justify it soon. We say that an A-module M is **coherent** if (i) it is finitely generated, and (ii) whenever we have a map $A^p \to M$ (not necessarily surjective!), the kernel is finitely generated.

Clearly coherent implies finitely presented, which in turn implies finitely generated.

14.6.1. Proposition. — *If A is Noetherian, then these three definitions are the same.*

Before proving this, we take this as an excuse to develop some algebraic background.

14.6.2. *Noetherian conditions for modules.* If A is any ring, not necessarily Noetherian, we say an A-module is Noetherian if it satisfies the ascending chain condition for submodules. Thus for example A is a Noetherian ring if and only if it is a Noetherian A-module.

14.6.B. EXERCISE. Show that if M is a Noetherian A-module, then any submodule of M is a finitely generated A-module.

14.6.C. EXERCISE. If $0 \to M' \to M \to M'' \to 0$ is exact, show that M' and M'' are Noetherian if and only if M is Noetherian. (Hint: Given an ascending chain in M, we get two simultaneous ascending chains in M' and M''. Possible further hint: prove that if $M' \longrightarrow M \xrightarrow{\phi} M''$ is exact, and $N \subset N' \subset M$, and $N \cap M' = N' \cap M'$ and $\phi(N) = \phi(N')$, then $N = N'$.)

14.6.D. EXERCISE. Show that if A is a Noetherian ring, then $A^{\oplus n}$ is a Noetherian A-module.

14.6.E. EXERCISE. Show that if A is a Noetherian ring and M is a finitely generated A-module, then M is a Noetherian module. Hence by Exercise 14.6.B, any submodule of a finitely generated module over a Noetherian ring is finitely generated.

Proof of Proposition 14.6.1. As we observed earlier, coherent implies finitely presented implies finitely generated. So suppose M is finitely generated. Take any $A^p \xrightarrow{\alpha} M$. Then $\ker \alpha$ is a submodule of a finitely generated module over A, and is thus finitely generated by Exercise 14.6.E. Thus M is coherent. $\square$

Hence most people can think of these three notions as the same thing.

14.6.3. Proposition. — *The coherent A-modules form an abelian subcategory of the category of A-modules.*

The proof in general is given in §14.8 in a series of short exercises. You should read this only if you are particularly curious.

Proof if A is Noetherian. Recall from our discussion at the start of §14.4 that we must check three things:

(i) The 0-sheaf is coherent.
(ii) The category of coherent modules is closed under finite sums.
(iii) The category of coherent modules is closed under kernels and cokernels

The first two are clear. For (iii), suppose that $f : M \to N$ is a map of finitely generated modules. Then $\operatorname{coker} f$ is finitely generated (it is the image of N), and $\ker f$ is too (it is a submodule of a finitely generated module over a Noetherian ring, Exercise 14.6.E). $\square$

14.6.F. $\star$ EASY EXERCISE (ONLY IMPORTANT FOR NON-NOETHERIAN PEOPLE). Show A is coherent as an A-module if and only if the notion of finitely presented agrees with the notion of coherent.

14.6.G. EXERCISE. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. (The "coherent" case is the tricky one.)

14.6.H. EXERCISE. If $(f_1, \dots, f_n) = A$, and M_{f_i} is finitely generated (resp. finitely presented, coherent) A_{f_i}-module for all i, then M is a finitely generated (resp. finitely presented, coherent) A-module. Hint for the finitely presented case: Exercise 14.6.A.

14.6.4. Definition. A quasicoherent sheaf $\mathcal{F}$ is **finite type** (resp. **finitely presented**, **coherent**) if for every affine open $\operatorname{Spec} A$, $\Gamma(\operatorname{Spec} A, \mathcal{F})$ is a finitely generated (resp. finitely presented, coherent) A-module. Note that coherent sheaves are always finite type, and that on a locally Noetherian scheme, all three notions are the same (by Proposition 14.6.1). By Proposition 14.6.3 implies that the coherent sheaves on X form an abelian category, which we denote Coh_X. Coherence is basically only interesting if $\mathcal{O}_X$ is coherent.

Thanks to the Affine Communication Lemma 6.3.2, and the two previous exercises 14.6.G and 14.6.H, it suffices to check this on the open sets in a single affine cover. Notice that locally free sheaves are always finite type, and if $\mathcal{O}_X$ is coherent, locally free sheaves on X are coherent. (If $\mathcal{O}_X$ is not coherent, then coherence is a pretty useless notion on X.)

I want to say a few words on the notion of coherence. Proposition 14.6.3 is a good motivation for this definition: it gives a small (in a non-technical sense) abelian category in which we can think about vector bundles.

There are two sorts of people who should care about the details of this definition, rather than living in a Noetherian world where coherent means finite type. Complex geometers should care. They consider complex-analytic spaces with the classical topology. One can define the notion of coherent $\mathcal{O}_X$-module in a way analogous to this (see [**S-FAC**, Def. 2]). Then Oka's theorem states that the structure sheaf is coherent, and this is very hard [**GR**, §2.5].

The second sort of people who should care are the sort of arithmetic people who may need to work with non-Noetherian rings. For example, the ring of *adeles* is non-Noetherian.

Warning: it is common in the later literature to incorrectly define coherent as finitely generated. Please only use the correct definition, as the wrong definition causes confusion. Besides doing this for the reason of honesty, it will also help you see what hypotheses are actually necessary to prove things. And that always helps you remember what the proofs are — and hence why things are true.

14.7 Pleasant properties of finite type and coherent sheaves

We begin with the fact that $\mathcal{H}om$ behaves reasonably if the source is coherent.

14.7.A. EXERCISE. (a) Suppose $\mathcal{F}$ is a coherent sheaf on X, and $\mathcal{G}$ is a quasicoherent sheaf on X. Show that $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is a quasicoherent sheaf. (Hint: Describe it on affine open sets, and show that it behaves well with respect to localization with respect to f. To show that $\mathrm{Hom}_A(M,N)_f \cong \mathrm{Hom}_{A_f}(M_f,N_f)$, use Exercise 2.6.G. Up to here, you need only the fact that $\mathcal{F}$ is locally finitely presented.)
(b) If further $\mathcal{G}$ is coherent and $\mathcal{O}_X$ is coherent, show that $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is also coherent. Show that $\mathcal{H}om$ is a left-exact functor in both variables (cf. Exercise 3.5.G). (We remark that the left-exactness fact has nothing to do with quasicoherence — it is true even for $\mathcal{O}_X$-modules, as remarked in §3.5.1.)

14.7.1. *Duals of coherent sheaves.* In particular, if $\mathcal{F}$ is coherent, its **dual** $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F},\mathcal{O})$ is too. This generalizes the notion of duals of vector bundles in Exercise 14.1.C. Your argument there generalizes to show that there is always a natural morphism $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$. Unlike in the vector bundle case, this is not always an isomorphism. (For an example, let $\mathcal{F}$ be the coherent sheaf associated to $k[t]/(t)$ on $\mathbb{A}^1 = \operatorname{Spec} k[t]$, and show that $\mathcal{F}^\vee = 0$.) Coherent sheaves for which the "double dual" map is an isomorphism are called **reflexive sheaves**, but we won't use this notion. The canonical map $\mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}_X$ is called the *trace* map — can you see why?

14.7.B. EXERCISE. Suppose

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{14.7.1.1}$$

is an exact sequence of quasicoherent sheaves on a scheme X, where $\mathcal{H}$ is a locally free quasicoherent sheaf, and suppose $\mathcal{E}$ is a quasicoherent sheaf. Show that the exact sequence (3.5.0.2) is also exact on the right:

$$0 \to \mathcal{H}om(\mathcal{H},\mathcal{E}) \to \mathcal{H}om(\mathcal{G},\mathcal{E}) \to \mathcal{H}om(\mathcal{F},\mathcal{E}) \to 0$$

is an exact sequence. (Hint: this is local, so you can assume that X is affine, say $\operatorname{Spec} A$, and $\mathcal{H} = \widetilde{A^{\oplus n}}$, so (14.7.1.1) can be written as $0 \to M \to N \to A^{\oplus n} \to 0$. Show that this exact sequence splits, so we can write $N = M \oplus A^{\oplus n}$ in a way that respects the exact sequence.) In particular, if $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{O}_X$ are all coherent, then we have an exact sequence of coherent sheaves

$$0 \to \mathcal{H}^\vee \to \mathcal{G}^\vee \to \mathcal{F}^\vee \to 0.$$

14.7.C. EXERCISE (THE SUPPORT OF A FINITE TYPE QUASICOHERENT SHEAF IS CLOSED). This exercise is partially an excuse to discuss the useful notion of "support". Suppose s is a section of a sheaf $\mathcal{F}$ of abelian groups. Define the **support** of s by

$$\operatorname{Supp} s := \{p \in X \ : \ s_p \neq 0 \text{ in } \mathcal{F}_p\}.$$

Define the **support** of $\mathcal{F}$ by $\operatorname{Supp} \mathcal{F} = \{p \in X \ : \ \mathcal{F}_p \neq 0\}$ (cf. Exercise 3.6.F(b)) — the union of "all the supports of sections on various open sets". (Support is a stalk-local notion, and hence behaves well with respect to restriction to open sets, or to stalks. Warning: Support is where the *germ(s)* are nonzero, not where the *value(s)* are nonzero.) Show that the support of a finite type quasicoherent sheaf on a scheme X is a closed subset. (Hint: Reduce to the case X affine. Choose a finite set of generators of the corresponding module.) Show that the support of a quasicoherent sheaf need not be closed. (Hint: If $A = \mathbb{C}[t]$, then $\mathbb{C}[t]/(t-a)$ is an

A-module supported at a. Consider $\oplus_{a \in \mathbb{C}} \mathbb{C}[t]/(t-a)$. Be careful: this example won't work if $\oplus$ is replaced by $\prod$.)

14.7.2. *Remark.* In particular, if X is a locally Noetherian scheme, the sheaf of nilpotents (Exercise 14.3.G) is coherent and in particular finite, and thus has closed support. This makes precise the statement that in good (Noetherian) situations, the fuzz on a scheme is supported on a closed subset, promised in §5.2.1.

We next come to a geometric interpretation of Nakayama's lemma, which is why I consider Nakayama's Lemma a geometric fact (with an algebraic proof).

14.7.D. USEFUL EXERCISE: GEOMETRIC NAKAYAMA (GENERATORS OF A FIBER GENERATE A FINITE TYPE QUASICOHERENT SHEAF NEARBY). Suppose X is a scheme, and $\mathcal{F}$ is a finite type quasicoherent sheaf. Show that if $U \subset X$ is a neighborhood of $x \in X$ and $a_1, \dots, a_n \in \mathcal{F}(U)$ so that the images $\overline{a}_1, \dots, \overline{a}_n \in \mathcal{F}_x$ generate $\mathcal{F}|_x$ (defined as $\mathcal{F}_x \otimes \kappa(x)$, §5.3.5), then there is an affine neighborhood $x \subset \operatorname{Spec} A \subset U$ of x such that "$a_1|_{\operatorname{Spec} A}, \dots, a_n|_{\operatorname{Spec} A}$ generate $\mathcal{F}|_{\operatorname{Spec} A}$" in the following senses:

(i) $a_1|_{\operatorname{Spec} A}, \dots, a_n|_{\operatorname{Spec} A}$ generate $\mathcal{F}(\operatorname{Spec} A)$ as an A-module;
(ii) for any $y \in \operatorname{Spec} A$, $a_1, \dots, a_n$ generate the stalk $\mathcal{F}|_{\operatorname{Spec} A}$ as an $\mathcal{O}_{X,y}$-module (and hence for any $y \in \operatorname{Spec} A$, the fibers $a_1|_y, \dots, a_n|_y$ generate the fiber $\mathcal{F}|_y$ as a $\kappa(y)$-vector space).

In particular, if $\mathcal{F}_x \otimes \kappa(x) = 0$, then there exists a neighborhood V of x such that $\mathcal{F}|_V = 0$.

14.7.E. USEFUL EXERCISE (LOCAL FREENESS OF A COHERENT SHEAF IS A STALK-LOCAL PROPERTY; AND LOCALLY FREE STALKS IMPLY LOCAL FREENESS NEARBY). Suppose $\mathcal{F}$ is a coherent sheaf on scheme X. Show that if $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for some $x \in X$, then $\mathcal{F}$ is locally free in some open neighborhood of X. Hence $\mathcal{F}$ is locally free if and only if $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all $x \in X$. Hint: Find an open neighborhood U of x, and n elements of $\mathcal{F}(U)$ that generate $\mathcal{F}|_x$ and hence by Nakayama's lemma they generate $\mathcal{F}_x$. Use Geometric Nakayama, Exercise 14.7.D, show that the sections generate $\mathcal{F}_y$ for all y in some neighborhood Y of x in U. Thus you have described a surjection $\mathcal{O}_Y^{\oplus n} \to \mathcal{F}|_Y$. Show that the kernel this map is finite type, and hence has closed support (say $Z \subset Y$), which does not contain x. Thus $\mathcal{O}_{Y \setminus Z}^{\oplus n} \to \mathcal{F}|_{Y \setminus Z}$ is an isomorphism.

This is enlightening in a number of ways. It shows that for coherent sheaves, local freeness is a stalk-local condition. Furthermore, on an integral scheme, any coherent sheaf $\mathcal{F}$ is automatically free over the generic point (do you see why?), so every coherent sheaf on an integral scheme is locally free over a dense open subset. And any coherent sheaf that is 0 at the generic point of an irreducible scheme is necessarily 0 on a dense open subset. The last two sentences show the utility of generic points; such statements would have been more mysterious in classical algebraic geometry.

14.7.F. EXERCISE. Show that torsion-free coherent sheaves on a nonsingular (hence implicitly locally Noetherian) curve are locally free. (Although "torsion sheaf" is has not yet been defined, you should also be able to make sense out of the statement: any coherent sheaf is a direct sum of a torsion-free sheaf and a torsion sheaf.)

To answer the previous exercise, use Useful Exercise 14.7.E (local freeness can be checked at stalks) to reduce to the discrete valuation ring case, and recall Remark 13.4.16, the structure theorem for finitely generated modules over a principal ideal domain A: any such module can be written as the direct sum of principal modules $A/(a)$. For discrete valuation rings, this means that the summands are of the form A or $A/\mathfrak{m}^k$. Hence:

14.7.3. Proposition. — *If M is a finitely generated module over a discrete valuation ring, then M is torsion-free if and only if M is free.*

(Exercise 25.2.B is closely related.)

Proposition 14.7.3 is false without the finite generation hypothesis: consider $M = K(A)$ for a suitably general ring A. It is also false if we give up the "dimension 1" hypothesis: consider $(x, y) \subset \mathbb{C}[x, y]$. And it is false if we give up the "nonsingular" hypothesis: consider $(x, y) \subset \mathbb{C}[x, y]/(xy)$. (These examples require some verification.)

14.7.4. Rank of a quasicoherent sheaf at a point.

Suppose $\mathcal{F}$ is a quasicoherent sheaf on a scheme X, and p is a point of X. The vector space $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p = \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} \kappa(p)$ can be interpreted as the fiber of the sheaf at the point, where $\mathfrak{m}$ is the maximal ideal corresponding to p, and $\kappa(p)$ is as usual the residue field at p. A section of $\mathcal{F}$ over an open set containing p can be said to take on a value at that point, which is an element of this vector space. The **rank** of a quasicoherent sheaf $\mathcal{F}$ at a point p is $\dim_{\kappa(p)} \mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$ (possibly infinite). More explicitly, on any affine set $\operatorname{Spec} A$ where $p = [\mathfrak{p}]$ and $\mathcal{F}(\operatorname{Spec} A) = M$, then the rank is $\dim_{K(A/\mathfrak{p})} M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$. Note that this definition of rank is consistent with the notion of rank of a locally free sheaf. In the locally free case, the rank is a (locally) constant function of the point. The converse is sometimes true, see Exercise 14.7.J below.

If X is irreducible, and $\mathcal{F}$ is a quasicoherent (usually coherent) sheaf on X on X, then $\operatorname{rank} \mathcal{F}$ (with no mention of a point) by convention means at the generic point.

14.7.G. EXERCISE. Consider the coherent sheaf $\mathcal{F}$ on $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ corresponding to the module $k[t]/(t)$. Find the rank of $\mathcal{F}$ at every point of $\mathbb{A}^1$. Don't forget the generic point!

14.7.H. EXERCISE. Show that at any point, $\operatorname{rank}(\mathcal{F} \oplus \mathcal{G}) = \operatorname{rank}(\mathcal{F}) + \operatorname{rank}(\mathcal{G})$ and $\operatorname{rank}(\mathcal{F} \otimes \mathcal{G}) = \operatorname{rank} \mathcal{F} \operatorname{rank} \mathcal{G}$ at any point. (Hint: Show that direct sums and tensor products commute with ring quotients and localizations, i.e. $(M \oplus N) \otimes_R (R/I) \cong M/IM \oplus N/IN$, $(M \otimes_R N) \otimes_R (R/I) \cong (M \otimes_R R/I) \otimes_{R/I} (N \otimes_R R/I) \cong M/IM \otimes_{R/I} N/IM$, etc.)

If $\mathcal{F}$ is finite type, then the rank is finite, and by Nakayama's lemma, the rank is the minimal number of generators of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module.

14.7.I. IMPORTANT EXERCISE. If $\mathcal{F}$ is a finite type quasicoherent sheaf on X, show that $\operatorname{rank}(\mathcal{F})$ is an upper semicontinuous function on X. Hint: generators at a point

p are generators nearby by Geometric Nakayama's Lemma, Exercise 14.7.D. (The example in Exercise 14.7.C shows the necessity of the finite type hypothesis.)

14.7.J. IMPORTANT HARD EXERCISE.
(a) If X is reduced, $\mathcal{F}$ is a finite type quasicoherent sheaf on X, and the rank is constant, show that $\mathcal{F}$ is locally free. Then use upper semicontinuity of rank (Exercise 14.7.I) to show that finite type quasicoherent sheaves on an integral scheme are locally free on a dense open set. (By examining your proof, you will see that the Integrality hypothesis can be relaxed. In fact it can be removed completely — reducedness is all that is necessary.) Hint: Reduce to the case where X is affine. Then show it in a neighborhood of a closed point p as follows. (You will have have to show that this suffices, using the affine assumption. But note that closed points aren't necessarily dense in an affine scheme, see for example Exercise 4.4.J) Suppose $n = \operatorname{rank} \mathcal{F}$. Choose n generators of the fiber $\mathcal{F}|_p$ (a basis as an $\kappa(p)$-vector space). By Geometric Nakayama's Lemma 14.7.D, we can find a smaller neighborhood $p \in \operatorname{Spec} A \subset X$, with $\mathcal{F}|_{\operatorname{Spec} A} = \tilde{M}$, so that the chosen generators $\mathcal{F}|_p$ lift to generators $m_1, \dots, m_n$ of M. Let $\phi : A^n \to M$ with $(r_1, \dots, r_n) \mapsto \sum r_i m_i$. If $\ker \phi \neq 0$, then suppose $(r_1, \dots, r_n)$ is in the kernel, with $r_1 \neq 0$. As $r_1 \neq 0$, there is some $\mathfrak{p}$ where $r_1 \notin \mathfrak{p}$ — here we use the reduced hypothesis. Then r_1 is invertible in $A_\mathfrak{p}$, so $M_\mathfrak{p}$ has fewer than n generators, contradicting the constancy of rank.
(b) Show that part (a) can be false without the condition of X being reduced. (Hint: $\operatorname{Spec} k[x]/x^2$, $M = k$.)

You can use the notion of rank to help visualize finite type quasicoherent sheaves, or even quasicoherent sheaves. For example, I think of a coherent sheaf as generalizing a finite rank vector bundle as follows: to each point there is an associated vector space, and although the ranks can jump, they fit together in families as well as one might hope. You might try to visualize the example of Example 14.7.G. Nonreducedness can fit into the picture as well — how would you picture the coherent sheaf on $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$ corresponding to $k[\epsilon]/(\epsilon)$? How about $k[\epsilon]/(\epsilon^2) \oplus k[\epsilon]/(\epsilon)$?

14.7.5. *Degree of a finite morphism at a point.* Suppose $\pi : X \to Y$ is a finite morphism. Then $\pi_* \mathcal{O}_X$ is a finite type (quasicoherent) sheaf on Y, and the rank of this sheaf at a point p is called the **degree** of the finite morphism at p. By Exercise 14.7.I, the degree of π is a upper semicontinuous function on Y. The degree can jump: consider the closed immersion of a point into a line corresponding to $k[t] \to k$ given by $t \mapsto 0$. It can also be constant in cases that you might initially find surprising — see Exercise 10.3.3, where the degree is always 2, but the 2 is obtained in a number of different ways.

14.7.K. EXERCISE. Suppose $\pi : X \to Y$ is a finite morphism. By unwinding the definition, verify that the degree of π at p is the dimension of the space of functions of the scheme-theoretic preimage of p, considered as a vector space over the residue field $\kappa(p)$. In particular, the degree is zero if and only if $\pi^{-1}(p)$ is empty.

14.8 $\star\star$ Coherent modules over non-Noetherian rings

This section is intended for people who might work with non-Noetherian rings, or who otherwise might want to understand coherent sheaves in a more general setting. Read this only if you really want to!

Suppose A is a ring. Recall the definition of when an A-module M is finitely generated, finitely presented, and coherent. The reason we like coherence is that coherent modules form an abelian category. Here are some accessible exercises working out why these notions behave well. Some repeat earlier discussion in order to keep this section self-contained.

The notion of coherence of a module is only interesting in the case that a ring is coherent over itself. Similarly, coherent sheaves on a scheme X will be interesting only when $\mathcal{O}_X$ is coherent ("over itself"). In this case, coherence is clearly the same as finite presentation. An example where non-Noetherian coherence comes up is the ring $R\langle x_1, \dots, x_n \rangle$ of "restricted power series" over a valuation ring R of a non-discretely valued K (for example, a completion of the algebraic closure of $\mathbb{Q}_p$). This is relevant to Tate's theory of non-archimedean analytic geometry over K. The importance of the coherence of the structure sheaf underlines the importance of Oka's theorem in complex geometry.

14.8.A. EXERCISE. Show that coherent implies finitely presented implies finitely generated. (This was discussed in the previous section.)

14.8.B. EXERCISE. Show that 0 is coherent.

Suppose for problems 14.8.C–14.8.I that

$$0 \to M \to N \to P \to 0 \tag{14.8.0.1}$$

is an exact sequence of A-modules. In thise series of problems, we will show that if two of $\{M, N, P\}$ are coherent, the third is as well, which will prove very useful.

14.8.1. Hint †. The following hint applies to several of the problems: try to write

$$\begin{tikzcd} 0 \arrow[r] & A^p \arrow[r] \arrow[d] & A^{p+q} \arrow[r] \arrow[d] & A^q \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & N \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

and possibly use the Snake Lemma 2.7.5.

14.8.C. EXERCISE. Show that N finitely generated implies P finitely generated. (You will only need right-exactness of (14.8.0.1).)

14.8.D. EXERCISE. Show that M, P finitely generated implies N finitely generated. (Possible hint: †.) (You will only need right-exactness of (14.8.0.1).)

14.8.E. EXERCISE. Show that N, P finitely generated need not imply M finitely generated. (Hint: if I is an ideal, we have $0 \to I \to A \to A/I \to 0$.)

14.8.F. EXERCISE. Show that N coherent, M finitely generated implies M coherent. (You will only need left-exactness of (14.8.0.1).)

14.8.G. EXERCISE. Show that N, P coherent implies M coherent. Hint for (i):

$$\begin{tikzcd}
 & A^q \arrow[d] \arrow[dr] & & & \\
 & & A^p \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[dr] & 0 \\
 & 0 & 0 & & 0
\end{tikzcd}$$

(You will only need left-exactness of (14.8.0.1).)

14.8.H. EXERCISE. Show that M finitely generated and N coherent implies P coherent. (Hint for (ii): †.)

14.8.I. EXERCISE. Show that M, P coherent implies N coherent. (Hint: †.)

14.8.J. EXERCISE. Show that a finite direct sum of coherent modules is coherent.

14.8.K. EXERCISE. Suppose M is finitely generated, N coherent. Then if $\phi : M \to N$ is any map, then show that $\operatorname{Im} \phi$ is coherent.

14.8.L. EXERCISE. Show that the kernel and cokernel of maps of coherent modules are coherent.

At this point, we have verified that coherent A-modules form an abelian subcategory of the category of A-modules. (Things you have to check: 0 should be in this set; it should be closed under finite sums; and it should be closed under taking kernels and cokernels.)

14.8.M. EXERCISE. Suppose M and N are coherent submodules of the coherent module P. Show that $M + N$ and $M \cap N$ are coherent. (Hint: consider the right map $M \oplus N \to P$.)

14.8.N. EXERCISE. Show that if A is coherent (as an A-module) then finitely presented modules are coherent. (Of course, if finitely presented modules are coherent, then A is coherent, as A is finitely presented!)

14.8.O. EXERCISE. If M is finitely presented and N is coherent, show that $\operatorname{Hom}(M, N)$ is coherent. (Hint: Hom is left-exact in its first argument.)

14.8.P. EXERCISE. If M is finitely presented, and N is coherent, show that $M \otimes N$ is coherent.

14.8.Q. EXERCISE. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. (Hint: localization is exact, Exercise 2.6.F(a).) This exercise is repeated from Exercise 14.6.G to make this section self-contained.

14.8.R. EXERCISE. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is finitely generated for all i, then M is too. (Hint: Say M_{f_i} is generated by $m_{ij} \in M$ as an A_{f_i}-module. Show that the m_{ij} generate M. To check surjectivity $\oplus_{i,j} A \to M$, it suffices to check "on $D(f_i)$" for all i.)

14.8.S. EXERCISE. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is coherent for all i, then M is too. (Hint: if $\phi : A^2 \to M$, then $(\ker \phi)_{f_i} = \ker(\phi_{f_i})$, which is finitely generated for all i. Then apply the previous exercise.)

CHAPTER 15

Line bundles: Invertible sheaves and divisors

We next describe convenient and powerful ways of working with and classifying line bundles (invertible sheaves). We begin with a fundamental example, the line bundles $\mathcal{O}(n)$ on projective space, §15.1. We then introduce Weil divisors (formal sums of codimension 1 subsets), and use them to determine $\operatorname{Pic} X$ in a number of circumstances, §15.2. We finally discuss sheaves of ideals that happen to be invertible (effective Cartier divisors), §15.3. A central theme is that line bundles are closely related to "codimension 1 information".

15.1 Some line bundles on projective space

We now describe an important family of invertible sheaves on projective space over a field k.

As a warm-up, we begin with the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_k}(1)$ on $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$. The subscript $\mathbb{P}^1_k$ refers to the space on which the sheaf lives, and is often omitted when it is clear from the context. We describe the invertible sheaf $\mathcal{O}(1)$ using transition functions. It is trivial on the usual affine open sets $U_0 = D(x_0) = \operatorname{Spec} k[x_{1/0}]$ and $U_1 = D(x_1) = \operatorname{Spec} k[x_{0/1}]$. (We continue to use the convention $x_{i/j}$ for describing coordinates on patches of projective space, see §5.4.9.) Thus the data of a section over U_0 is a polynomial in $x_{1/0}$. The transition function from U_0 to U_1 is multiplication by $x_{0/1} = x_{1/0}^{-1}$. The transition function from U_1 to U_0 is hence multiplication by $x_{1/0} = x_{0/1}^{-1}$.

This information is summarized below:

open cover	$U_0 = \operatorname{Spec} k[x_{1/0}]$	$U_1 = \operatorname{Spec} k[x_{0/1}]$
trivialization and transition functions	$k[x_{1/0}]$	$k[x_{0/1}]$

$$\begin{tikzcd} k[x_{1/0}] \arrow[r, bend left, "\times x_{0/1} = x_{1/0}^{-1}"] & k[x_{0/1}] \arrow[l, bend left, "\times x_{1/0} = x_{0/1}^{-1}"] \end{tikzcd}$$

To test our understanding, let's compute the global sections of $\mathcal{O}(1)$. This will generalize our hands-on calculation that $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \cong k$ (Example 5.4.6). A global section is a polynomial $f(x_{1/0}) \in k[x_{1/0}]$ and a polynomial $g(x_{0/1}) \in k[x_{0/1}]$ such that $f(1/x_{0/1})x_{0/1} = g(x_{0/1})$. A little thought will show that f must be linear: $f(x_{1/0}) = a x_{1/0} + b$, and hence $g(x_{0/1}) = a + b x_{0/1}$. Thus

$$\dim \Gamma(\mathbb{P}^1_k, \mathcal{O}(1)) = 2 \neq 1 = \dim \Gamma(\mathbb{P}^1_k, \mathcal{O}).$$

Thus $\mathcal{O}(1)$ is not isomorphic to $\mathcal{O}$, and we have constructed our first (proved) example of a nontrivial line bundle!

We next define more generally $\mathcal{O}_{\mathbb{P}^1_k}(n)$ on $\mathbb{P}^1_k$. It is defined in the same way, except that the transition functions are the nth powers of those for $\mathcal{O}(1)$.

open cover	$U_0 = \operatorname{Spec} k[x_{1/0}]$	$U_1 = \operatorname{Spec} k[x_{0/1}]$

trivialization and transition functions
$$\begin{tikzcd} k[x_{1/0}] \arrow[r, bend left, "\times x_{0/1}^n = x_{1/0}^{-n}"] & k[x_{0/1}] \arrow[l, bend left, "\times x_{1/0}^n = x_{0/1}^{-n}"] \end{tikzcd}$$

In particular, thanks to the explicit transition functions, we see that $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ (with the obvious meaning if n is negative: $(\mathcal{O}(1)^{\otimes(-n)})^\vee$). Clearly also $\mathcal{O}(m) \otimes \mathcal{O}(n) = \mathcal{O}(m+n)$.

15.1.A. IMPORTANT EXERCISE. Show that $\dim \Gamma(\mathbb{P}^1, \mathcal{O}(n)) = n+1$ if $n \geq 0$, and 0 otherwise.

15.1.1. *Example.* Long ago (§3.5.I), we warned that sheafification was necessary when tensoring $\mathcal{O}_X$-modules: if $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules on a ringed space, then it is not necessarily true that $\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{G}(X) \cong (\mathcal{F} \otimes \mathcal{G})(X)$. We now have an example: let $X = \mathbb{P}^1_k$, $\mathcal{F} = \mathcal{O}(1)$, $\mathcal{G} = \mathcal{O}(-1)$.

15.1.B. EXERCISE. Show that if $m \neq n$, then $\mathcal{O}(m) \not\cong \mathcal{O}(n)$. Hence conclude that we have an injection of groups $\mathbb{Z} \hookrightarrow \operatorname{Pic} \mathbb{P}^1_k$ given by $n \mapsto \mathcal{O}(n)$.

It is useful to identify the global sections of $\mathcal{O}(n)$ with the homogeneous polynomials of degree n in x_0 and x_1, i.e. with the degree n part of $k[x_0, x_1]$. Can you see this from your solution to Exercise 15.1.A? We will see that this identification is natural in many ways. For example, we will later see that the definition of $\mathcal{O}(n)$ doesn't depend on a choice of affine cover, and this polynomial description is also independent of cover. As an immediate check of the usefulness of this point of view, ask yourself: where does the section $x_0^3 - x_0x_1^2$ of $\mathcal{O}(3)$ vanish? The section $x_0 + x_1$ of $\mathcal{O}(1)$ can be multiplied by the section x_0^2 of $\mathcal{O}(2)$ to get a section of $\mathcal{O}(3)$. Which one? Where does the rational section $x_0^4(x_1+x_0)/x_1^7$ of $\mathcal{O}(-2)$ have zeros and poles, and to what order? (We saw the notion of zeros and poles in Definition 13.4.7, and will meet them again in §15.2, but you should intuitively answer these questions already.)

We now define the invertible sheaf $\mathcal{O}_{\mathbb{P}^m_k}(n)$ on the projective space $\mathbb{P}^m_k$. On the usual affine open set $U_i = \operatorname{Spec} k[x_{0/i}, \dots, x_{m/i}]/(x_{i/i} - 1) = \operatorname{Spec} A_i$, it is trivial, so sections (as an A_i-module) are isomorphic to A_i. The transition function from U_i to U_j is multiplication by $x_{i/j}^n = x_{j/i}^{-n}$.

$U_i = \operatorname{Spec} k[x_{0/i}, \dots, x_{m/i}]/(x_{i/i} - 1)$	$U_j = \operatorname{Spec} k[x_{0/j}, \dots, x_{m/j}]/(x_{j/j} - 1)$

$$\begin{tikzcd} k[x_{0/i}, \dots, x_{m/i}]/(x_{i/i} - 1) \arrow[r, bend left, "\times x_{i/j}^n = x_{j/i}^{-n}"] & \operatorname{Spec} k[x_{0/j}, \dots, x_{m/j}]/(x_{j/j} - 1) \arrow[l, bend left, "\times x_{j/i}^n = x_{i/j}^{-n}"] \end{tikzcd}$$

Note that these transition functions clearly satisfy the cocycle condition.

15.1.C. Essential exercise. Show that $\dim_k \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(n)) = \binom{m+n}{n}$.

As in the case of $\mathbb{P}^1$, sections of $\mathcal{O}(n)$ on $\mathbb{P}^m_k$ are naturally identified with homogeneous degree n polynomials in our $m+1$ variables. Thus $x+y+2z$ is a section of $\mathcal{O}(1)$ on $\mathbb{P}^2$. It isn't a function, but we know where this section vanishes — precisely where $x+y+2z=0$.

Also, notice that for fixed m, $\binom{m+n}{n}$ is a polynomial in n of degree m *for* $n \geq 0$ (or better: for $n \geq -m-1$). This should be telling you that this function "wants to be a polynomial," but won't succeed without assistance. We will later define $h^0(\mathbb{P}^m_k, \mathcal{O}(n)) := \dim_k \Gamma(\mathbb{P}^m_k, \mathcal{O}(n))$, and later still we will define higher cohomology groups, and we will define the *Euler characteristic* $\chi(\mathbb{P}^m_k, \mathcal{O}(n)) := \sum_{i=0}^{\infty}(-1)^i h^i(\mathbb{P}^m_k, \mathcal{O}(n))$ (cohomology will vanish in degree higher than n). We will discover the moral that the Euler characteristic is better-behaved than h^0, and so we should now suspect (and later prove, see Theorem 20.1.2) that this polynomial is in fact the Euler characteristic, and the reason that it agrees with h^0 for $n \geq 0$ because all the other cohomology groups should vanish.

We finally note that we can define $\mathcal{O}(n)$ on $\mathbb{P}^m_A$ for any ring A: the above definition applies without change.

15.2 Line bundles and Weil divisors

The notion of Weil divisors gives a great way of understanding and classifying line bundles, at least on Noetherian normal schemes. Some of what we discuss will apply in more general circumstances, and the expert is invited to consider generalizations by judiciously weakening hypotheses in various statements. Before we get started, I want to warn you: this is one of those topics in algebraic geometry that is hard to digest — learning it changes the way in which you think about line bundles. But once you become comfortable with the imperfect dictionary to divisors, it becomes second nature.

For the rest of this section, we consider only *Noetherian schemes*. We do this because we want to discuss codimension 1 subsets, and also have decomposition into irreducibles components. We will also use Hartogs' lemma, which requires Noetherianness.

Define a **Weil divisor** as a formal sum of codimension 1 irreducible closed subsets of X. In other words, a Weil divisor is defined to be an object of the form

$$\sum_{Y \subset X \text{ codimension } 1} n_Y [Y]$$

the n_Y are integers, all but a finite number of which are zero. Weil divisors obviously form an abelian group, denoted $\operatorname{Weil} X$.

For example, if X is a curve, the Weil divisors are linear combination of closed points.

We say that $[Y]$ is an **irreducible** (Weil) divisor. A Weil divisor is said to be **effective** if $n_Y \geq 0$ for all Y. In this case we say $D \geq 0$, and by $D_1 \geq D_2$ we mean $D_1 - D_2 \geq 0$. The **support** of a Weil divisor D is the subset $\cup_{n_Y \neq 0} Y$. If $U \subset X$ is an open set, there is a natural restriction map $\operatorname{Weil} X \to \operatorname{Weil} U$, where $\sum n_Y [Y] \mapsto \sum_{Y \cap U \neq \varnothing} n_Y [Y \cap U]$.

Suppose now that X is *regular in codimension* 1 (and Noetherian). We add this hypothesis because we will use properties of discrete valuation rings. Assume also that X is *reduced*. (This is only so we can talk about rational functions without worrying about them being defined at embedded points. Feel free to relax this hypothesis.) Suppose that $\mathcal{L}$ is an invertible sheaf, and s a rational section not vanishing everywhere on any irreducible component of X. (Rational sections are given by a section over a dense open subset of X, with the obvious equivalence, §14.1.7.) Then s determines a Weil divisor

$$\operatorname{div}(s) := \sum_Y \operatorname{val}_Y(s)[Y]$$

called the **divisor of zeros and poles** (cf. Definition 13.4.7). To determine the valuation $\operatorname{val}_Y(s)$ of s along Y, take any open set U containing the generic point of Y where $\mathcal{L}$ is trivializable, along with any trivialization over U; under this trivialization, s is a nonzero rational function on U, which thus has a valuation. Any two such trivializations differ by a unit (transition functions are units), so this valuation is well-defined. Note that $\operatorname{val}_Y(s) = 0$ for all but finitely many Y, by Exercise 13.4.G. The map div is a group homomorphism

$$\operatorname{div} : \{(\mathcal{L}, s)\} \to \operatorname{Weil} X.$$

(Be sure you understand how $\{(\mathcal{L}, s)\}$ forms a group!) A unit has no poles or zeros, so div descends to a group homomorphism

$$\operatorname{div} : \{(\mathcal{L}, s)\}/\Gamma(X, \mathcal{O}_X)^\times \to \operatorname{Weil} X. \tag{15.2.0.1}$$

15.2.A. Easier Exercise. (a) *(divisors of rational functions)* Verify that on $\mathbb{A}^1_k$, $\operatorname{div}(x^3/(x+1)) = 3[(x)] - [(x+1)]$ ("$= 3[0] - [-1]$").
(b) *(divisor of a rational sections of a nontrivial invertible sheaf)* On $\mathbb{P}^1_k$, there is a rational section of $\mathcal{O}(1)$ "corresponding to" $x^2/(x+y)$. Figure out what this means, and calculate $\operatorname{div}(x^2/(x+y))$.

Homomorphism (15.2.0.1) will be the key to determining all the line bundles on many X. Note that any invertible sheaf will have such a rational section (for each irreducible component, take a non-empty open set not meeting any other irreducible component; then shrink it so that $\mathcal{L}$ is trivial; choose a trivialization; then take the union of all these open sets, and choose the section on this union corresponding to 1 under the trivialization). We will see that in reasonable situations, this map div will be injective, and often an isomorphism. Thus by forgetting the rational section (taking an appropriate quotient), we will have described the Picard group of all line bundles. Let's put this strategy into action.

15.2.1. Proposition. — *If X is normal and Noetherian then the map* div *is injective.*

Proof. Suppose $\operatorname{div}(\mathcal{L}, s) = 0$. Then s has no poles. Hence by Hartogs' lemma for invertible sheaves (Exercise 14.1.I), s is a regular section. Now s vanishes nowhere, so s gives an isomorphism $\times s : \mathcal{O}_X \to \mathcal{L}$. (More precisely, on an open set U, the bijection $\mathcal{O}_X(U) \to \mathcal{L}(U)$ is multiplication by $s|_U$, and the inverse is division by $s|_U$. This behaves well with respect to restriction maps, and hence gives an isomorphism of sheaves.) $\square$

Motivated by this, we try to find an inverse to div, or at least to determine the image of div.

15.2.2. Important Definition. Assume now that X is irreducible (purely to avoid making (15.2.2.1) look uglier — but feel free to relax this, see Exercise 15.2.B). Suppose D is a Weil divisor. Define the sheaf $\mathcal{O}_X(D)$ by

$$\Gamma(U, \mathcal{O}_X(D)) := \{t \in K(X)^\times \ : \ \operatorname{div}|_U t + D|_U \geq 0\} \cup \{0\}. \tag{15.2.2.1}$$

(Here $\operatorname{div}|_U t$ means take the divisor of t considered as a rational function on U, i.e. consider just the irreducible divisors of U.) The subscript X in $\mathcal{O}_X(D)$ is omitted when it is clear from context. The sections of $\mathcal{O}_X(D)$ over U are the rational functions on U that have poles and zeros constrained by D. A positive co-efficient in D allows a pole of that order; a negative coefficients demands a zero of that order. Away from the support of D, this is (isomorphic to) the structure sheaf (by algebraic Hartogs' theorem 12.3.10).

15.2.B. LESS IMPORTANT EXERCISE. Generalize this definition to the case when X is not necessarily irreducible. (This is just a question of language. Once you have done this, feel free to drop this hypothesis in the rest of this section.)

15.2.C. EASY EXERCISE. Verify that $\mathcal{O}_X(D)$ is a quasicoherent sheaf. (Hint: the distinguished affine criterion for quasicoherence of Exercise 14.3.D.)

In good situations, $\mathcal{O}_X(D)$ is an invertible sheaf. For example, let $X = \mathbb{A}^1_k$. Consider

$$\mathcal{O}_X\left(-2[(x)] + [(x-1)] + [(x-2)]\right),$$

often written $\mathcal{O}(-2[0] + [1] + [2])$ for convenience. Then $3x^3/(x-1)$ is a global section; it has the required two zeros at $x = 0$ (and even one to spare), and takes advantage of the allowed pole at $x = 1$, and doesn't have a pole at $x = 2$, even though one is allowed. (Unimportant aside: the statement remains true in characteristic 2, although the explanation requires editing.)

15.2.D. EASY EXERCISE. (This is a consequence of later discussion as well, but you should be able to do this by hand.)
(a) Show that any global section of $\mathcal{O}_{\mathbb{A}^1_k}(-2[(x)] + [(x-1)] + [(x-2)])$ is a $k[x]$-multiple of $x^2/(x-1)(x-2)$.
(b) Extend the argument of (a) to give an isomorphism

$$\mathcal{O}_{\mathbb{A}^1_k}(-2[(x)] + [(x-1)] + [(x-2)]) \cong \mathcal{O}_{\mathbb{A}^1_k}.$$

More generally, in good circumstances, $\mathcal{O}_X(D)$ is an invertible sheaf, as shown in the next several exercises. (In fact the $\mathcal{O}_X(D)$ construction can be useful even if $\mathcal{O}_X(D)$ is *not* an invertible sheaf, but this won't concern us here. An example of an $\mathcal{O}_X(D)$ that is not an invertible sheaf is given in Exercise 15.2.G.)

15.2.E. IMPORTANT EXERCISE. Suppose $\mathcal{L}$ is an invertible sheaf, and s is a nonzero rational section of $\mathcal{L}$.
(a) Describe an isomorphism $\mathcal{O}(\operatorname{div} s) \cong \mathcal{L}$. Hint: let U be an open set on which $\mathcal{O}(\operatorname{div} s) \cong \mathcal{O}$. Show that such U cover X. For each such U, define $\phi_U : \mathcal{O}(\operatorname{div} s)(U) \to \mathcal{L}(U)$ sending a rational function t to st. Show that this is an isomorphism (with

the obvious inverse map of division by s). Explain why the ϕ_U glue (this should be pretty clear), and argue that this map is a sheaf isomorphism.
(b) Let σ be the map from $K(X)$ to the rational sections of $\mathcal{L}$, where $\sigma(t)$ is the rational section of $\mathcal{O}_X(D) \cong \mathcal{L}$ defined via (15.2.2.1). Show that the isomorphism of (a) can be chosen such that $\sigma(1) = s$. (Hint: the map in part (a) sends 1 to s.)

15.2.3. *Definition.* If D is a Weil divisor on (Noetherian normal irreducible) X such $D = \operatorname{div} s$ for some rational function s, we say that D is **principal**. Principal divisors clearly form a subgroup of $\operatorname{Weil} X$; denote this group or principal divisors $\operatorname{Prin} X$. If X can be covered with open sets U_i such that on U_i, D is principal, we say that D is **locally principal**.

15.2.4. *Important observation.* As a consequence of Exercise 15.2.E(a) (taking $\mathcal{L} = \mathcal{O}$), if D is principal, then $\mathcal{O}(D) \cong \mathcal{O}$. (Diagram (15.2.6.1) will imply that the converse holds: if $\mathcal{O}(D) \cong \mathcal{O}$, then D is principal.) Thus if D is *locally* prinicipal, $\mathcal{O}_X(D)$ is *locally* isomorphic to $\mathcal{O}_X$, so $\mathcal{O}_X(D)$ is an invertible sheaf.

15.2.F. IMPORTANT EXERCISE. Show the converse: if $\mathcal{O}_X(D)$ is an invertible sheaf, show that D is locally principal. Hint: use $\sigma(1)$, where σ was defined in Exercise 15.2.E(b).

15.2.5. *Remark.* In definition (15.2.2.1), it may seem cleaner to consider those s such that $\operatorname{div} s \geq D|_U$. The reason for the convention comes from our desire that $\operatorname{div} \sigma(1) = D$.

15.2.G. LESS IMPORTANT EXERCISE: A WEIL DIVISOR THAT IS NOT LOCALLY PRINCIPAL. Let $X = \operatorname{Spec} k[x, y, z]/(xy - z^2)$, a cone, and let D be the ruling $z = x = 0$. Show that D is not locally principal. (Hint: consider the stalk at the origin. Use the Zariski tangent space, see Problem 13.1.3.) In particular $\mathcal{O}_X(D)$ is not an invertible sheaf.

15.2.H. IMPORTANT EXERCISE. If X is Noetherian and factorial, show that for any Weil divisor D, $\mathcal{O}(D)$ is invertible. (Hint: It suffices to deal with the case where D is irreducible, and to cover X by open sets so that on each open set U there is a function whose divisor is $[Y \cap U]$. One open set will be $X - Y$. Next, we find an open set U containing an arbitrary $x \in Y$, and a function on U. As $\mathcal{O}_{X,x}$ is a unique factorization domain, the prime corresponding to 1 is codimension 1 and hence principal by Lemma 12.2.2. Let $f \in K(X)$ be a generator. It is regular at x, and it has a finite number of zeros and poles, and through x, $[Y]$ is the only zero. Let U be X minus all the others zeros and poles.)

15.2.I. EXERCISE (THE EXAMPLE OF §15.1). Let $D = \{x_0 = 0\}$ be a hyperplane divisor on $\mathbb{P}^n_k$. Show that $\mathcal{O}_{\mathbb{P}^n_k}(mD) \cong \mathcal{O}_{\mathbb{P}^n_k}(m)$. For this reason, $\mathcal{O}(1)$ is sometimes called the **hyperplane class** in $\operatorname{Pic} X$. (Of course, x_0 can be replaced by any linear form.)

15.2.6. The class group. We can now get a handle on the Picard group. Define the **class group** of X, $\operatorname{Cl} X$, by $\operatorname{Weil} X / \operatorname{Prin} X$. By taking the quotient of the inclusion (15.2.0.1) by $\operatorname{Prin} X$, we have the inclusion $\operatorname{Pic} X \hookrightarrow \operatorname{Cl} X$. This is summarized in the convenient and enlightening diagram

(15.2.6.1)
$$\begin{tikzcd} & \{(\mathcal{L}, s)\}/\Gamma(X, \mathcal{O}_X)^\times \arrow[r, hook, "\mathrm{div}"] \arrow[d, "/\{(\mathcal{O}_X, s)\}"] & \mathrm{Weil}\, X \arrow[d, "/\,\mathrm{Prin}\, X"] \\ \mathrm{Pic}\, X \arrow[r, equal] & \{\mathcal{L}\} \arrow[r, hook] & \mathrm{Cl}\, X \end{tikzcd}$$

This diagram is very important, and although it is short to state, it takes time to internalize. (If X is Noetherian and regular in codimension 1 but not necessarily normal, our arguments show that we have a similar diagram, except the horizontal maps are not necessarily inclusions.)

In particular, if A is a unique factorization domain, then all Weil divisors on $\mathrm{Spec}\, A$ are principal by Lemma 12.2.2, so $\mathrm{Cl}\,\mathrm{Spec}\, A = 0$, and hence $\mathrm{Pic}\,\mathrm{Spec}\, A = 0$.

As $k[x_1, \dots, x_n]$ has unique factorization, $\mathrm{Cl}(\mathbb{A}^n_k) = 0$, so $\boxed{\mathrm{Pic}(\mathbb{A}^n_k) = 0}$. Geometers might find this believable: "$\mathbb{C}^n$ is a contractible manifold, and hence should have no nontrivial line bundles". (Aside: for this reason, you might expect that $\mathbb{A}^n_k$ also has no vector bundles. This is the Quillen-Suslin Theorem, formerly known as Serre's conjecture, part of Quillen's work leading to his 1978 Fields Medal. For a short proof by Vaserstein, see [**L**, p. 850].)

Removing subset of X of codimension greater 1 doesn't change the class group, as it doesn't change the Weil divisor group or the principal divisors. (Warning: it *can* affect the Picard group, Exercise 15.2.P.)

Removing a subset of codimension 1 changes the Weil divisor group in a controllable way. For example, suppose Z is an irreducible codimension 1 subset of X. Then we clearly have an exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Z]} \mathrm{Weil}\, X \longrightarrow \mathrm{Weil}(X - Z) \longrightarrow 0.$$

When we take the quotient by principal divisors, we lose exactness on the left, and get an **excision exact sequence for class groups**:

(15.2.6.2)
$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \mathrm{Cl}\, X \longrightarrow \mathrm{Cl}(X - Z) \longrightarrow 0.$$

(Do you see why?)

For example, if X is an open subscheme of $\mathbb{A}^n$, $\mathrm{Pic}\, X = \{0\}$.

As another application, let $X = \mathbb{P}^n_k$, and Z be the hyperplane $x_0 = 0$. We have

$$\mathbb{Z} \longrightarrow \mathrm{Cl}\, \mathbb{P}^n_k \longrightarrow \mathrm{Cl}\, \mathbb{A}^n_k \longrightarrow 0$$

from which $\mathrm{Cl}\, \mathbb{P}^n_k$ is generated by the class $[Z]$, and $\mathrm{Pic}\, \mathbb{P}^n_k$ is a subgroup of this.

By Exercise 15.2.I, $[Z] \mapsto \mathcal{O}(1)$, and as $\mathcal{O}(n)$ is nontrivial for $n \neq 0$ (Exercise 15.1.B), $[Z]$ is not torsion in $\mathrm{Cl}\, \mathbb{P}^n_k$. Hence $\mathrm{Pic}\, \mathbb{P}^n_k \hookrightarrow \mathrm{Cl}\, \mathbb{P}^n_k$ is an isomorphism, and $\boxed{\mathrm{Pic}\, \mathbb{P}^n_k \cong \mathbb{Z}}$, with generator $\mathcal{O}(1)$. The **degree** of an invertible sheaf on $\mathbb{P}^n$ is defined using this: define $\deg \mathcal{O}(d)$ to be d.

We have gotten good mileage from the fact that the Picard group of the spectrum of a unique factorization domain is trivial. More generally, Exercise 15.2.H gives us:

15.2.7. Proposition. — *If X is Noetherian and factorial, then for any Weil divisor D, $\mathcal{O}(D)$ is invertible, and hence the map $\mathrm{Pic}\, X \to \mathrm{Cl}\, X$ is an isomorphism.*

This makes the connection to the class group in number theory precise, see Exercise 14.1.K; see also §15.2.10. (I want to think this through and edit this.)

15.2.8. *Mild but important generalization: twisting line bundles by divisors.* The above constructions can be extended, with $\mathcal{O}_X$ replaced by an arbitrary invertible sheaf, as follows. Let $\mathcal{L}$ be an invertible sheaf on a normal Noetherian scheme X. Then define $\mathcal{L}(D)$ by $\mathcal{O}_X(D) \otimes \mathcal{L}$.

15.2.J. EASY EXERCISE. (a) Show that sections of $\mathcal{L}(D)$ can be interpreted as rational sections of $\mathcal{L}$ have zeros and poles constrained by D, just as in (15.2.2.1):

$$\Gamma(U, \mathcal{L}(D)) := \{t \text{ rational section of } \mathcal{L} \ : \ \operatorname{div}|_U t + D|_U \geq 0\} \cup \{0\}.$$

(b) Suppose D_1 and D_2 are locally principal. Show that $(\mathcal{O}(D_1))(D_2) \cong \mathcal{O}(D_1 + D_2)$.

15.2.9. Fun examples: hypersurface complements, and quadric surfaces.

We can now actually calculate some Picard and class groups. First, a useful observation: notice that you can restrict invertible sheaves on X to any subscheme Y, and this can be a handy way of checking that an invertible sheaf is not trivial. Effective Cartier divisors (§9.1.2) sometimes restrict too: if you have effective Cartier divisor on X, then it restricts to a closed subscheme on Y, locally cut out by one equation. If you are fortunate and this equation doesn't vanish on any associated point of Y, then you get an effective Cartier divisor on Y. You can check that the restriction of effective Cartier divisors corresponds to restriction of invertible sheaves.

15.2.K. EXERCISE: A TORSION PICARD GROUP. Suppose that Y is an irreducible degree d hypersurface of $\mathbb{P}^n_k$. Show that $\operatorname{Pic}(\mathbb{P}^n_k - Y) \cong \mathbb{Z}/d$. (For differential geometers: this is related to the fact that $\pi_1(\mathbb{P}^n_k - Y) \cong \mathbb{Z}/d$.) Hint: (15.2.6.2).

The next two exercises explore consequences of Exercise 15.2.K, and provide us with some examples promised in Exercise 6.4.M.

15.2.L. EXERCISE. Keeping the same notation, assume $d > 1$ (so $\operatorname{Pic}(\mathbb{P}^n - Y) \neq 0$), and let $H_0, \dots, H_n$ be the $n+1$ coordinate hyperplanes on $\mathbb{P}^n$. Show that $\mathbb{P}^n - Y$ is affine, and $\mathbb{P}^n - Y - H_i$ is a distinguished open subset of it. Show that the $\mathbb{P}^n - Y - H_i$ form an open cover of $\mathbb{P}^n - Y$. Show that $\operatorname{Pic}(\mathbb{P}^n - Y - H_i) = 0$. Then by Exercise 15.2.Q, each $\mathbb{P}^n - Y - H_i$ is the Spec of a unique factorization domain, but $\mathbb{P}^n - Y$ is not. Thus the property of being a unique factorization domain is not an affine-local property — it satisfies only one of the two hypotheses of the Affine Communication Lemma 6.3.2.

15.2.M. EXERCISE. Keeping the same notation as the previous exercise, show that on $\mathbb{P}^n - Y$, H_i (restricted to this open set) is an effective Cartier divisor that is not cut out by a single equation. (Hint: Otherwise it would give a trivial element of the class group.)

15.2.N. EXERCISE: PICARD GROUP OF $\mathbb{P}^1 \times \mathbb{P}^1$. Let $X = \mathbb{P}^1_k \times_k \mathbb{P}^1_k \cong \operatorname{Proj} k[w, x, y, z]/(wz - xy)$, a smooth quadric surface (Figure 9.2) (see Example 10.5.2). Show that $\operatorname{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z}$ as follows: Show that if $L = \{\infty\} \times \mathbb{P}^1 \subset X$ and $M = \mathbb{P}^1 \times \{\infty\} \subset X$, then

$X - L - M \cong \mathbb{A}^2$. This will give you a surjection $\mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \operatorname{Cl} X$. Show that $\mathcal{O}(L)$ restricts to $\mathcal{O}$ on L and $\mathcal{O}(1)$ on M. Show that $\mathcal{O}(M)$ restricts to $\mathcal{O}$ on M and $\mathcal{O}(1)$ on L. (This exercise takes some time, but is enlightening.)

15.2.O. EXERCISE. Show that irreducible smooth projective surfaces (over k) can be birational but not isomorphic. Hint: show $\mathbb{P}^2$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ using the Picard group. (Aside: we will see in Exercise 22.2.D that the Picard group of the "blown up plane" is $\mathbb{Z}^2$, but in Exercise 22.2.E we will see that the blown up plane is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, using a little more information in the Picard group.)

This is unlike the case for curves: birational irreducible smooth projective curves (over k) must be isomorphic, as we will see in Theorem 18.4.3. Nonetheless, any two surfaces are related in a simple way: if X and X' are projective, nonsingular, and birational, then X can be sequentially blown up at judiciously chosen points, and X' can too, such that the two results are isomorphic. (Blowing up will be discussed in Chapter 19.)

15.2.P. EXERCISE: PICARD GROUP OF THE CONE. Let $X = \operatorname{Spec} k[x, y, z]/(xy - z^2)$, a cone, where $\operatorname{char} k \neq 2$. (The characteristic hypothesis is not necessary for the result, but is included so you can use Exercise 6.4.H to show normality of X.) Show that $\operatorname{Pic} X = \{1\}$, and $\operatorname{Cl} X \cong \mathbb{Z}/2$. (Hint: show that the ruling $Z = \{x = z = 0\}$ generates $\operatorname{Cl} X$ by showing that its complement $D(x)$ is isomorphic to an open subset of $\mathbb{A}^2_k$. Show that $2[Z] = \operatorname{div}(x)$ and hence principal, and that Z is not principal, Exercise 15.2.G. (Remark: you know enough to show that $X - \{(0, 0, 0)\}$ is factorial. So although the class group is insensitive to removing loci of codimension greater than 1, §15.2.6, this is not true of the Picard group.)

The Picard group of the "blown up projective plane" will be computed in Exercise 22.2.D.

15.2.10. More on class groups and unique factorization.

As mentioned in §6.4.5, there are few commonly used means of checking that a ring is a unique factorization domain. The next exercise is one of them, and it is useful. For example, it implies the classical fact that for rings of integers in number fields, the class group is the obstruction to unique factorization (see Exercise 14.1.K and Proposition 15.2.7).

15.2.Q. EXERCISE. Suppose that A is a Noetherian integral domain. Show that A is a unique factorization domain if and only if A is integrally closed and $\operatorname{Cl} \operatorname{Spec} A = 0$. (One direction is easy: we have already shown that unique factorization domains are integrally closed in their fraction fields. Also, Lemma 12.2.2 shows that all codimension 1 primes of a unique factorization domain are principal, so that implies that $\operatorname{Cl} \operatorname{Spec} A = 0$. It remains to show that if A is integrally closed and $\operatorname{Cl} \operatorname{Spec} A = 0$, then all codimension 1 prime ideals are principal, as this characterizes unique factorization domains (Proposition 12.3.5). Hartogs' theorem 12.3.10 may arise in your argument.) This is the third important characterization of unique factorization domains promised in §6.4.5.

My final favorite method of checking that a ring is a unique factorization domain (§6.4.5) is Nagata's Lemma. It is also the least useful.

15.2.R. $\star\star$ EXERCISE (NAGATA'S LEMMA). Suppose A is a Noetherian domain, $x \in A$ an element such that (x) is prime and $A_x = A[1/x]$ is a unique factorization domain. Then A is a unique factorization domain. (Hint: Exercise 15.2.Q. Use the short exact sequence $[(x)] \to \operatorname{Cl} \operatorname{Spec} A \to \operatorname{Cl} A_x \to 0$ (15.2.6.2) to show that $\operatorname{Cl} \operatorname{Spec} A = 0$. Show that $A[1/x]$ is integrally closed, then show that A is integrally closed as follows. Suppose $T^n + a_{n-1}T^{n-1} + \cdots + a_0 = 0$, where $a_i \in A$, and $T \in K(A)$. Then by integral closure of A_x, we have that $T = r/x^m$, where if $m > 0$, then $r \notin x$. Then we quickly get a contradiction if $m > 0$.)

This leads to a fun algebra fact promised in Remark 13.3.3. Suppose k is an algebraically closed field of characteristic not 2. Let $A = k[x_1, \dots, x_n]/(x_1^2 + \cdots + x_m^2)$ where $m \leq n$. When $m \leq 2$, we get some special behavior. (If $m = 0$, we get affine space; if $m = 1$, we get a nonreduced scheme; if $m = 2$, we get a reducible scheme that is the union of two affine spaces.) If $m \geq 3$, we have verified that $\operatorname{Spec} A$ is normal, in Exercise 6.4.I(b).

In fact, if $m \geq 3$, then A is a unique factorization domain *unless* $m = 4$ (Exercise 6.4.K; see also Exercise 13.1.D). The failure at 4 comes from the geometry of the quadric surface: we have checked that in $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$, there is a codimension 1 prime ideal — the cone over a line in a ruling — that is not principal.

We already understand the case $m = 3$: $A = k[x, y, z, w_1, \dots, w_{n-3}]/(x^2 + y^2 - z^2)$ is a unique factorization domain, as it is normal (basically Exercise 6.4.I(b)) and has class group 0 (by essentially the same argument as for Exercise 15.2.P).

15.2.S. EXERCISE (THE CASE $m \geq 5$). Suppose that k is algebraically closed of characteristic not 2. Show that if $m \geq 3$, then $A = k[a, b, x_1, \dots, x_n]/(ab - x_1^2 - \cdots - x_m^2)$ is a unique factorization domain, by using Nagata's Lemma with $x = a$.

15.3 $\star$ Effective Cartier divisors "=" invertible ideal sheaves

We now give a completely different means of describing invertible sheaves on a scheme. One advantage of this over Weil divisors is that it can give line bundles on generically nonreduced schemes (if a scheme is nonreduced everywhere, it can't be regular at any codimension 1 prime). But we won't use this so it is less important.

Suppose $D \hookrightarrow X$ is a closed subscheme such that corresponding ideal sheaf $\mathcal{I}$ is an invertible sheaf. Then $\mathcal{I}$ is locally trivial; suppose U is a trivializing affine open set $\operatorname{Spec} A$. Then the closed subscheme exact sequence (14.5.5.1)

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

corresponds to

$$0 \to I \to A \to A/I \to 0$$

with $I \cong A$ as A-modules. Thus I is generated by a single element, say a, and this exact sequence starts as

$$0 \longrightarrow A \xrightarrow{\times a} A$$

As multiplication by a is injective, a is not a zerodivisor. We conclude that D is locally cut out by a single equation, that is not a zerodivisor. This was the definition of *effective Cartier divisor* given in §9.1.2. This argument is clearly reversible, so we have a quick new definition of effective Cartier divisor (an ideal sheaf $\mathcal{I}$ that is an invertible sheaf — or equivalently, the corresponding closed subscheme).

15.3.A. EASY EXERCISE. Show that a is unique up to multiplication by a unit.

In the case where X is locally Noetherian, we can use the language of associated points, so we can restate this definition as: D is locally cut out by a single equation, not vanishing at any associated point of X.

We now define an invertible sheaf corresponding to D. The seemingly obvious definition would be to take $\mathcal{I}_D$, but instead we define the invertible sheaf $\mathcal{O}(D)$ corresponding to an effective Cartier divisor to be the *dual*: $\mathcal{I}_D^\vee$. (The reason for the dual is Exercise 15.3.B.) The ideal sheaf $\mathcal{I}_D$ is sometimes denoted $\mathcal{O}(-D)$. We have an exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

The invertible sheaf $\mathcal{O}(D)$ has a canonical section s_D: Tensoring $0 \to \mathcal{I} \to \mathcal{O}$ with $\mathcal{I}^\vee$ gives us $\mathcal{O} \to \mathcal{I}^\vee$. (Easy unimportant fact: instead of tensoring $\mathcal{I} \to \mathcal{O}$ with $\mathcal{I}^\vee$, we could have dualized $\mathcal{I} \to \mathcal{O}$, and we would get the same section.)

15.3.B. IMPORTANT AND SURPRISINGLY TRICKY EXERCISE. Recall that a section of a locally free sheaf on X cuts out a closed subscheme of X (Exercise 14.1.H). Show that this section s_D cuts out D. (Compare this to Remark 15.2.5.)

This construction is "invertible".

15.3.C. EXERCISE. Suppose $\mathcal{L}$ is an invertible sheaf, and s is a section that is not locally a zerodivisor. (Make sense of this! In particular, if X is locally Noetherian, this means "s does not vanish at an associated point".) Show that $s = 0$ cuts out an effective Cartier divisor D, and $\mathcal{O}(D) \cong \mathcal{L}$.

15.3.D. EXERCISE. Suppose $\mathcal{I}$ and $\mathcal{J}$ are invertible ideal sheaves (hence corresponding to effective Cartier divisors, say D and D' respectively). Show that $\mathcal{I}\mathcal{J}$ is an invertible ideal sheaf. (We define the **product of two quasicoherent ideal sheaves** $\mathcal{I}\mathcal{J}$ as you might expect: on each affine, we take the product of the two corresponding ideals. To make sure this is well-defined, we need only check that if A is a ring, and $f \in A$, and $I, J \subset A$ are two ideals, then $(IJ)_f = I_f J_f$ in A_f.) We define the corresponding Cartier divisor to be $D + D'$. Verify that $\mathcal{O}(D + D') \cong \mathcal{O}(D) \otimes \mathcal{O}(D')$.

We thus have an important correspondence between *effective Cartier divisors* (closed subschemes whose ideal sheaves are invertible, or equivalently locally cut out by one non-zerodivisor, or in the locally Noetherian case, locally cut out by one equation not vanishing at an associated point) and *ordered pairs* $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf, and s is a section that is not locally a zerodivisor (or in the locally Noetherian case, not vanishing at an associated point). The effective Cartier divisors form an abelian semigroup. We have a map of semigroups, from effective Cartier divisors to invertible sheaves with sections not locally zerodivisors (and hence also to the Picard group of invertible sheaves).

We get lots of invertible sheaves, by taking differences of two effective Cartier divisors. In fact we "usually get them all" — it is very hard to describe an invertible sheaf on a finite type k-scheme that is not describable in such a way. For example, there are none if the scheme is nonsingular or even factorial (basically by Proposition 15.2.7 for factoriality; and nonsingular schemes are factorial by the Auslander-Buchsbaum theorem 13.3.1).

CHAPTER 16

Quasicoherent sheaves on projective A-schemes

The first two sections of this chapter are relatively straightforward, and the last two are trickier.

16.1 The quasicoherent sheaf corresponding to a graded module

We now describe quasicoherent sheaves on a projective A-scheme. Recall that a projective A-scheme is produced from the data of $\mathbb{Z}^{\geq 0}$-graded ring $S_\bullet$, with $S_0 = A$, and S_+ finitely generated as an A-module. The resulting scheme is denoted $\operatorname{Proj} S_\bullet$.

Let $X = \operatorname{Proj} S_\bullet$. Suppose $M_\bullet$ is a graded $S_\bullet$ module, *graded by* $\mathbb{Z}$. (While reading the next section, you may wonder why we don't grade by $\mathbb{Z}^+$. You will see that it doesn't matter. A $\mathbb{Z}$-grading will make things cleaner when we produce an $M_\bullet$ from a quasicoherent sheaf on $\operatorname{Proj} S_\bullet$.) We define the quasicoherent sheaf $\widetilde{M_\bullet}$ as follows. (I will avoid calling it $\tilde{M}$, as this might cause confusion with the affine case; but $\widetilde{M_\bullet}$ is *not* graded in any way.) For each f of positive degree, we define a quasicoherent sheaf $\widetilde{M_\bullet(f)}$ on the distinguished open $D(f) = \{p : f(p) \neq 0\}$ by

$$\widetilde{M_\bullet}(f) := \widetilde{(M_f)_0}.$$

As in (5.5.3.1), the subscript 0 means "the 0-graded piece". We have obvious isomorphisms of the restriction of $\widetilde{M_\bullet(f)}$ and $\widetilde{M_\bullet(g)}$ to $D(fg)$, satisfying the cocycle conditions. (Think through this yourself, to be sure you agree with the word "obvious"!) By Exercise 3.7.D, these sheaves glue together to a single sheaf on $\widetilde{M_\bullet}$ on X. We then discard the temporary notation $\widetilde{M_\bullet(f)}$.

This is clearly quasicoherent, because it is quasicoherent on each $D(f)$, and quasicoherence is local.

16.1.A. EXERCISE. Show that the stalk of $\widetilde{M_\bullet}$ at a point corresponding to homogeneous prime $\mathfrak{p} \subset S_\bullet$ is isomorphic $((M_\bullet)_\mathfrak{p})_0$.

16.1.B. UNIMPORTANT EXERCISE. Use the previous exercise to give an alternate definition of $\widetilde{M_\bullet}$ in terms of "compatible stalks" (cf. Exercise 5.5.J).

Given a map of graded modules $\phi : M_\bullet \to N_\bullet$, we we get an induced map of sheaves $\widetilde{M_\bullet} \to \widetilde{N_\bullet}$. Explicitly, over $D(f)$, the map $M_\bullet \to N_\bullet$ induces $M_\bullet[1/f] \to N_\bullet[1/f]$, which induces $\phi_f : (M_\bullet[1/f])_0 \to (N_\bullet[1/f])_0$; and this behaves well with respect to restriction to smaller distinguished open sets, i.e. the following diagram commutes.

$$\begin{array}{ccc} (M_\bullet[1/f])_0 & \xrightarrow{\phi_f} & (N_\bullet[1/f])_0 \\ \downarrow & & \downarrow \\ (M_\bullet[1/(fg)])_0 & \xrightarrow{\phi_{fg}} & (N_\bullet[1/(fg)])_0. \end{array}$$

Thus $\sim$ is a functor from the category of graded $S_\bullet$-modules to the category of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$. We shall soon see (Exercise 16.1.D) that this isn't an isomorphism (or equivalence), but it is close. The relationship is akin to that between presheaves and sheaves, and the sheafification functor.

16.1.C. EASY EXERCISE. Show that $\sim$ is an exact functor. (Hint: everything in the construction is exact.)

16.1.D. EXERCISE. Show that if $M_\bullet$ and $M'_\bullet$ agree in high enough degrees, then $\widetilde{M_\bullet} \cong \widetilde{M'_\bullet}$. Then show that the map from graded $S_\bullet$-modules (up to isomorphism) to quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ (up to isomorphism) is not a bijection. (Really: show this isn't an equivalence of categories.)

16.1.E. EXERCISE. Describe a map of S_0-modules $M_0 \to \Gamma(\widetilde{M_\bullet}, X)$. (This foreshadows the "saturation map" of §16.4.5 that takes a graded module to its saturation, see Exercise 16.4.C.)

16.1.1. Graded ideals of $S_\bullet$ give closed subschemes of $\operatorname{Proj} S_\bullet$. Recall that a graded ideal $I_\bullet \subset S_\bullet$ yields a closed subscheme $\operatorname{Proj} S_\bullet/I_\bullet \hookrightarrow \operatorname{Proj} S_\bullet$. For example, suppose $S_\bullet = k[w,x,y,z]$, so $\operatorname{Proj} S_\bullet \cong \mathbb{P}^3$. The ideal $I_\bullet = (wz-xy, x^2-wy, y^2-xz)$ yields our old friend, the twisted cubic (defined in Exercise 9.2.A)

16.1.F. EXERCISE. Show that if the functor $\sim$ is applied to the exact sequence of graded $S_\bullet$-modules

$$0 \to I_\bullet \to S_\bullet \to S_\bullet/I_\bullet \to 0$$

we obtain the closed subscheme exact sequence (14.5.5.1) for $\operatorname{Proj} S_\bullet/I_\bullet \hookrightarrow \operatorname{Proj} S_\bullet$.

We will soon see (Exercise 16.4.H) that all closed subschemes of $\operatorname{Proj} S_\bullet$ arise in this way.

16.2 Invertible sheaves (line bundles) on projective A-schemes

Suppose that $S_\bullet$ is generated in degree 1 (not a huge assumption, by Exercise 7.4.G). Suppose $M_\bullet$ is a graded $S_\bullet$-module. Define the graded module $M(n)_\bullet$ by $M(n)_m := M_{n+m}$. Thus the quasicoherent sheaf $\widetilde{M(n)_\bullet}$ satisfies

$$\Gamma(D(f), \widetilde{M(n)_\bullet}) = ((M_\bullet)_f)_n$$

where here the subscript means we take the nth graded piece. (These subscripts are admittedly confusing!)

16.2.A. EXERCISE. If $S_\bullet = k[x_0, \dots, x_m]$, so $\operatorname{Proj} S_\bullet = \mathbb{P}^m_k$, show $\widetilde{S_\bullet(n)} \cong \mathcal{O}(n)$ using transition functions (cf. §15.1).

16.2.B. IMPORTANT EXERCISE. If $S_\bullet$ is generated in degree 1, show that $\mathcal{O}_{\operatorname{Proj} S_\bullet}(n)$ is an invertible sheaf.

If $\mathcal{F}$ is a quasicoherent sheaf on $\operatorname{Proj} S_\bullet$, define $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}(n)$. This is often called **twisting $\mathcal{F}$ by $\mathcal{O}(n)$ or by n**. More generally, if $\mathcal{L}$ is an invertible sheaf, then $\mathcal{F} \otimes \mathcal{L}$ is often called **twisting $\mathcal{F}$ by $\mathcal{L}$**.

16.2.C. EXERCISE. Show that $\widetilde{M_\bullet}(n) \cong \widetilde{M(n)_\bullet}$.

16.2.D. EXERCISE. Use transition functions to show that $\mathcal{O}(m+n) \cong \mathcal{O}(m) \otimes \mathcal{O}(n)$ on any $\operatorname{Proj} S_\bullet$ where $S_\bullet$ is generated in degree 1.

16.2.1. *Unimportant remark.* Even if $S_\bullet$ is not generated in degree 1, then by Exercise 7.4.G, $S_{d\bullet}$ is generated in degree 1 for some d. In this case, we may define the invertible sheaves $\mathcal{O}(dn)$ for $n \in \mathbb{Z}$. This does *not* mean that we *can't* define $\mathcal{O}(1)$; this depends on $S_\bullet$. For example, if $S_\bullet$ is the polynomial ring $k[x, y]$ with the usual grading, except without linear terms (so $S_\bullet = k[x^2, xy, y^2, x^3, x^2y, xy^2, y^3]$), then $S_{2\bullet}$ and $S_{3\bullet}$ are both generated in degree 1, meaning that we may define $\mathcal{O}(2)$ and $\mathcal{O}(3)$. There is good reason to call their "difference" $\mathcal{O}(1)$.

16.3 Globally generated, base-point-free, and (very) ample line bundles

Throughout this section, $S_\bullet$ will be a finitely generated graded ring over A, generated in degree 1. We will prove the following result.

16.3.1. Theorem. — *Any coherent sheaf $\mathcal{F}$ on* $\operatorname{Proj} S_\bullet$ *can be presented in the form*

$$\oplus_{\mathit{finite}} \mathcal{O}(-n) \to \mathcal{F} \to 0.$$

Because we can work with the line bundles $\mathcal{O}(-n)$ in a hands-on way, this result will give us great control over all coherent sheaves (and in particular, vector bundles) on $\operatorname{Proj} S_\bullet$. As just a first example, it will allow us to show that every coherent sheaf on a projective k-scheme has a finite-dimensional space of global sections (Corollary 20.1.4). (This fact will grow up to be the fact that the higher pushforward of coherent sheaves under proper morphisms are also coherent, see Theorem 20.7.1(d) and Grothendieck's Coherence Theorem 20.8.1.)

Rather than proceeding directly to a proof, we use this as an excuse to introduce notions that are useful in wider circumstances (global generation, base-point-freeness, ampleness), and their interrelationships. But first we use it as an excuse to mention an important result.

16.3.2. *The Hilbert Syzygy Theorem.*

Given any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$, Theorem 16.3.1 a surjection $\phi : \oplus_{\text{finite}} \mathcal{O}(-m) \to \mathcal{F} \to 0$. The kernel of the surjection is also coherent, so iterating this construction, we can construct an infinite resolution of $\mathcal{F}$ by a direct sum of line bundles:

$$\cdots \oplus_{\text{finite}} \mathcal{O}(m_{2,j}) \to \oplus_{\text{finite}}\mathcal{O}(m_{1,j}) \to \oplus_{\text{finite}}\mathcal{O}(m_{0,j}) \to \mathcal{F} \to 0.$$

The Hilbert Syzygy Theorem states that there is in fact a *finite* resolution, of length at most n. (The Hilbert Syzygy Theorem in fact states more.) Because we won't use this, we don't give a proof, but [**E**, Ch. 19] has an excellent discussion. See the comments after Theorem 4.6.5 for the original history of this result.

16.3.3. Globally generated sheaves. Suppose X is a scheme, and $\mathcal{F}$ is an $\mathcal{O}$-module. The most important definition of this section is the following: $\mathcal{F}$ is **globally generated** (or **generated by global sections**) if it admits a surjection from a free sheaf on X:

$$\mathcal{O}^{\oplus I} \twoheadrightarrow \mathcal{F}.$$

Here I is some index set. The global sections in question are the images of the $|I|$ sections corresponding to 1 in the various summands of $\mathcal{O}_X^{\oplus I}$; those images generate the stalks of $\mathcal{F}$. We say $\mathcal{F}$ is **finitely globally generated** (or **generated by a finite number of global sections**) if the index set I can be taken to be finite.

More definitions in more detail: we say that $\mathcal{F}$ is **globally generated at a point** p (or sometimes **generated by global sections at** p) if we can find $\phi : \mathcal{O}^{\oplus I} \to \mathcal{F}$ that is surjective on stalks at p:

$$\mathcal{O}_p^{\oplus I} \xrightarrow{\phi_p} \mathcal{F}_p.$$

(It would be more precise to say that the stalk of $\mathcal{F}$ at p is generated by global sections of $\mathcal{F}$.) Note that $\mathcal{F}$ is *globally generated* if it is globally generated at all points p. (Exercise 3.4.E showed that isomorphisms can be checked on the level of stalks. An easier version of the same argument shows that surjectivity can also be checked on the level of stalks.) Notice that we can take a single index set for all of X, by taking the union of all the index sets for each p.

16.3.A. EASY EXERCISE (REALITY CHECK). Show that every quasicoherent sheaf on every affine scheme is globally generated. Show that every finite type quasicoherent sheaf on every affine scheme is generated by a finite number of global sections.

16.3.B. EASY EXERCISE. Show that if quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$ are globally generated at a point p, then so is $\mathcal{F} \otimes \mathcal{G}$.

16.3.C. EASY BUT IMPORTANT EXERCISE. Suppose $\mathcal{F}$ is a finite type quasicoherent sheaf on X.
(a) Show that $\mathcal{F}$ is globally generated at p if and only if "the fiber of $\mathcal{F}$ is generated by global sections at p", i.e. the map from global sections to the fiber $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$ is surjective, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$. (Hint: Geometric Nakayama, Exercise 14.7.D.)
(b) Show that if $\mathcal{F}$ is globally generated at p, then "$\mathcal{F}$ is globally generated near p": there is an open neighborhood U of p such that $\mathcal{F}$ is globally generated at every point of U.
(c) Suppose further that X is a quasicompact scheme. Show that if $\mathcal{F}$ is globally generated at all closed points of X, then $\mathcal{F}$ is globally generated at all points of X. (Note that nonempty quasicompact schemes *have* closed points, Exercise 6.1.E.)

16.3.D. EASY EXERCISE. If $\mathcal{F}$ is a finite type quasicoherent sheaf on X, and X is quasicompact, show that $\mathcal{F}$ is globally generated if and only if it is generated by a *finite number* of global sections.

16.3.E. EASY EXERCISE. An invertible sheaf $\mathcal{L}$ on X is globally generated if and only if for any point $x \in X$, there is a section of $\mathcal{L}$ not vanishing at x. See Theorem 17.4.1 for why we care.

16.3.4. *Definitions.* If $\mathcal{L}$ is an invertible sheaf on X, then those points where all sections of $\mathcal{L}$ vanish are called the **base points** of $\mathcal{L}$, and the set of base points is called the **base locus** of $\mathcal{L}$; it is a closed subset of X. (We can refine this to a closed subscheme: by taking the scheme-theoretic intersection of the vanishing loci of the sections of $\mathcal{L}$, we obtain the **scheme-theoretic base locus**.) The complement of the **base locus** is the **base-point-free locus**. If $\mathcal{L}$ has no base-points, it is **base-point-free**. By the previous discussion, (i) the base-point-free locus is an open subset of X, and (ii) $\mathcal{L}$ is generated by global sections if and only if it is base-point free. By Exercise 16.3.B, the tensor of two base-point-free line bundles is base-point-free.

(Remark: we will later see in Exercise 20.2.H that if X is a k-scheme, and $\mathcal{L}$ is an invertible sheaf on X, and K/k is any field extension, then $\mathcal{L}$ is base-point-free if and only if it is "base-point-free after base change to K". You could reasonably prove this now.)

16.3.5. *Base-point-free line bundles and maps to projective space.* Recall Exercise 7.3.M, which shows that $n+1$ functions on a scheme X with no common zeros yield a map to $\mathbb{P}^n$. This notion generalizes.

16.3.F. EXERCISE. Suppose $s_0, \dots, s_n$ are n sections of an invertible sheaf $\mathcal{L}$ on a scheme X, with no common zero. Define a corresponding map to $\mathbb{P}^n$:

$$X \xrightarrow{[s_0;\cdots;s_n]} \mathbb{P}^n$$

Hint: If U is an open subset on which $\mathcal{L}$ is trivial, choose a trivialization, then translate the s_i into functions using this trivialization, and use Exercise 7.3.M to obtain a morphism $U \to \mathbb{P}^n$. Then show that all of these maps (for different U and different trivializations) "agree".

(In Theorem 17.4.1, we will see that this yields *all* maps to projective space.) Note that this exercise works over $\mathbb{Z}$, although many readers will just work over a particular base such as a given field k. Here is some convenient classical language which is used in this case.

16.3.6. *Definitions.* A **linear series** on a k-scheme X is a k-vector space V (usually finite-dimensional), an invertible sheaf $\mathcal{L}$, and a linear map $\lambda : V \to \Gamma(X, \mathcal{L})$. Such a linear series is often called "V", with the rest of the data left implicit. If the map λ is an isomorphism, it is called a **complete linear series**, and is often written $|\mathcal{L}|$. The language of base-points (Definition 16.3.4) readily translates to this situation. For example, given a linear series, any point $x \in X$ on which all elements of the linear series V vanish, we say that x is a **base-point** of V. If V has no base-points, we say that it is **base-point-free**. The union of base-points is called the **base locus** of the linear series. One can similarly define the **base scheme** of the linear series.

As a reality check, you should understand why, an $n+1$-dimensional linear series on a k-scheme X with base-point-free locus U defines a morphism $U \to \mathbb{P}^n_k$.

16.3.7. *Serre's Theorem A.* We are now able to state a celebrated result of Serre.

16.3.8. Serre's Theorem A. — *Suppose $S_\bullet$ is generated in degree 1, and finitely generated over $A = S_0$. Let $\mathcal{F}$ be any finite type quasicoherent sheaf on $\operatorname{Proj} S_\bullet$. Then there exists some n_0 such that for all $n \geq n_0$, $\mathcal{F}(n)$ can be generated by a finite number of global sections.*

We could now prove Serre's Theorem A directly, but will continue to use this as an excuse to introduce more ideas; it will be a consequence of Theorem 16.3.11. Before getting to Theorem 16.3.11, we note that Theorem 16.3.1 follows from Theorem 16.3.8 as follows.

16.3.9. *Proof of Theorem 16.3.1 given Theorem 16.3.8.* Suppose we have m global sections $s_1, \dots, s_m$ of $\mathcal{F}(n)$ that generate $\mathcal{F}(n)$. This gives a map

$$\oplus_m \mathcal{O} \longrightarrow \mathcal{F}(n)$$

given by $(f_1, \dots, f_m) \mapsto f_1 s_1 + \cdots + f_m s_m$ on any open set. Because these global sections generate $\mathcal{F}$, this is a surjection. Tensoring with $\mathcal{O}(-n)$ (which is exact, as tensoring with any locally free sheaf is exact, Exercise 14.1.E) gives the desired result. $\square$

16.3.10. Very ampleness and ampleness.

We next introduce the notions of very ampleness and ampleness of line bundles on proper A-schemes. Suppose $\pi : X \to \operatorname{Spec} A$ is a proper morphism, and $\mathcal{L}$ is an invertible sheaf on X. The case when A is a field is the one of most immediate interest.

We say that $\mathcal{L}$ is **very ample over** A or π**-very ample**, or **relatively very ample** if $X = \operatorname{Proj} S_\bullet$ where $S_\bullet$ is a finitely generated graded ring over A generated in degree 1 (Definition 5.5.3, and $\mathcal{L} \cong \mathcal{O}_{\operatorname{Proj} S_\bullet}(1)$. One often just says **very ample** if the structure morphism is clear form the context. Note that the existence of a very ample line bundle implies that π is projective.

16.3.G. Easy exercise (very ample implies base-point-free). Show that a very ample invertible sheaf $\mathcal{L}$ on a proper A-scheme must be base-point-free.

16.3.H. Exercise (very ample $\otimes$ base-point-free is very ample, hence very ample $\otimes$ very ample is very ample). Suppose $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a proper A-scheme X, and $\mathcal{L}$ is very ample over A and $\mathcal{M}$ is base-point-free, then $\mathcal{L} \otimes \mathcal{M}$ is very ample. (Hint: $\mathcal{L}$ gives a closed immersion $X \hookrightarrow \mathbb{P}^m$, and $\mathcal{M}$ gives a morphism $X \to \mathbb{P}^n$. Show that the product map $X \to \mathbb{P}^m \times \mathbb{P}^n$ is a closed immersion, using the Cancellation Theorem 11.1.19 for closed immersions on $X \to \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$. Finally, consider the composition $X \hookrightarrow \mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{mn+m+n}$, where the last closed immersion is the Segre morphisms.)

16.3.I. Exercise (very ample $\boxtimes$ very ample is very ample). Suppose X and Y are proper A-schemes, and $\mathcal{L}$ (resp. $\mathcal{M}$) is a very ample invertible sheaf on X (resp. Y). If $\pi_1 : X \times_A Y \to X$ and $\pi_2 : X \times_A Y \to Y$ are the usual projections, show that

$\pi_1^*\mathcal{L} \otimes \pi_2^*\mathcal{M}$ is very ample on $X \times_A Y$. (The notion $\boxtimes$ is often used for this notion: $\mathcal{L} \boxtimes \mathcal{M} := \pi_1^*\mathcal{L} \otimes \pi_2^*\mathcal{M}$. The notation is used more generally when $\mathcal{L}$ and $\mathcal{M}$ are quasicoherent sheaves, or indeed just sheaves on ringed spaces.)

We say that $\mathcal{L}$ is **ample over** A or π**-ample**, or **relatively ample** if one of the following equivalent conditions holds.

16.3.11. Theorem. — *Suppose $\pi : X \to \operatorname{Spec} A$ is proper, and $\mathcal{L}$ is an invertible sheaf on X. The following are equivalent.*

(a) *For some $N > 0$, $\mathcal{L}^{\otimes N}$ is very ample over A.*
(a') *For all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample over A.*
(b) *For all finite type quasicoherent sheaves $\mathcal{F}$, there is an n_0 such that for $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated.*
(c) *As f runs over the section of $\mathcal{L}^{\otimes n}$ $(n > 0)$, the open subsets $X_f = \{x \in X : f(x) \neq 0\}$ form a base for the topology of X.*
(c') *As f runs over the section of $\mathcal{L}^{\otimes n}$ $(n > 0)$, those open subsets X_f which are affine form a base for the topology of X.*

(Variants of this Theorem 16.3.11 in the "absolute" and "relative" settings will be given in Theorems 16.3.15 and 18.3.6 respectively.)

Properties (a) and (a') relate to projective geometry, and property (b) relates to global generation (stalks). Properties (c) and (c') are somehow more topological, and while they may seem odd, they will provide the connection between (a)/(a') and (b). Note that (c) and (c') make no reference to the structure morphism π. In Theorem 20.6.1, we will meet a cohomological criterion (due, unsurprisingly, to Serre) later. Kodaira also gives a criterion for ampleness in the complex category: if X is a complex projective variety, then an invertible sheaf $\mathcal{L}$ on X is ample if and only if it admits a Hermitian metric with curvature positive everywhere.

The different flavor of these conditions gives some indication that ampleness is better-behaved than very ampleness in a number of ways. We mention without proof another property: if $f : X \to T$ is a finitely presented proper morphism, then those points on T where the fiber is ample forms an open subset of T (see [**EGA**, III_1.4.7.1] in the locally Noetherian case, and [**EGA**, IV_3.9.5.4] in general). We won't use this fact, but it is good to know.

Before getting to the proof, we give some sample applications. First, the fact that (a) implies (b) gives Serre's Theorem A (Theorem 16.3.8).

16.3.J. IMPORTANT EXERCISE. Suppose $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a proper A-scheme X, and $\mathcal{L}$ is ample. Show that $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is very ample for $n \gg 0$. (Hint: use both (a) and (b) of Theorem 16.3.11, and Exercise 16.3.H.)

16.3.K. IMPORTANT EXERCISE. Show that every line bundle on a projective A-scheme X is the difference of two very ample line bundles. More precisely, for any invertible sheaf $\mathcal{L}$ on X, we can find two very ample invertible sheaves $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{L} \cong \mathcal{M} \otimes \mathcal{N}^\vee$. (Hint: use the previous Exercise.)

16.3.L. EXERCISE (AMPLE $\otimes$ AMPLE IS AMPLE, AMPLE $\otimes$ BASE-POINT-FREE IS AMPLE. Suppose $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a proper A-scheme X, and $\mathcal{L}$ is ample. Show that if $\mathcal{M}$ is ample or base-point-free, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

16.3.M. LESS IMPORTANT EXERCISE. Solve Exercise 16.3.I with "very ample" replaced by "ample".

16.3.12. *Proof of Theorem 16.3.11 in the case* X *is Noetherian.* **Note:** Noetherian hypotheses are used at only one point in the proof, and we explain how to remove them, and give a reference for the details.

Obviously, *(a′) implies (a)*.

Clearly *(c′) implies (c)*. We now show that *(c) implies (c′)*. Suppose we have a point x in an open subset U of X. We seek an *affine* X_f containing x and contained in U. By shrinking U, we may assume that U is affine. From (c), U contains some X_f. But this X_f is affine, as it is the complement of the vanishing locus of a section of a line bundle on an affine scheme (Exercise 8.3.F), so (c′) holds. Note for future reference that the equivalence of (c) and (c′) did not require the hypothesis of properness.

We next show that *(a) implies (c)*. Given a closed subset $Z \subset X$, and a point x of the complement $X \setminus Z$, we seek a section of some $\mathcal{L}^{\otimes N}$ that vanishes on Z and not on x. The existence of such a section follows from the fact that $V(I(Z)) = Z$ (Exercise 5.5.E(c)): there is some element of $I(Z)$ that does not vanish on x.

We next show that *(b) implies (c)*. Suppose we have a point x in an open subset U of X. We seek a section of $\mathcal{L}^{\otimes N}$ that doesn't vanish at x, but vanishes on $X \setminus U$. Let $\mathcal{I}$ be the sheaf of ideals of functions vanishing on $X \setminus U$ (the quasicoherent sheaf of ideals cutting out $X \setminus U$, with reduced structure). As X is Noetherian, $\mathcal{I}$ is finite type, so by (b), $\mathcal{I} \otimes \mathcal{L}^{\otimes N}$ is generated by global sections for some N, so there is some section of it not vanishing at x. (*Noetherian note:* This is the only part of the argument where we use Noetherian hypotheses. They can be removed as follows. Show that for a quasicompact quasiseparated scheme, every ideal sheaf is generated by its finite type subideal sheaves. Indeed, any quasicoherent sheaf on a quasicompact quasiseparated scheme is the union of its finite type quasicoherent subsheaves, see [**EGA′**, (6.9.9)] or [**GW**, Cor. 10.50]. One of these finite type ideal sheaves doesn't vanish at x; use this as $\mathcal{I}$ instead.)

We now have to start working harder.

We next show that *(c′) implies (b)*. We wish to show that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for $n \gg 0$.

We first show that (c′) implies that for some N, $\mathcal{L}^{\otimes N}$ is globally generated, as follows. For each closed point $x \in X$, there is some $f \in \Gamma(X, \mathcal{L}^{\otimes N(x)})$ not vanishing at x, so $x \in X_f$. (Don't forget that quasicompact schemes have closed points, Exercise 6.1.E!) As x varies, these X_f cover all of X. Use quasicompactness of X to select a finite number of these X_f that cover X. To set notation, say these are $X_{f_1}, \dots, X_{f_n}$, where $f_i \in \Gamma(X, \mathcal{L}^{\otimes N_i})$. By replacing f_i with $f_i^{\otimes (\prod_j N_j)/N_i}$, we may assume that they are all sections of the *same* power $\mathcal{L}^{\otimes N}$ of $\mathcal{L}$ ($N = \prod_j N_j$). Then $\mathcal{L}^{\otimes N}$ is generated by these global sections.

We next show that it suffices to show that for all finite type quasicoherent sheaves $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^{\otimes mN}$ is globally generated for $m \gg 0$. For if we knew this, we could apply it to $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}$, ..., $\mathcal{F} \otimes \mathcal{L}^{\otimes (N-1)}$ (a finite number of times), and the result would follow. For this reason, we can replace $\mathcal{L}$ by $\mathcal{L}^{\otimes N}$. In other words, to show that (c′) implies (b), we may also assume the additional hypothesis that $\mathcal{L}$ is globally generated.

For each closed point x, choose an affine neighborhood of the form X_f, using (c′). Then $\mathcal{F}|_{X_f}$ is generated by a finite number of global sections (Easy Exercise 16.3.A). By Exercise 14.3.H, each of these generators can be expressed as a quotient of a section (over X) of $\mathcal{F} \otimes \mathcal{L}^{\otimes M(x)}$ by $f^{M(x)}$. (Note: we can take a single $M(x)$ for each x.) Then $\mathcal{F} \otimes \mathcal{L}^{\otimes M(x)}$ is globally generated at x by a finite number of global sections. By Exercise 16.3.C(b), $\mathcal{F} \otimes \mathcal{L}^{\otimes M(x)}$ is globally generated at all points in some neighborhood U_x of x. As $\mathcal{L}$ is also globally generated, this implies that $\mathcal{F} \otimes \mathcal{L}^{\otimes M'}$ is globally generated at all points of U_x for $M' \geq M(x)$ (cf. Easy Exercise 16.3.B). From quasicompactness of X, a finite number of these U_x cover X, so we are done (by taking the maximum of these $M(x)$).

Our penultimate step is to show that *(c′) implies (a)*. Choose a cover of (quasicompact) X by n affine open subsets $X_{a_1}, \dots, X_{a_n}$, where $a_1, \dots, a_n$ are all sections of powers of $\mathcal{L}$. By replacing each section with a suitable power, we may assume that they are all sections of the *same* power of $\mathcal{L}$, say $\mathcal{L}^{\otimes N}$. Say $X_{a_i} = \operatorname{Spec} A_i$, where (using that π is finite type) $A_i = \operatorname{Spec} B[a_{i1}, \dots, a_{ij_i}]/I_i$. By Exercise 14.3.H, each a_{ij} is of the form $s_{ij}/a_i^{m_{ij}}$, where $s_{ij} \in \Gamma(X, \mathcal{L}^{\otimes m_{ij}})$ (for some m_{ij}). Let $m = \max_{i,j} m_{ij}$. Then for each i, j, $a_{ij} = (s_{ij} a_i^{m - m_{ij}})/a_i^m$. For convenience, let $b_i = a_i^m$, and $b_{ij} = s_{ij} a_i^{m - m_{ij}}$; these area all global sections of $\mathcal{L}^{\otimes mN}$. Now consider the linear series generated by the b_i and b_{ij}. As the $D(b_i) = X_{a_i}$ cover X, this linear series is base-point-free, and hence (by Exercise 16.3.F) gives a morphism to $\mathbb{P}^Q$ (where $Q = \sharp b_i + \sharp b_{ij} - 1$). Let $x_1, \dots, x_n, \dots, x_{ij}, \dots$ be the projective coordinates on $\mathbb{P}^Q$, so $f^* x_i = b_i$, and $f^* x_{ij} = b_{ij}$. Then the morphism of affine schemes $X_{a_i} \to D(x_i)$ is a closed immersion, as the associated maps of rings is a surjection (the generator a_{ij} of A_i is the image of x_{ij}/x_i).

At this point, we note for future reference that we have shown the following. If $X \to \operatorname{Spec} A$ is finite type, and $\mathcal{L}$ satisfies (c)=(c′), then X is an open immersion into a projective A-scheme. (We did not use separatedness.) We conclude our proof that (c′) implies (a) by using properness to show that the image of this open immersion into a projective A-scheme is in fact closed, so X is a projective A-scheme.

Finally, we note that *(a) and (b) together imply (a′)*: if $\mathcal{L}^{\otimes N}$ is very ample (from (a)), and $\mathcal{L}^{\otimes n}$ is base-point-free for $n \geq n_0$ (from (b)), then $\mathcal{L}^{\otimes n}$ is very ample for $n \geq n_0 + N$ by Exercise 16.3.H. □

16.3.13. ⋆⋆ *Semiample line bundles.* Just as an invertible sheaf is ample if some tensor power of it is very ample, an invertible sheaf is said to be **semiample** if some tensor power of it is base-point-free. We won't use this notion.

16.3.14. ⋆ **Ampleness in the absolute setting.** (We will not use this section in any serious way later.) Note that global generation is already an absolute notion, i.e. is defined for a quasicoherent sheaf on a scheme, with no reference to any morphism. An examination of the proof of Theorem 16.3.11 shows that ampleness may similarly be interpreted in an absolute setting. We make this precise. Suppose $\mathcal{L}$ is an invertible sheaf on a *quasicompact* scheme X. We say that $\mathcal{L}$ is **ample** if as f runs over the section of $\mathcal{L}^{\otimes n}$ ($n > 0$), the open subsets $X_f = \{x \in X : f(x) \neq 0\}$ form a base for the topology of X. (We emphasize that quasicompactness in X is part of the condition of ampleness of $\mathcal{L}$.) For example, (i) if X is an affine scheme, every invertible sheaf is ample, and (ii) if X is a projective A-scheme, $\mathcal{O}(1)$ is ample.

16.3.N. EASY EXERCISE (PROPERTIES OF ABSOLUTE AMPLENESS). (a) Fix a positive integer n. Show that $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ is ample.
(b) Show that if $Z \hookrightarrow X$ is a closed immersion, and $\mathcal{L}$ is ample on X, then $\mathcal{L}|_Z$ is ample on Z.

The following result will give you some sense of how ampleness behaves. We will not use it, and hence omit the proof (which is given in [**Stacks**, tag 01Q3]. However, many parts of the proof are identical to (or generalize) the corresponding arguments in Theorem 16.3.11. The labeling of the statements parallels the labelling of the statements in Theorem 16.3.11.

16.3.15. Theorem (cf. Theorem 16.3.11). — *Suppose $\mathcal{L}$ is an invertible sheaf on a quasicompact scheme X. The following are equivalent.*

- (b) X *is quasiseparated, and for every finite type quasicoherent sheaf $\mathcal{F}$, there is an n_0 such that for $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated.*
- (c) *As f runs over the section of $\mathcal{L}^{\otimes n}$ ($n > 0$), the open subsets $X_f = \{x \in X : f(x) \neq 0\}$ form a base for the topology of X (i.e. $\mathcal{L}$ is ample).*
- (c′) *As f runs over the section of $\mathcal{L}^{\otimes n}$ ($n > 0$), those open subsets X_f which are affine form a base for the topology of X.*
- (d) *Let $S_\bullet$ be the graded ring $\oplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$. (Warning: it needn't be finitely generated.) Then the open sets X_s with $s \in S_+$ cover X, and the associated map $X \to \operatorname{Proj} S$ is an open immersion. (Warning: $\operatorname{Proj} S$ is not necessarily finite type.)*

Part (d) implies that X is separated (and thus quasiseparated).

16.3.16. $\star$ **Transporting global generation, base-point-freeness, and ampleness to the relative situation.**

These notions can be "relativized". We could do this right now, but we wait until §18.3.5, when we will have defined the notion of a projective morphism, and thus a "relatively very ample" line bundle.

16.4 $\star$ Quasicoherent sheaves and graded modules

(This section answers some fundamental questions, but it is surprisingly tricky. You may wish to skip this section, or at least the proofs, on first reading, unless you have a particular need for them.)

Throughout this section, $S_\bullet$ is a finitely generated graded algebra *generated in degree* 1, so in particular $\mathcal{O}(n)$ is defined for all n.

We know how to get quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ from graded $S_\bullet$-modules. We will now see that we can get them all in this way. We will define a functor $\Gamma_\bullet$ from (the category of) quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ to (the category of) graded $S_\bullet$-modules that will attempt to reverse the $\sim$ construction. They are not quite inverses, as $\sim$ can turn two different graded modules into the same quasicoherent sheaf (see for example Exercise 16.1.D). But we will see a natural isomorphism $\widetilde{\Gamma_\bullet(\mathcal{F})} \cong \mathcal{F}$. In fact $\Gamma_\bullet(\widetilde{M_\bullet})$ is a better ("saturated") version of $M_\bullet$, and there is a saturation functor $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$ that is akin to groupification and sheafification — it is adjoint to the forgetful functor from saturated graded modules to graded modules. And thus we come to the fundamental relationship between $\sim$ and $\Gamma_\bullet$: they are an adjoint pair.

$$\begin{tikzcd}
 & \text{graded } S_\bullet\text{-modules} \arrow[dl, "\sim"'] \arrow[dd, bend right, "\text{saturate}"'] \\
QCoh_{\operatorname{Proj} S_\bullet} \arrow[dr, "\text{equivalence}", "\Gamma_\bullet"'] & \\
 & \text{saturated graded } S_\bullet\text{-modules} \arrow[uu, bend right, "\text{forget}"']
\end{tikzcd}$$

We now make some of this precise, but as little as possible to move forward. In particular, we will show that every quasicoherent sheaf on a projective A-scheme arises from a graded module (Corollary 16.4.2), and that every closed subscheme of $\operatorname{Proj} S_\bullet$ arises from a graded ideal $I_\bullet \subset S_\bullet$ (Exercise 16.4.H).

16.4.1. Definition of $\Gamma_\bullet$. When you do Essential Exercise 15.1.C (on global sections of $\mathcal{O}_{\mathbb{P}^m_k}(n)$), you will suspect that in good situations,

$$M_n \cong \Gamma(\operatorname{Proj} S_\bullet, \tilde{M}(n)).$$

Motivated by this, we define

$$\Gamma_n(\mathcal{F}) := \Gamma(\operatorname{Proj} S_\bullet, \mathcal{F}(n)).$$

16.4.A. EXERCISE. Describe a morphism of S_0-modules $M_n \to \Gamma(\operatorname{Proj} S_\bullet, \widetilde{M(n)_\bullet})$, extending the $n = 0$ case of Exercise 16.1.E.

16.4.B. EXERCISE. Show that $\Gamma_\bullet(\mathcal{F})$ is a graded $S_\bullet$-module. (Hint: consider $S_n \to \Gamma(\operatorname{Proj} S_\bullet, \mathcal{O}(n))$.)

16.4.C. EXERCISE. Show that the map $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$ arising from the previous two exercises is a map of $S_\bullet$-modules. We call this the **saturation map**.

16.4.D. EXERCISE. (a) Show that the saturation map need not be injective, nor need it be surjective. (Hint: $S_\bullet = k[x]$, $M_\bullet = k[x]/x^2$ or $M_\bullet = xk[x]$.)
(b) On the other hand, show that if $M_\bullet$ is finitely generated, then the saturation map is an isomorphism in large degree. In other words, show that there exists an n_0 such that $M_n \to \Gamma(\operatorname{Proj} S_\bullet, \widetilde{M(n)_\bullet})$, is an isomorphism for $n \geq n_0$.

16.4.E. EXERCISE. Show that $\Gamma_\bullet$ gives a functor from the category of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ to the category of graded $S_\bullet$-modules. In other words, if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$, describe the natural map $\Gamma_\bullet \mathcal{F} \to \Gamma_\bullet \mathcal{G}$, and show that such maps respect the identity and composition.

Now that we have defined the saturation map $M_\bullet \to \Gamma_\bullet \widetilde{M_\bullet}$, we will describe a map $\widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}$. While subtler to define, it will have the advantage of being an isomorphism.

16.4.F. EXERCISE. Define the natural map $\widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}$ as follows. First describe the map on sections over $D(f)$. Note that sections of the left side are of the form m/f^n where $m \in \Gamma_{n \deg f}(\mathcal{F})$, and $m/f^n = m'/f^{n'}$ if there is some N with $f^N(f^{n'}m - f^n m') = 0$. Sections on the right are implicitly described in Exercise 14.3.H. Show that your map behaves well on overlaps $D(f) \cap D(g) = D(fg)$.

16.4.G. EXERCISE. Show that the natural map $\widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}$ is an isomorphism, by showing that it is an isomorphism of sections over $D(f)$ for any f. First show surjectivity, using Exercise 14.3.H to show that any section of $\mathcal{F}$ over $D(f)$ is of the form m/f^n where $m \in \Gamma_{n \deg f}(\mathcal{F})$. Then verify that it is injective.

16.4.2. Corollary. — *Every quasicoherent sheaf on a projective* A*-scheme arises from the* $\sim$ *construction.*

16.4.H. EXERCISE. Show that each closed subscheme of $\operatorname{Proj} S_\bullet$ arises from a graded ideal $I_\bullet \subset S_\bullet$. (Hint: Suppose Z is a closed subscheme of $\operatorname{Proj} S_\bullet$. Consider the exact sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_{\operatorname{Proj} S_\bullet} \to \mathcal{O}_Z \to 0$. Apply $\Gamma_\bullet$, and then $\sim$. Be careful: $\Gamma_\bullet$ is left-exact, but not necessarily exact.)

For the first time, we see that every closed subscheme of a projective scheme is cut out by homogeneous equations. This is the analogue of the fact that every closed subscheme of an affine scheme is cut out by equations. It is disturbing that it is so hard to prove this fact.

16.4.I. $\star$ EXERCISE ($\Gamma_\bullet$ AND $\sim$ ARE ADJOINT FUNCTORS). Describe a natural bijection $\operatorname{Hom}(M_\bullet, \Gamma_\bullet \mathcal{F}) \cong \operatorname{Hom}(\widetilde{M_\bullet}, \mathcal{F})$, as follows.

(a) Show that maps $M_\bullet \to \Gamma_\bullet \mathcal{F}$ are the "same" as maps $((M_\bullet)_f)_0 \to ((\Gamma_\bullet \mathcal{F})_f)_0$ as f varies through S_+, that are "compatible" as f varies, i.e. if $D(g) \subset D(f)$, there is a commutative diagram

$$\begin{tikzcd} ((M_\bullet)_f)_0 \arrow[r] \arrow[d] & ((\Gamma_\bullet \mathcal{F})_f)_0 \arrow[d] \\ ((M_\bullet)_g)_0 \arrow[r] & ((\Gamma_\bullet \mathcal{F})_g)_0 \end{tikzcd}$$

More precisely, give a bijection between $\operatorname{Hom}(M_\bullet, \Gamma_\bullet \mathcal{F})$ and the set of compatible maps

$$\left(\operatorname{Hom}((M_\bullet)_f)_0 \to ((\Gamma_\bullet \mathcal{F})_f)_0 \right)_{f \in S_+}.$$

(b) Describe a bijection between the set of compatible maps $(\operatorname{Hom}((M_\bullet)_f)_0 \to ((\Gamma_\bullet \mathcal{F})_f)_0)_{f \in S_+}$ and the set of compatible maps $\Gamma(D(f), \widetilde{M_\bullet}) \to \Gamma(D(f), \mathcal{F})$.

16.4.3. *Remark.* We will show later (in Exercise 20.1.D) that under Noetherian hypotheses, if $\mathcal{F}$ is a coherent sheaf on $\operatorname{Proj} S_\bullet$, then $\Gamma_\bullet \mathcal{F}$ is a coherent $S_\bullet$-module. Thus the close relationship between quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ and graded $S_\bullet$-modules respects coherence.

16.4.4. *The special case* $M_\bullet = S_\bullet$. We have a saturation map $S_\bullet \to \Gamma_\bullet \widetilde{S_\bullet}$, which is a map of $S_\bullet$-modules. But $\Gamma_\bullet \widetilde{S_\bullet}$ has the structure of a graded ring (basically because we can multiply sections of $\mathcal{O}(m)$ by sections of $\mathcal{O}(n)$ to get sections of $\mathcal{O}(m+n)$, see Exercise 16.2.D).

16.4.J. EXERCISE. Show that the map of graded rings $S_\bullet \to \Gamma_\bullet \widetilde{S_\bullet}$ induces (via the construction of Essential Exercise 7.4.0.1) an isomorphism $\operatorname{Proj} \Gamma_\bullet \widetilde{S_\bullet} \to \operatorname{Proj} S_\bullet$, and under this isomorphism, the respective $\mathcal{O}(1)$'s are identified.

This addresses the following question: to what extent can we recover $S_\bullet$ from $(\operatorname{Proj} S_\bullet, \mathcal{O}(1))$? The answer is: we cannot recover $S_\bullet$, but we can recover its "saturation". And better yet: given a projective A-scheme $\pi : X \to \operatorname{Spec} A$, along with $\mathcal{O}(1)$, we obtain it as a Proj of a graded algebra in a canonical way, via

$$X \cong \operatorname{Proj}\left(\oplus_{n \geq 0} \Gamma(X, \mathcal{O}(n))\right).$$

There is one last worry you might have, which is assuaged by the following exercise.

16.4.K. EXERCISE. Suppose $X = \operatorname{Proj} S_\bullet \to \operatorname{Spec} A$ is a projective A-scheme. Show that $(\oplus_{n \geq 0} \Gamma(X, \mathcal{O}(n)))$ is a finitely generated A-algebra. (Hint: $S_\bullet$ and $(\oplus_{n \geq 0} \Gamma(X, \mathcal{O}(n)))$ agree in sufficiently high degrees, by Exercise 16.4.D.)

16.4.5. $\star$ **Saturated $S_\bullet$-modules.** We end with a remark: different graded $S_\bullet$-modules give the same quasicoherent sheaf on $\operatorname{Proj} S_\bullet$, but the results of this section show that there is a "best" (saturated) graded module for each quasicoherent sheaf, and there is a map from each graded module to its "best" version, $M_\bullet \to \Gamma_\bullet \widetilde{M_\bullet}$. A module for which this is an isomorphism (a "best" module) is called *saturated*. We won't use this term later.

This "saturation" map $M_\bullet \to \Gamma_\bullet \widetilde{M_\bullet}$ is analogous to the sheafification map, taking presheaves to sheaves. For example, the saturation of the saturation equals the saturation.

There is a bijection between saturated quasicoherent sheaves of ideals on $\operatorname{Proj} S_\bullet$ and closed subschemes of $\operatorname{Proj} S_\bullet$.

CHAPTER 17

Pushforwards and pullbacks of quasicoherent sheaves

17.1 Introduction

Suppose $B \to A$ is a morphism of rings. Then there is an obvious functor $Mod_A \to Mod_B$: if M is an A-module, you can create a B-module M_B by simply treating it as a B-module. There is an equally obvious functor $Mod_B \to Mod_A$: if N is a B-module, you can create an A-module $N \otimes_B A$. These functors are adjoint: we have isomorphisms

$$\operatorname{Hom}_A(N \otimes_B A, M) \cong \operatorname{Hom}_B(N, M_B)$$

functorial in both arguments. These constructions behave well with respect to localization (in an appropriate sense), and hence work (often) in the category of quasicoherent sheaves on schemes (and indeed always in the category of $\mathcal{O}$-modules on ringed spaces, see Remark 17.3.9, although we won't particularly care). The easier construction ($M \mapsto M_B$) will turn into our old friend pushforward. The other ($N \mapsto A \otimes_B N$) will be a relative of pullback, whom I'm reluctant to call an "old friend".

17.2 Pushforwards of quasicoherent sheaves

The main moral of this section is that in "reasonable" situations, the pushforward of a quasicoherent sheaf is quasicoherent, and that this can be understood in terms of one of the module constructions defined above. We begin with a motivating example:

17.2.A. EXERCISE. Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes, and suppose M is an A-module, so $\tilde{M}$ is a quasicoherent sheaf on $\operatorname{Spec} A$. Give an isomorphism $f_*\tilde{M} \to \widetilde{M_B}$. (Hint: There is only one reasonable way to proceed: look at distinguished open sets.)

In particular, $f_*\tilde{M}$ is quasicoherent. Perhaps more important, this implies that the pushforward of a quasicoherent sheaf under an affine morphism is also quasicoherent.

17.2.B. EXERCISE. If $\pi : X \to Y$ is an affine morphism, show that π_* is an exact functor $QCoh_X \to QCoh_Y$.

The following result, proved earlier, generalizes the fact that the pushforward of a quasicoherent sheaf under an affine morphism is also quasicoherent.

17.2.1. Theorem (Exercise 14.3.I). — *Suppose $\pi : X \to Y$ is a quasicompact quasiseparated morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X. Then $\pi_* \mathcal{F}$ is a quasicoherent sheaf on Y.*

Coherent sheaves don't always push forward to coherent sheaves. For example, consider the structure morphism $f : \mathbb{A}^1_k \to \operatorname{Spec} k$, corresponding to $k \mapsto k[t]$. Then $f_* \mathcal{O}_{\mathbb{A}^1_k}$ is the $k[t]$, which is not a finitely generated k-module. But in good situations, coherent sheaves do push forward. For example:

17.2.C. EXERCISE. Suppose $f : X \to Y$ is a finite morphism of Noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on X, show that $f_* \mathcal{F}$ is a coherent sheaf. Hint: Show first that $f_* \mathcal{O}_X$ is finite type. (Noetherian hypotheses are stronger than necessary, see Remark 20.1.6, but this suffices for most purposes.)

Once we define cohomology of quasicoherent sheaves, we will quickly prove that if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_k$, then $\Gamma(\mathbb{P}^n_k, \mathcal{F})$ is a finite-dimensional k-module, and more generally if $\mathcal{F}$ is a coherent sheaf on $\operatorname{Proj} S_\bullet$, then $\Gamma(\operatorname{Proj} S_\bullet, \mathcal{F})$ is a coherent A-module (where $S_0 = A$). This is a special case of the fact the "pushforwards of coherent sheaves by projective morphisms are also coherent sheaves". (The notion of projective morphism, a relative version of $\operatorname{Proj} S_\bullet \to \operatorname{Spec} A$, will be defined in §18.3.)

More generally, pushforwards of coherent sheaves by proper morphisms are also coherent sheaves (Theorem 20.8.1).

17.3 Pullbacks of quasicoherent sheaves

The notion of the pullback of a quasicoherent sheaf can be confusing on first (and second) glance. I will try to introduce it in two ways. One is directly in terms of thinking of quasicoherent sheaves in terms of modules over rings corresponding to affine open sets, and is suitable for direct computation. The other is elegant and functorial in terms of adjoints, and applies to ringed spaces in general. Both perspectives have advantages and disadvantages, and it is worth seeing both.

We note here that pullback to a closed subscheme or an open subscheme is often called **restriction**.

17.3.1. Construction/description of the pullback. Suppose $\pi : X \to Y$ is a morphism of schemes, and $\mathcal{G}$ is a quasicoherent sheaf on Y. We want to define the pullback quasicoherent sheaf $\pi^* \mathcal{G}$ on X in terms of affine open sets on X and Y. Suppose $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$ are affine open sets, with $\pi(\operatorname{Spec} A) \subset \operatorname{Spec} B$. Suppose $\mathcal{G}|_{\operatorname{Spec} B} \cong \tilde{N}$. Perhaps motivated by the fact that pullback should relate to tensor product, we want

$$\Gamma(\operatorname{Spec} A, \pi^* \mathcal{G}) = N \otimes_B A.$$

Our main goal will be to show that the A-module on the right is independent of our choice of $\operatorname{Spec} B$. Then we are largely done with the construction of $\pi^* \mathcal{G}$, as $N \otimes_B A$ behaves well with respect to localization at some $f \in A$ (cf. Exercise 14.3.D characterizing quasicoherent sheaves in terms of distinguished restrictions). True, not every $\operatorname{Spec} A$ has image contained in some $\operatorname{Spec} B$. (Can you think of an example? Hint: $\mathbb{A}^2 - \{(0,0)\} \to \mathbb{P}^1$.) But we can cover X with such $\operatorname{Spec} A$ — choose a cover of Y by $\operatorname{Spec} B_u$'s, and for each B_i, cover $\pi^{-1}(\operatorname{Spec} B_i)$ with $\operatorname{Spec} A_{ij}$. (To make this work, we have to be careful about what we mean by the sentence "this is independent of our choice of $\operatorname{Spec} B$." We sort this out by Exercise 17.3.D.)

17.3.2. We begin this project by *fixing* an affine open subset $\operatorname{Spec} B \subset Y$, and use it to define sections over *any* affine open subset $\operatorname{Spec} A \subset \pi^{-1}(\operatorname{Spec} B)$. To avoid confusion, let $\phi = \pi|_{\pi^{-1}(\operatorname{Spec} B)}$. We show that this gives us a quasicoherent sheaf $\phi^*\mathcal{G}$ on $\pi^{-1}(\operatorname{Spec} B)$, by showing that these sections behave well with respect to distinguished restrictions (Exercise 14.3.D again). First, note that if $\operatorname{Spec} A_f \subset \operatorname{Spec} A$ is a distinguished open set, then

$$\Gamma(\operatorname{Spec} A_f, \phi^*\mathcal{G}) = N \otimes_B A_f = (N \otimes_B A)_f = \Gamma(\operatorname{Spec} A, \phi^*\mathcal{G})_f$$

where "=" means "canonical isomorphism". Define the restriction map $\Gamma(\operatorname{Spec} A, \phi^*\mathcal{G}) \to \Gamma(\operatorname{Spec} A_f, \phi^*\mathcal{G})$,

$$\Gamma(\phi^*\mathcal{G}, \operatorname{Spec} A) \to \Gamma(\phi^*\mathcal{G}, \operatorname{Spec} A) \otimes_A A_f, \tag{17.3.2.1}$$

by $\alpha \mapsto \alpha \otimes 1$ (of course). Thus $\phi^*\mathcal{G}$ is (or: extends to) a quasicoherent sheaf on $\pi^{-1}(\operatorname{Spec} B)$.

We have now defined a quasicoherent sheaf on $\pi^{-1}(\operatorname{Spec} B)$, for all affine open $\operatorname{Spec} B \subset Y$. We want to show that this construction, as $\operatorname{Spec} B$ varies, glues into a single quasicoherent sheaf on X.

You are welcome to do this gluing appropriately, for example using the distinguished affine base of Y. This can get a little confusing, so we will follow an alternate universal property approach, yielding a construction that parallels the elegance of our construction of the fibered product.

17.3.3. Universal property definition of pullback. If $\pi : X \to Y$, and $\mathcal{G}$ is a quasicoherent sheaf on Y, we temporarily abuse notation, and redefine the pullback $\pi^*\mathcal{G}$ using the following adjointness universal property: for any $\mathcal{O}_X$-module $\mathcal{F}$, there is a bijection $\operatorname{Hom}_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F}) \leftrightarrow \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{F})$, and these bijections are functorial in $\mathcal{F}$. By universal property nonsense, this determines $\pi^*\mathcal{G}$ up to unique isomorphism; we just need to make sure that it exists. (Notice that we avoid worrying about whether the pushforward of a quasicoherent sheaf is quasicoherent by just working in a larger category.)

17.3.A. IMPORTANT EXERCISE. If Y is affine, then the construction of the quasicoherent sheaf in §17.3.2 satisfies this universal property of pullback of $\mathcal{G}$. Thus calling this sheaf $\pi^*\mathcal{G}$ is justified. (Hint: Interpret both sides of the alleged bijection explicitly. The adjointness in the ring/module case should turn up.)

We next show that if $\pi^*\mathcal{G}$ satisfies the universal property (for the morphism $\pi : X \to Y$), then if $j : V \hookrightarrow Y$ is any open subset, and $U = \pi^{-1}(V) \hookrightarrow X$, then $\pi^*\mathcal{G}|_U$ satisfies the universal property for $\pi|_U : U \to V$, so $\pi^*\mathcal{G}|_U$ deserves to be called $\pi|_U^*(\mathcal{G}|_V)$ (or more precisely, we have a canonical isomorphism). You will notice that we really need to work with $\mathcal{O}$-modules, not just with quasicoherent sheaves.

17.3.4. To do this, we introduce a new construction on sheaves. Suppose W is an open subset of a topological space Z, with inclusion $k : W \hookrightarrow Z$, and $\mathcal{H}$ is an $\mathcal{O}_W$-module. Define the **extension by zero** of $\mathcal{H}$ (over Z), denoted $k_! \mathcal{H}$, as follows: for open set $U \subset Z$, $k_!\mathcal{H}(U) = \mathcal{H}(U)$ if $U \subset W$, and 0 otherwise (with the obvious restriction maps). Note that $k_!\mathcal{H}$ is an $\mathcal{O}_Z$-module, and $k_!\mathcal{H}|_W$ and $\mathcal{H}$ are canonically isomorphic.

17.3.B. EASY EXERCISE. If $\mathcal{H}'$ is an $\mathcal{O}_Z$-module, describe an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_W}(\mathcal{H}'|_W, \mathcal{H}) \leftrightarrow \mathrm{Hom}_{\mathcal{O}_W}(\mathcal{H}', k_!\mathcal{H}),$$

functorial in $\mathcal{H}$ and $\mathcal{H}'$.

17.3.C. EASIER EXERCISE. Continuing the notation $i : U \hookrightarrow X$, $j : V \hookrightarrow Y$ above, if $\mathcal{F}'$ is an $\mathcal{O}_X$ describe a bijection $\mathrm{Hom}_{\mathcal{O}_U}(\pi^*\mathcal{G}|_U, \mathcal{F}') \leftrightarrow \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{G}|_V, (\pi|_U)_*\mathcal{F}')$, functorial in $\mathcal{F}'$. Hint: Justify the isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_U}(\pi^*\mathcal{G}|_U, \mathcal{F}') &\cong \mathrm{Hom}_{\mathcal{O}_X}(\pi^*\mathcal{G}, i_!\mathcal{F}') \\ &\cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \pi_* i_!\mathcal{F}') \\ &\cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, j_!(\pi|_U)_*\mathcal{F}') \\ &\cong \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{G}|_V, (\pi|_U)_*\mathcal{F}'). \end{aligned}$$

Hence show/conclude that the pullback exists if Y is an open subset of an affine scheme.

17.3.D. EXERCISE. Show that the pullback always exists, following the idea behind the construction of the fibered product.

The following is immediate from the universal property.

17.3.5. Proposition. — *Suppose $\pi : X \to Y$ is a quasicompact, quasiseparated morphism. Then pullback is left-adjoint to pushforward for quasicoherent sheaves: there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{F}), \tag{17.3.5.1}$$

natural in both arguments.

The "quasicompact and quasiseparated" hypotheses are just to ensure that π_* indeed sends $QCoh_X$ to $QCoh_Y$ (Theorem 14.3.I).

We have now described a quasicoherent sheaf $\pi^*\mathcal{G}$ on X whose behavior on affines mapping to affines was as promised. This is all you will need to prove the following useful properties of the pullback.

17.3.6. Theorem. — *Suppose $\pi : X \to Y$ is a morphism of schemes, and $\mathcal{G}$ is a quasicoherent sheaf on Y.*

(1) *(pullback preserves the structure sheaf) There is a canonical isomorphism $\pi^*\mathcal{O}_Y \cong \mathcal{O}_X$.*
(2) *(pullback preserves finite type quasicoherent sheaves) If $\mathcal{G}$ is a finite type quasicoherent sheaf, so is $\pi^*\mathcal{G}$. Hence if X is locally Noetherian, and $\mathcal{G}$ is coherent, then so is $\pi^*\mathcal{G}$. (It is not always true that the pullback of a coherent sheaf is coherent, and the interested reader can think of a counterexample.)*
(3) *(pullback preserves vector bundles) If $\mathcal{G}$ is locally free sheaf of rank r, then so is $\pi^*\mathcal{G}$. (In particular, the pullback of an invertible sheaf is invertible.)*

(4) *(functoriality in the morphism) If* $\phi : W \to X$ *is a morphism of schemes, then there is a canonical isomorphism* $\phi^*\pi^*\mathcal{G} \cong (\pi \circ \phi)^*\mathcal{G}$.
(5) *(functoriality in the quasicoherent sheaf)* π^* *is a functor* $QCoh_Y \to QCoh_X$.
(6) *(pulling back a section) Hence as a section of* $\mathcal{G}$ *is the data of a map* $\mathcal{O}_Y \to \mathcal{G}$, *by (1) and (5), if* $s : \mathcal{O}_Y \to \mathcal{G}$ *is a section of* $\mathcal{G}$ *then there is a natural section* $\pi^* s : \mathcal{O}_X \to \pi^*\mathcal{G}$ *of* $\pi^*\mathcal{G}$. *The pullback of the locus where* s *vanishes is the locus where the pulled-back section* $\pi^* s$ *vanishes.*
(7) *(pullback on stalks) If* $\pi : X \to Y$, $\pi(x) = y$, *then pullback induces an isomorphism*

$$(\pi^*\mathcal{G})_x \xrightarrow{\sim} \mathcal{G}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} .$$

(8) *(pullback on fibers) Pullback of fibers are given as follows: if* $\pi : X \to Y$, *where* $\pi(x) = y$, *then*

$$\pi^*\mathcal{G}/\mathfrak{m}_{X,x}\pi^*\mathcal{G} \cong (\mathcal{G}/\mathfrak{m}_{Y,y}\mathcal{G}) \otimes_{\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}.$$

(9) *(pullback preserves tensor product)* $\pi^*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{G}') = \pi^*\mathcal{G} \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}'$. *(Here* $\mathcal{G}'$ *is also a quasicoherent sheaf on* Y.*)*
(10) *Pullback is a right-exact functor.*

All of the above are interconnected in obvious ways that you should be able to prove by hand. (As just one example: the stalk of a pulled back section, (6), is the expected element of the pulled back stalk, (7).) In fact much more is true, that you should be able to prove on a moment's notice, such as for example that the pullback of the symmetric power of a locally free sheaf is naturally isomorphic to the symmetric power of the pullback, and similarly for wedge powers and tensor powers.

17.3.E. IMPORTANT EXERCISE. Prove Theorem 17.3.6. Possible hints: You may find it convenient to do right-exactness (10) early; it is related to right-exactness of $\otimes$. For the tensor product fact (8), show that $(M \otimes_B A) \otimes (N \otimes_B A) \cong (M \otimes N) \otimes_B A$, and that this behaves well with respect to localization. The proof of the fiber fact (8) is as follows. Given a ring map $B \to A$ with $[\mathfrak{m}] \mapsto [\mathfrak{n}]$, show that $(N \otimes_B A) \otimes_A (A/\mathfrak{m}) \cong (N \otimes_B (B/\mathfrak{n})) \otimes_{B/\mathfrak{n}} (A/\mathfrak{m})$ by showing both sides are isomorphic to $N \otimes_B (A/\mathfrak{m})$.

17.3.F. UNIMPORTANT EXERCISE. Verify that the following is a example showing that pullback is not left-exact: consider the exact sequence of sheaves on $\mathbb{A}^1$, where p is the origin:

$$0 \to \mathcal{O}_{\mathbb{A}^1}(-p) \to \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}|_p \to 0.$$

(This is the closed subscheme exact sequence for $p \in \mathbb{A}^1$, and corresponds to the exact sequence of $k[t]$-modules $0 \to tk[t] \to k[t] \to k \to 0$. Warning: here $\mathcal{O}|_p$ is not the stalk $\mathcal{O}_p$; it is the structure sheaf of the scheme p.) Restrict to p.

17.3.G. EXERCISE (THE PUSH-PULL FORMULA, CF. EXERCISE 20.7.B). Suppose $f : Z \to Y$ is any morphism, and $\pi : X \to Y$ as usual is quasicompact and separated.

Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Suppose

(17.3.6.1)
$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Z \arrow[r, "f"] & Y \end{tikzcd}$$

is a commutative diagram. Describe is a natural morphism $f^*\pi_* \to \pi'_*(f')^*\mathcal{F}$ of sheaves on Z. (Possible hint: first do the special case where (17.3.6.1) is a fiber diagram.)

By applying the above exercise in the special case where Z is a point y of Y, we see that there is a natural map from the fiber of the pushforward to the sections over the fiber:

(17.3.6.2)
$$\pi_*\mathcal{F} \otimes K(y) \to H^0(\pi^{-1}(y), \mathcal{F}|_{\pi^{-1}(y)}).$$

One might hope that $\pi_*\mathcal{F}$ "glues together" the fibers $H^0(\pi^{-1}(y), \mathcal{F}|_{\pi^{-1}(y)})$, and this is too much to ask, but at least there is a map (17.3.6.2). (In fact, under just the right circumstances, (17.3.6.2) is an isomorphism; more on this later.)

17.3.H. EXERCISE (PROJECTION FORMULA, TO BE GENERALIZED IN EXERCISE 20.7.E). Suppose $\pi : X \to Y$ is quasicompact and separated, and $\mathcal{E}$, $\mathcal{F}$ are quasicoherent sheaves on X and Y respectively.
(a) Describe a natural morphism $(\pi_*\mathcal{E}) \otimes \mathcal{F} \to \pi_*(\mathcal{E} \otimes \pi^*\mathcal{F})$. (Hint: the FHHF Theorem, Exercise 2.6.H.)
(b) If $\mathcal{F}$ is locally free, show that this natural morphism is an isomorphism. (Hint: what if $\mathcal{F}$ is free?)

17.3.7. *Remark: flatness.* Given $\pi : X \to Y$, if the functor π^* from quasicoherent sheaves on Y to quasicoherent sheaves on X is also left-exact (hence exact), we will say that π is a **flat morphism**. This is an incredibly important notion, and we will come back to it in Chapter 25.

17.3.8. *Remark: pulling back ideal sheaves.* There is one subtlety in pulling back quasicoherent ideal sheaves. Suppose $i : X \hookrightarrow Y$ is a closed immersion, and $\pi : Y' \to Y$ is an arbitrary morphism. Let $X' := X \times_Y Y'$. As "closed immersion pull back" (§10.2.1), the pulled back map $i' : X' \to Y'$ is a closed immersion. Now π^* induces canonical isomorphisms $\pi^*\mathcal{O}_Y \cong \mathcal{O}_{Y'}$ and $\pi^*\mathcal{O}_X \cong \mathcal{O}_{X'}$, but it is *not* always true that $\pi^*\mathcal{I}_{X/Y} = \mathcal{I}_{X'/Y'}$. (Exercise 17.3.F yields an example.) This is because the application of π^* to the closed subscheme exact sequence $0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to \mathcal{O}_X \to 0$ yields something that is a priori only left-exact: $\pi^*\mathcal{I}_{X/Y} \to \mathcal{O}_{Y'} \to \mathcal{O}_{X'} \to 0$. Thus, as $\mathcal{I}_{X'/Y'}$ is the kernel of $\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$, we see that $\mathcal{I}_{X'/Y}$ is the image of $\pi^*\mathcal{I}_{X/Y}$ in $\mathcal{O}_{Y'}$. We can also see this explicitly from Exercise 10.2.B: affine-locally, the ideal of the pullback is generated by the pullback of the ideal.

Note also that if π is flat (Remark 17.3.7), then $\pi^*\mathcal{I}_{X/Y} \to \mathcal{I}_{X'/Y'}$ *is* an isomorphism.

17.3.9. $\star\star$ *Pullback for ringed spaces.* (This is conceptually important but distracting for our exposition; we encourage the reader to skip this, at least on the first reading.) Pullbacks and pushforwards may be defined in the category of $\mathcal{O}$-modules on ringed spaces. We define pushforward in the usual way (Exercise 7.2.B), and then define the pullback of an $\mathcal{O}$-module using the adjoint property. Then one must show that it exists.

Here is a construction that always works in the category of ringed spaces. Suppose we have a morphism of ringed spaces $\pi : X \to Y$, and an $\mathcal{O}_Y$-module $\mathcal{G}$. Then $\pi^{-1}\mathcal{G}$ is a $\pi^{-1}\mathcal{O}_Y$-module (on the topological space X), and $\mathcal{O}_X$ is also an $\pi^{-1}\mathcal{O}_Y$-module (this module structure is part of the definition of morphism of ringed space). Then define

$$\pi^*\mathcal{G} = \pi^{-1}\mathcal{G} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X. \tag{17.3.9.1}$$

The interested reader is welcome to show that this definition, applied to quasicoherent sheaves, is the same as ours.

17.3.I. EXERCISE. Show that π^* and π_* are adjoint functors between the category of $\mathcal{O}_X$-modules and the category of $\mathcal{O}_Y$-modules. Hint: Justify the following equalities.

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}(\pi^{-1}\mathcal{G} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X, \mathcal{F}) &= \mathrm{Hom}_{\pi^{-1}\mathcal{O}_Y}(\pi^{-1}\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{F}) \end{aligned}$$

Once one defines quasicoherent sheaves on a ringed space, one may show that the pullback of a quasicoherent sheaf is quasicoherent, but we won't need this fact.

17.4 Invertible sheaves and maps to projective schemes

Theorem 17.4.1, the converse or completion to Exercise 16.3.F, will give one reason why line bundles are crucially important: they tell us about maps to projective space, and more generally, to quasiprojective A-schemes. Given that we have had a hard time naming any non-quasiprojective schemes, they tell us about maps to essentially all schemes that are interesting to us.

17.4.1. Important theorem. — *For a fixed scheme X, maps $X \to \mathbb{P}^n$ are in bijection with the data $(\mathcal{L}, s_0, \dots, s_n)$, where $\mathcal{L}$ is an invertible sheaf and $s_0, \dots, s_n$ are sections of $\mathcal{L}$ with no common zeros, up to isomorphisms of this data.*

(This works over $\mathbb{Z}$ or indeed any base.) Informally: morphisms to $\mathbb{P}^n$ correspond to $n+1$ sections of a line bundle, not all vanishing at any point, modulo global sections of $\mathcal{O}_X^*$, as multiplication by a unit gives an automorphism of $\mathcal{L}$. This is one of those important theorems in algebraic geometry that is easy to prove, but quite subtle in its effect on how one should think. It takes some time to properly digest.

17.4.2. The theorem describes all morphisms to projective space, and hence by the Yoneda philosophy, this can be taken as the *definition* of projective space: it defines projective space up to unique isomorphism. *Projective space $\mathbb{P}^n$ (over $\mathbb{Z}$) is the moduli space of a line bundle $\mathcal{L}$ along with $n+1$ sections with no common zeros.* (Can you give an analogous definition of projective space over X, $\mathbb{P}^n_X$?)

Every time you see a map to projective space, you should immediately simultaneously keep in mind the invertible sheaf and sections.

Maps to projective schemes can be described similarly. For example, if $Y \hookrightarrow \mathbb{P}^2_k$ is the curve $x_2^2 x_0 = x_1^3 - x_1 x_0^2$, then maps from a scheme X to Y are given by an invertible sheaf on X along with three sections s_0, s_1, s_2, with no common zeros, satisfying $s_2^2 s_0 - s_1^3 + s_1 s_0^2 = 0$. We make this precise in Exercise 17.4.A.

Here more precisely is the correspondence of Theorem 17.4.1. If you have $n+1$ sections, then away from the intersection of their zero-sets, we have a morphism. Conversely, if you have a map to projective space $f : X \to \mathbb{P}^n$, then we have $n+1$ sections of $\mathcal{O}_{\mathbb{P}^n}(1)$, corresponding to the hyperplane sections, $x_0, \dots, x_{n+1}$. then $f^* x_0, \dots, f^* x_{n+1}$ are sections of $f^* \mathcal{O}_{\mathbb{P}^n}(1)$, and they have no common zero.

So to prove this, we just need to show that these two constructions compose to give the identity in either direction.

Proof. Given $n+1$ sections $s_0, \dots, s_n$ of an invertible sheaf. We get trivializations on the open sets where each section doesn't vanish. The transition functions are precisely s_i/s_j on $U_i \cap U_j$. We pull back $\mathcal{O}(1)$ by this map to projective space, This is trivial on the distinguished open sets. Furthermore, $f^* D(x_i) = D(s_i)$. Moreover, $s_i/s_j = f^*(x_i/x_j)$. Thus starting with the $n+1$ sections, taking the map to the projective space, and pulling back $\mathcal{O}(1)$ and taking the sections $x_0, \dots, x_n$, we recover the s_i's. That's one of the two directions.

Correspondingly, given a map $f : X \to \mathbb{P}^n$, let $s_i = f^* x_i$. The map $[s_0; \cdots; s_n]$ is precisely the map f. We see this as follows. The preimage of U_i is $D(s_i) = D(f^* x_i) = f^* D(x_i)$. So the right open sets go to the right open sets. And $D(s_i) \to D(x_i)$ indeed corresponds to the ring map $f^* : x_j/x_i \mapsto s_j/s_i$. □

17.4.3. *Remark: Extending Theorem 17.4.1 to rational maps.* Suppose $s_0, \dots, s_n$ are sections of an invertible sheaf $\mathcal{L}$ on a scheme X. Then Theorem 17.4.1 yields a morphism $X - V(s_1, \dots, s_n) \to \mathbb{P}^n$. In particular, if X is integral, and the s_i are not all 0, this data yields a rational map $X \dashrightarrow \mathbb{P}^n$.

17.4.A. IMPORTANT EXERCISE. Suppose $S_\bullet$ is a finitely generated graded A-algebra, generated in degree 1. If Y is an A-scheme, give a bijection between A-morphisms $Y \to \operatorname{Proj} S_\bullet$ and the following data (up to isomorphism):

- maps of graded rings $f : S_\bullet \to \oplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$, where $\mathcal{L}$ is an invertible sheaf globally generated by $f(S_1)$,
- where two such maps are considered the same if they agree in sufficiently high degree (i.e. if the two maps agree in degree higher than n_0 for some n_0).

(It will take some thought to extract this from Theorem 17.4.1. Your bijection will be functorial in Y.)

17.4.B. EXERCISE (AUTOMORPHISMS OF PROJECTIVE SPACE). Show that all the automorphisms of projective space $\mathbb{P}^n_k$ correspond to $(n+1) \times (n+1)$ invertible matrices over k, modulo scalars (also known as $PGL_{n+1}(k)$). (Hint: Suppose $f : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism. Show that $f^* \mathcal{O}(1) \cong \mathcal{O}(1)$. Show that $f^* : \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ is an isomorphism.)

Exercise 17.4.B will be useful later, especially for the case $n = 1$. In this case, these automorphisms are called *fractional linear transformations*. (For experts: why did I not state that previous exercise over an arbitrary base ring A? Where does the argument go wrong in that case?)

17.4.C. EXERCISE. Show that $\mathrm{Aut}(\mathbb{P}^1_k)$ is strictly three-transitive on k-points, i.e. given two triplets (p_1, p_2, p_3) and (q_1, q_2, q_3) each of distinct (k-)points of $\mathbb{P}^1$, there is precisely one automorphism of $\mathbb{P}^1$ sending p_i to q_i ($i = 1, 2, 3$).

Here are more examples of these ideas in action.

17.4.4. *Example: the tautological rational map from affine space to projective space.* Consider the $n+1$ functions $x_0, \dots, x_n$ on $\mathbb{A}^{n+1}$ (otherwise known as $n+1$ sections of the trivial bundle). They have no common zeros on $\mathbb{A}^{n+1} - 0$. Hence they determine a morphism $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$. (We discussed this morphism in Exercise 7.3.E, but now we don't need tedious gluing arguments.)

17.4.5. *Example: the Veronese embedding is $|\mathcal{O}_{\mathbb{P}^n}(d)|$.* Consider the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n$. We have checked that the number of sections of this line bundle are $\binom{n+m}{m}$, and they correspond to homogeneous degree m polynomials in the projective coordinates for $\mathbb{P}^n$. Also, they have no common zeros (as for example the subset of sections x_0^m, x_1^m, …, x_n^m have no common zeros). Thus the complete linear series is base-point-free, and determines a morphism $\mathbb{P}^n \to \mathbb{P}^{\binom{n+m}{m}-1}$. This is the Veronese embedding (Definition 9.2.8). For example, if $n = 2$ and $m = 2$, we get a map $\mathbb{P}^2 \to \mathbb{P}^5$.

Remark 9.2.8 showed that this is a closed immersion. The following is a more general method of checking that maps to projective space are closed immersion.

17.4.D. LESS IMPORTANT EXERCISE. Suppose $\pi : X \to \mathbb{P}^n_A$ corresponds to an invertible sheaf $\mathcal{L}$ on X, and sections $s_0, \dots, s_n$. Show that π is a closed immersion if and only if

(i) each open set X_{s_i} is affine, and
(ii) for each i, the map of rings $A[y_0, \dots, y_n] \to \Gamma(X_{s_i}, \mathcal{O})$ given by $y_j \mapsto s_j/s_i$ is surjective.

17.4.6. *Example: Maps $\mathbb{P}^1 \to \mathbb{P}^n$.* Recall that the image of the Veronese morphism when $n = 1$ is called a *rational normal curve of degree* m (Exercise 9.2.J). Our map is $\mathbb{P}^1 \to \mathbb{P}^m$ given by $[x; y] \to [x^m; x^{m-1}y; \cdots; xy^{m-1}; y^m]$.

17.4.E. EXERCISE. If the image scheme-theoretically lies in a hyperplane of projective space, we say that it is *degenerate* (and otherwise, *non-degenerate*). Show that a base-point-free linear series V with invertible sheaf $\mathcal{L}$ is non-degenerate if and only if the map $V \to \Gamma(X, \mathcal{L})$ is an inclusion. Hence in particular a complete linear series is always non-degenerate.

17.4.F. EXERCISE. Suppose we are given a map $\pi : \mathbb{P}^1_k \to \mathbb{P}^n_k$ where the corresponding invertible sheaf on $\mathbb{P}^1_k$ is $\mathcal{O}(d)$. (We will later call this a *degree* d *map*.) Show that if $d < n$, then the image is degenerate. Show that if $d = n$ and the image is nondegenerate, then the image is isomorphic (via an automorphism of projective space, Exercise 17.4.B) to a rational normal curve.

17.4.G. EXERCISE: AN EARLY LOOK AT INTERSECTION THEORY, RELATED TO BÉZOUT'S THEOREM. A classical definition of the degree of a curve in projective space is as follow: intersect it with a "general" hyperplane, and count the number of points of intersection, with appropriate multiplicity. We interpret this in the case of $\pi : \mathbb{P}^1_k \to \mathbb{P}^n_k$. Show that there is a hyperplane H of $\mathbb{P}^n_k$ not containing $\pi(\mathbb{P}^1_k)$. Equivalently, $\pi^* H \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ is not 0. Show that the number of zeros of $\pi^* H$ is precisely d. (You will have to define "appropriate multiplicity".) What does it mean geometrically if π is a closed immersion, and $\pi^* H$ has a double zero? Can you make sense of this even if π is not a closed immersion?) Thus this classical notion of degree agrees with the notion of degree in Exercise 17.4.F. (See Exercise 9.2.E for another case of Bézout's theorem. Here we intersect a degree d curve with a degree 1 hyperplane; there we intersect a degree 1 curve with a degree d hyperplane. Exercise 20.5.M will give a common generalization.)

17.4.7. *Example: The Segre morphism revised.* The Segre morphism can also be interpreted in this way. This is a useful excuse to define some notation. Suppose $\mathcal{F}$ is a quasicoherent sheaf on a Z-scheme X, and $\mathcal{G}$ is a quasicoherent sheaf on a Z-scheme Y. Let π_X, π_Y be the projections from $X \times_Z Y$ to X and Y respectively. Then $\mathcal{F} \boxtimes \mathcal{G}$ is defined to be $\pi_X^* \mathcal{F} \otimes \pi_Y^* \mathcal{G}$. In particular, $\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a, b)$ is defined to be $\mathcal{O}_{\mathbb{P}^m}(a) \boxtimes \mathcal{O}_{\mathbb{P}^n}(b)$ (over any base Z). The Segre morphism $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ corresponds to the complete linear series for the invertible sheaf $\mathcal{O}(1,1)$.

When we first saw the Segre morphism in §10.5, we saw (in different language) that this complete linear series is base-point-free. We also checked by hand (§10.5.1) that it is a closed immersion, essentially by Exercise 17.4.D.

Recall that if $\mathcal{L}$ and $\mathcal{M}$ are both base-point-free invertible sheaves on a scheme X, then $\mathcal{L} \otimes \mathcal{M}$ is also base-point-free (Exercise 16.3.B, see also Definition 16.3.4). We may interpret this fact using the Segre morphism (under reasonable hypotheses on X). If $\phi_{\mathcal{L}} : X \to \mathbb{P}^M$ is a morphism corresponding to a (base-point-free) linear series based on $\mathcal{L}$, and $\phi_{\mathcal{M}} : X \to \mathbb{P}^N$ is a morphism corresponding to a linear series on $\mathcal{M}$, then the Segre morphism yields a morphism $X \to \mathbb{P}^M \times \mathbb{P}^N \to \mathbb{P}^{(M+1)(N+1)-1}$, which corresponds to a base-point-free series of sections of $\mathcal{L} \otimes \mathcal{M}$.

17.4.H. FUN EXERCISE. Show that any map from projective space to a smaller projective space is constant (over a field). Hint: show that if $m < n$ then m non-empty hypersurfaces in $\mathbb{P}^n$ have non-empty intersection. For this, use the fact that any non-empty hypersurface in $\mathbb{P}^n_k$ has non-empty intersection with any subscheme of dimension at least 1.

17.4.I. EXERCISE. Show that a base-point-free linear series V on X corresponding to $\mathcal{L}$ induces a morphism to projective space $X \to \mathbb{P}V^\vee = \operatorname{Proj} \oplus_n \mathcal{L}^{\otimes n}$. The resulting morphism is often written

$$X \xrightarrow{|V|} \mathbb{P}^n .$$

17.4.8. $\star\star$ **A proper nonprojective k-scheme — and gluing schemes along closed subschemes.**

We conclude by using what we have developed to describe an example of a scheme that is proper but not projective (promised in Remark 11.3.6). We use a construction that looks so fundamental that you may be surprised to find that we won't use it in any meaningful way later.

Fix an algebraically closed field k. For $i = 1, 2$, let $X_i \cong \mathbb{P}^3_k$, Z_i be a line in X_i, and Z'_i be a nonsingular conic in X_i disjoint from X_i (both Z_i and Z'_i isomorphic to $\mathbb{P}^1_k$). The construction of §17.4.9 will allow us to glue X_1 to X_2 so that Z_1 is identified with Z'_2 and Z'_1 is identified with Z_2. (You will be able to make this precise after reading §17.4.9.) The result, call it X, is proper, by Exercise 17.4.M.

Then X is not projective. For if it were, then it would be embedded in projective space by some invertible sheaf $\mathcal{L}$. If X is embedded, then X_1 is too, so $\mathcal{L}$ must restrict to an invertible sheaf on X_1 of the form $\mathcal{O}_{X_1}(n_1)$, where $n_1 > 0$. You can check that the restriction of $\mathcal{L}$ to Z_1 is $\mathcal{O}_{Z_1}(n_1)$, and the restriction of $\mathcal{L}$ to Z'_1 is $\mathcal{O}_{Z'_1}(2n_1)$. Symmetrically, the restriction of $\mathcal{L}$ to Z_2 is $\mathcal{O}_{Z_2}(n_2)$ for some $n_2 > 0$, and the restriction of $\mathcal{L}$ to Z'_2 is $\mathcal{O}_{Z'_2}(2n_2)$. But after gluing, $Z_1 = Z'_2$, and $Z'_1 = Z_2$, so we have $n_1 = 2n_2$ and $2n_1 = n_2$, which is impossible.

17.4.9. *Gluing two schemes together along isomorphic closed subschemes.*

It is straightforward to show that you can glue two schemes along isomorphic open subschemes. (More precisely, if X_1 and X_2 are schemes, with open subschemes U_1 and U_2 respectively, and an isomorphism $U_1 \cong U_2$, you can make sense of gluing X_1 and X_2 along $U_1 \cong U_2$. You should think this through.) You can similarly glue two schemes along isomorphic *closed* subschemes. We now make this precise. Suppose $Z_1 \hookrightarrow X_1$ and $Z_2 \hookrightarrow X_2$ are closed immersions, and $\phi : Z_1 \xrightarrow{\sim} Z_2$ is an isomorphism. We will explain how to glue X_1 to X_2 along ϕ. The result will be called $X_1 \coprod_\phi X_2$.

17.4.10. *Motivating example.* Our motivating example is if $X_i = \operatorname{Spec} A_i$ and $Z_i = \operatorname{Spec} A_i/I_i$, and ϕ corresponds to $\phi^\sharp : A_2/I_2 \xrightarrow{\sim} A_1/I_1$. Then the result will be $\operatorname{Spec} R$, where R is the ring of consisting of ordered pairs $(a_1, a_2) \in A_1 \times A_2$ that "agree via ϕ". More precisely, this is a fibered product of rings:

$$R := A_1 \times_{\phi^\sharp : A_1/I_1 \to A_2/I_2} A_2.$$

17.4.11. *The general construction, as a locally ringed space.* In our general situation, we might wish to cover X_1 and X_2 by open charts of this form. We would then have to worry about gluing and choices, so to avoid this, we instead first construct $X_1 \coprod_\phi X_2$ as a locally ringed space. As a topological space, the definition is clear: we glue the underlying sets together along the underlying sets of $Z_1 \cong Z_2$, and topologize it so that a subset of $X_1 \coprod_\phi X_2$ is open if and only if its restrictions to X_1 and X_2 are both open. For convenience, let Z be the image of Z_1 (or equivalently Z_2) in $X_1 \coprod_\phi X_2$. We next define the stalk of the structure sheaf at any point $x \in X_1 \coprod_\phi X_2$. If $x \in X_i \setminus Z = (X_1 \coprod_\phi X_2) \setminus X_{3-i}$ (hopefully the meaning of this is clear), we define the stalk as $\mathcal{O}_{X,x}$. If $x \in X_1 \cap X_2$, we define the stalk to consist of elements (s_1, s_2) $\mathcal{O}_{X_1,x} \times \mathcal{O}_{X_2,x}$ such that agree in $\mathcal{O}_{Z_1,x} \cong \mathcal{O}_{Z_2,x}$. The meaning of everything in this paragraph will be clear to you if you can do the following.

17.4.J. EXERCISE. Define the structure sheaf of $\mathcal{O}_{X_1 \coprod_\phi X_2}$ in terms of compatible germs. (What should it mean for germs to be compatible? Hint: for $z \in Z$, suppose we have open subsets U_1 of X_1 and U_2 of X_2, with $U_1 \cap Z = U_2 \cap Z$, so U_1 and

U_2 glue together to give an open subset U of $X_1 \coprod_\phi X_2$. Suppose we also have functions f_1 on X_1 and f_2 on U_2 that "agree on $U \cap Z$" — what does that mean? Then we declare that the germs of the "function on U obtained by gluing together f_1 and f_2" are compatible.) Show that the resulting ringed space is a locally ringed space.

We next want to show that the locally ringed space $X_1 \coprod_\phi X_2$ is a scheme. Clearly it is a scheme away from Z. We first verify a special case.

17.4.K. EXERCISE. Show that in Example 17.4.10 the construction of §17.4.11 indeed yields $\operatorname{Spec}(A_1 \times_{\phi^\sharp} A_2)$.

17.4.L. EXERCISE. In the general case, suppose $x \in Z$. Show that there is an affine open subset $\operatorname{Spec} A_i \subset X_i$ such that $Z \cap \operatorname{Spec} A_1 = Z \cap \operatorname{Spec} A_2$. Then use Exercise 17.4.J to show that $X_1 \coprod_\phi X_2$ is a scheme in a neighborhood of x, and thus a scheme.

17.4.12. *Remarks.* (a) As the notation suggests, this is a fibered coproduct in the category of schemes, and indeed in the category of locally ringed spaces. We won't need this fact, but you can prove it if you wish; it isn't hard. Unlike the situation for products, fibered coproducts don't exist in general in the category of schemes. Miraculously (and for reasons that are specific to schemes), the resulting cofibered diagram is *also* a fibered diagram. This has pleasant ramifications. For example, this construction "behaves well with respect to" (or "commutes with") base change; this can help with Exercise 17.4.M(a), but if you use it, you have to prove it.

(b) Here are some interesting questions to think through: Can we recover the gluing locus from the "glued scheme" $X_1 \coprod_\phi X_2$ and the two closed subschemes X_1 and X_2? (Yes.) When is a scheme the gluing of two closed subschemes along their scheme-theoretic intersection? (When their scheme-theoretic union is the entire scheme.)

(c) You might hope that if you have a single scheme X with two disjoint closed subschemes W' and W'', and an isomorphism $W' \to W''$, then you should be able to glue X to itself along $W' \to W''$. This construction doesn't work, and indeed it may not be possible. You can still make sense of the quotient as an *algebraic space*, which I will not define here.

17.4.M. EXERCISE. We continue to use the notation X_i, ϕ, etc. Suppose we are working in the category of A-schemes.

(a) If X_1 and X_2 are universally closed, show that $X_1 \coprod_\phi X_2$ is as well.
(b) If X_1 and X_2 are separated, show that $X_1 \coprod_\phi X_2$ is as well.
(c) If X_1 and X_2 are finite type over A, show that $X_1 \coprod_\phi X_2$ is as well. (Hint: Reduce to the "affine" case of the Motivating Example 17.4.10. Choose generators $x_1, \dots, x_n$ of A_1, and $y_1, \dots, y_n$, such that x_i modulo I_1 agrees with y_i modulo I_2 via ϕ. Choose generators $g_1, \dots, g_m$ of I_2. Show that (x_i, y_i) and $(0, g_i)$ generate $R \subset A_1 \times A_2$, as follows. Suppose $(a_1, a_2) \in R$. Then there is some polynomial m such that $a_1 = m(x_1, \dots, x_n)$. Hence $(a_1, a_2) - m((x_1, y_1), \dots, (x_n, y_n)) = (0, a_2')$ for

some $a_2' \in I_2$. Then a_2' can be written as $\sum_{i=1}^m \ell_i(y_1, \dots, y_n) g_i$. But then $(0, a_2') = \sum_{i=1}^m \ell_i((x_1, y_1), \dots, (x_n, y_n))(0, g_i)$.)

Thus if X_1 and X_2 are proper, so is $X_1 \coprod_\phi X_2$.

17.5 The Curve-to-projective Extension Theorem

We now use the main theorem of the previous section, Theorem 17.4.1, to prove something useful and concrete.

17.5.1. The Curve-to-projective Extension Theorem. — *Suppose C is a pure dimension 1 Noetherian scheme over a base S, and $p \in C$ is a nonsingular closed point of it. Suppose Y is a projective S-scheme. Then any morphism $C \setminus \{p\} \to Y$ extends to $C \to Y$.*

In practice, we will use this theorem when $S = k$, and C is a k-variety.

Note that if such an extension exists, then it is unique: the nonreduced locus of C is a closed subset (Exercise 9.3.F). Hence by replacing C by an open neighborhood of p that is reduced, we can use the Reduced-to-Separated theorem 11.2.1 that maps from reduced schemes to separated schemes are determined by their behavior on a dense open set. Alternatively, maps to a separated scheme can be extended over an effective Cartier divisor in at most one way (Exercise 11.2.C).

The following exercise show that the hypotheses are necessary.

17.5.A. EXERCISE. In each of the following cases, prove that the morphism $C \setminus \{p\} \to Y$ cannot be extended to a morphism $C \to Y$.

(a) *Projectivity of Y is necessary.* Suppose $C = \mathbb{A}_k^1$, $p = 0$, $Y = \mathbb{A}_k^1$, and $C \setminus \{p\} \to Y$ is given by "$t \mapsto 1/t$".
(b) *One-dimensionality of C is necessary.* Suppose $C = \mathbb{A}_k^2$, $p = (0, 0)$, $Y = \mathbb{P}_k^1$, and $C \setminus \{p\} \to Y$ is given by $(x, y) \mapsto [x; y]$.
(c) *Non-singularity of C is necessary.* Suppose $C = \operatorname{Spec} k[x, y]/(y^2 - x^3)$, $p = 0$, $Y = \mathbb{P}_k^1$, and $C \setminus \{p\} \to Y$ is given by $(x, y) \mapsto [x; y]$.

We remark that by combining this (easy) theorem with the (hard) valuative criterion of properness (Theorem 13.5.6), one obtains a proof of the properness of projective space bypassing the (tricky) Fundamental Theorem of Elimination Theory 8.4.5.

The central idea of the proof may be summarized as "clear denominators", as illustrated by the following motivating example. Suppose you have a morphism from $\mathbb{A}^1 - \{0\}$ to projective space, and you wanted to extend it to $\mathbb{A}^1$. Suppose the map was given by $t \mapsto [t^4 + t^{-3}; t^{-2} + 4t]$. Then of course you would "clear the denominators", and replace the map by $t \mapsto [t^7 + 1; t + t^4]$. Similarly, if the map was given by $t \mapsto [t^2 + t^3; t^2 + t^4]$, you would divide by t^2, to obtain the map $t \mapsto [1 + t; 1 + t^2]$.

Proof. We begin with some quick reductions. We can assume S is affine, say $\operatorname{Spec} R$ (by shrinking S and C). The nonreduced locus of C is closed and doesn't contain p (Exercise 9.3.F), so by replacing C by an appropriate neighborhood of p, we may assume that C is reduced and affine.

We next reduce to the case where $Y = \mathbb{P}^n_R$. Choose a closed immersion $Y \to \mathbb{P}^n_R$. If the result holds for $\mathbb{P}^n$, and we have a morphism $C \to \mathbb{P}^n$ with $C \setminus \{p\}$ mapping to Y, then C must map to Y as well. Reason: we can reduce to the case where the source is an affine open subset, and the target is $\mathbb{A}^n_R \subset \mathbb{P}^n_R$ (and hence affine). Then the functions vanishing on $Y \cap \mathbb{A}^n_R$ pull back to functions that vanish at the generic point of C and hence vanish everywhere on C (using reducedness of C), i.e. C maps to Y.

Choose a uniformizer $t \in \mathfrak{m} - \mathfrak{m}^2$ in the local ring of C at p. This is an element of $K(C)^\times$, with a finite number of poles (from Exercise 13.4.G on finiteness of number of zeros and poles). The complement of these finite number of points is an open neighborhood of p, so by replacing C by a smaller open affine neighborhood of p, we may assume that t is a function on C. Then $V(t)$ is also a finite number of points (including p), again from Exercise 13.4.G)so by replacing C by an open affine neighborhood of p in $C \setminus V(t) \cup p$, we may assume that p is only zero of the function t (and of course t vanishes to multiplicity 1 at p).

We have a map $C \setminus \{p\} \to \mathbb{P}^n_R$, which by Theorem 17.4.1 corresponds to a line bundle $\mathcal{L}$ on $C \setminus \{p\}$ and $n+1$ sections of it with no common zeros in $C \setminus \{p\}$. Let U be a nonempty open set of $C \setminus \{p\}$ on which $\mathcal{L} \cong \mathcal{O}$. Then by replacing C by $U \cup p$, we interpret the map to $\mathbb{P}^n$ as $n+1$ rational *functions* $f_0, \dots, f_n$, defined away away from p, with no common zeros away from p. Let $N = \min_i(\operatorname{val}_p f_i)$. Then $t^{-N} f_0, \dots, t^{-N} f_n$ are $n+1$ functions with no common zeros. Thus they determine a morphism $C \to \mathbb{P}^n$ extending $C \setminus \{p\} \to \mathbb{P}^n$ as desired. □

17.5.B. EXERCISE (USEFUL PRACTICE). Suppose X is a Noetherian k-scheme, and Z is an irreducible codimension 1 subvariety whose generic point is a nonsingular point of X (so the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring). Suppose $X \dashrightarrow Y$ is a rational map to a projective k-scheme. Show that the domain of definition of the rational map includes a dense open subset of Z. In other words, rational maps from Noetherian k-schemes to projective k-schemes can be extended over nonsingular codimension 1 sets. (We have seen this principle in action, see Exercise 7.5.J on the Cremona transformation.)

17.6 ⋆ The Grassmannian as a moduli space

In §7.7, we gave a preliminary description of the Grassmannian. We are now in a position to give a better definition.

We describe the "Grassmannian functor" of $G(k, n)$, then show that it is representable. The construction works over an arbitrary base scheme, so we work over the final object $\operatorname{Spec} \mathbb{Z}$. (You should think through what to change if you wish to work with, for example, complex schemes.) The functor is defined as follows. To a scheme B, we associate the set of *locally free rank* k *quotients of the rank* n *free sheaf, up to isomorphism*. An isomorphism of two such quotients $\phi : \mathcal{O}_B^{\oplus n} \to \mathcal{Q} \to 0$ and

$\phi' : \mathcal{O}_B^{\oplus n} \to \mathcal{Q}' \to 0$ is an isomorphism $\sigma : \mathcal{Q} \to \mathcal{Q}'$ such that the diagram

$$\begin{tikzcd} \mathcal{O}^{\oplus n} \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & \mathcal{Q} \arrow[d, "\sigma"] \\ & \mathcal{Q}' \end{tikzcd}$$

commutes. By Exercise 14.5.B(b), $\ker \phi$ is locally free of rank $n-k$. (Thus if you prefer, you can consider the functor to take B to short exact sequences $0 \to \mathcal{S} \to \mathcal{O}^{\oplus n} \to \mathcal{Q} \to 0$ of locally free sheaves over B.)

It may surprise you that we are considering rank k *quotients* of a rank n sheaf, not rank k *subobjects*, given that the Grassmannian should parametrize k-dimensional subspace of an n-dimensional space. This is done for several reasons. One is that the kernel of a surjective map of locally free sheaves must be locally free, while the cokernel of an injective map of locally free sheaves need not be locally free (Exercise 14.5.B(b) and (c) respectively). Another reason: we will later see that the geometric incarnation of this problem indeed translates to this. We can already see a key example here: if $k=1$, our definition yields one-dimensional quotients $\mathcal{O}^{\oplus n} \to \mathcal{L} \to 0$. But this is precisely the data of n sections of $\mathcal{L}$, with no common zeros, which by Theorem 17.4.1 (the functorial description of projective space) corresponds precisely to maps to $\mathbb{P}^n$, so the $k=1$ case parametrizes what we want.

We now show that the Grassmannian functor is representable for given n and k. Throughout the rest of this section, a k-subset is a subset of $\{1, \dots, n\}$ of size k.

17.6.A. EXERCISE. (a) Suppose I is a k-subset. Make the following statement precise: there is an open subfunctor $G(k,n)_I$ of $G(k,n)$ where the k sections of $\mathcal{Q}$ corresponding to I (of the n sections of $\mathcal{Q}$ coming from the surjection $\phi : \mathcal{O}^{\oplus n} \to \mathcal{Q}$) are linearly independent. Hint: in a trivializing neighborhood of $\mathcal{Q}$, where we can choose an isomorphism $\mathcal{Q} \xrightarrow{\sim} \mathcal{O}^{\oplus k}$, ϕ can be interpreted as a $k \times n$ matrix M, and this locus is where the determinant of the $k \times k$ matrix consisting of the I columns of M is nonzero. Show that this locus behaves well under transitions between trivializations.
(b) Show that these open subfunctors $G(k,n)_I$ cover the functor $G(k,n)$ (as I runs through the k-subsets).

Hence by Exercise 10.1.I, to show $G(k,n)$ is representable, we need only show that $G(k,n)_I$ is representable for arbitrary I. After renaming the summands of $\mathcal{O}^{\oplus n}$, without loss of generality we may as well assume $I = \{1, \dots, k\}$.

17.6.B. EXERCISE. Show that $G(k,n)_{\{1\dots,k\}}$ is represented by $\mathbb{A}^{nk}$ as follows. (You will have to make this precise.) Given a surjection $\phi : \mathcal{O}^{\oplus n} \to \mathcal{Q}$, let $\phi_i : \mathcal{O} \to \mathcal{Q}$ be the map from the ith summand of $\mathcal{O}^{\oplus n}$. (Really, ϕ_i is just a section of $\mathcal{Q}$.) For the open subfunctor $G(k,n)_I$, show that

$$\phi_1 \oplus \cdots \oplus \phi_k : \mathcal{O}^{\oplus k} \to \mathcal{Q}$$

is an isomorphism. For a scheme B, the bijection $G(k,n)_I(B) \leftrightarrow \mathbb{A}^{nk}$ is given as follows. Given an element $\phi \in G(k,n)_I(B)$, for $j \in \{k+1, \dots, n\}$, $\phi_j = a_{1j}\phi_1 + a_{2j}\phi_2 + \cdots + a_{kj}\phi_k$, where a_{ij} are functions on B. But $k(n-k)$ functions on B is the same as a map to $\mathbb{A}^{k(n-k)}$ (Exercise 7.6.C). Conversely, given $k(n-k)$ functions

a_{ij} $(1 \leq i \leq k < j \leq n)$, define a surjection $\phi : \mathcal{O}^{\oplus n} \to \oplus^{\oplus k}$ as follows: $(\phi_1 \dots, \phi_k)$ is the identity, and $\phi_j = a_{1j}\phi_1 + a_{2j}\phi_2 + \cdots + a_{kj}\phi_k$ for $j > k$.

You have now shown that $G(k,n)$ is representable, by covering it with $\binom{n}{k}$ copies of $\mathbb{A}^{k(n-k)}$. (You might wish to relate this to the description you gave in §7.7.) In particular, the Grassmannian over a field is smooth, and irreducible of dimension $k(n-k)$. (Once we define smoothness in general, the Grassmannian over any base will be smooth over that base, because $\mathbb{A}_B^{k(n-k)} \to B$ will always be smooth.)

17.6.1. The Plücker embedding.

By applying $\wedge^k$ to a surjection $\phi : \mathcal{O}^{\oplus n} \to \mathcal{Q}$ (over an arbitrary base B), we get a surjection $\wedge^k \phi : \mathcal{O}^{\oplus \binom{n}{k}} \to \det \mathcal{Q}$ (Exercise 14.5.G). But a surjection from a rank N free sheaf to a line bundle is the same as a map to $\mathbb{P}^{N-1}$ (Theorem 17.4.1).

17.6.C. EXERCISE. Use this to describe a map $P : \mathbb{G}(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$. (This is just a tautology: a natural transformation of functors induces a map of the representing schemes. This is Yoneda's Lemma, although if you didn't do Exercise 2.3.Y, you may wish to do it by hand. But once you do, you may as well go back to prove Yoneda's Lemma and do Exercise 2.3.Y, because the argument is just the same!)

17.6.D. EXERCISE. The projective coordinate on $\mathbb{P}^{\binom{n}{k}-1}$ corresponding to the Ith factor of $\mathcal{O}^{\oplus \binom{n}{k}}$ may be interpreted as the determinant of the map $\phi_I : \mathcal{O}^{\oplus k} \to \mathcal{Q}$, where the $\mathcal{O}^{\oplus k}$ consists of the summands of $\mathcal{O}^{\oplus n}$ corresponding to I. Make this precise.

17.6.E. EXERCISE. Show that the standard open set U_I of $\mathbb{P}^{\binom{n}{k}-1}$ corresponding to k-subset I (i.e. where the corresponding coordinate doesn't vanish) pulls back to the open subscheme $G(k,n)_I \subset G(k,n)$. Denote this map $P_I : G(k,n)_I \to U_I$.

17.6.F. EXERCISE. Show that P_I is a closed immersion as follows. We may deal with the case $I = \{1, \dots, k\}$. Note that $G(k,n)_I$ is affine — you described it $\operatorname{Spec} \mathbb{Z}[a_{ij}]_{1 \leq i \leq k < j \leq n}$ in Exercise 17.6.B. Also, U_I is affine, with coordinates $x_{I'/I}$, as I' varies over the other k-subsets. You want to show that the map

$$P_I^\sharp : \mathbb{Z}[x_{I'/I}]_{I' \subset \{1,\dots,n\}, |I'|=k\}}/(x_{I/I} - 1) \to \mathbb{Z}[a_{ij}]_{1 \leq i \leq k < j \leq n}$$

is a surjection. By interpreting the map $\phi : \mathcal{O}^{\oplus n} \to \mathcal{O}^{\oplus k}$ as a $k \times n$ matrix M whose left k columns are the identity matrix and whose remaining entries are a_{ij} $(1 \leq i \leq k < j \leq n)$, interpret $P_I^\sharp$ as taking $x_{I'/I}$ to the determinant of the $k \times k$ submatrix corresponding to the columns in I'. For each (i,j) (with $1 \leq i \leq k < j \leq n$), find some I' so that $x_{I'/I} \mapsto \pm a_{ij}$. (Let $I' = \{1, \dots, i-1, i+1, \dots, k, j\}$.)

Hence $G(k,n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$ is projective over $\mathbb{Z}$.

17.6.2. *Remark: The Plücker equations.* The equations of $G(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$ are particularly nice. There are quadratic relations among the $k \times k$ minors of a $k \times (n-k)$ matrix, called the Plücker relations. By our construction, they are equations satisfied by $G(k,n)$. It turns out that these equations cut out $G(k,n)$, and in fact generate the homogeneous ideal of $G(k,n)$, but this takes more work.

17.6.G. $\star\star$ EXERCISE (GRASSMANNIAN BUNDLES). Suppose $\mathcal{F}$ is a rank n locally free sheaf on a scheme X. Define the Grassmannian bundle $G(k, \mathcal{F})$ over X. Intuitively, if $\mathcal{F}$ is a varying family of n-dimensional vector spaces over X, $G(k, \mathcal{F})$ should parametrize k-dimensional quotients of the fibers. You may want to define the functor first, and then show that it is representable. Your construction will behave well under base change.

CHAPTER 18

Relative versions of Spec and Proj, and projective morphisms

In this chapter, we will use universal properties to define two useful constructions, $\mathcal{S}$pec of a sheaf of algebras $\mathcal{A}$, and $\mathcal{P}$roj of a sheaf of graded algebras $\mathcal{A}_\bullet$ on a scheme X. These will both generalize (globalize) our constructions of Spec of A-algebras and Proj of graded A-algebras. We will see that affine morphisms are precisely those of the form $\mathcal{S}\mathrm{pec}\,\mathcal{A} \to X$, and so we will *define* projective morphisms to be those of the form $\mathcal{P}\mathrm{roj}\,\mathcal{A}_\bullet \to X$.

In both cases, our plan is to make a notion we know well over a ring work more generally over a scheme. The main issue is how to glue the constructions over each affine open subset together. The slick way we will proceed is to give a universal property, then show that the affine construction satisfies this universal property, then that the universal property behaves well with respect to open subsets, then to use the idea that let us glue together the fibered product (or normalization) together to do all the hard gluing work. The most annoying part of this plan is finding the right universal property, especially in the $\mathcal{P}$roj case.

18.1 Relative Spec of a (quasicoherent) sheaf of algebras

Given an A-algebra, B, we can take its Spec to get an affine scheme over Spec A: $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$. We will now see universal property description of a globalization of that notation. Consider an arbitrary scheme X, and a quasicoherent sheaf of algebras $\mathcal{B}$ on it. We will define how to take Spec of this sheaf of algebras, and we will get a scheme $\mathcal{S}\mathrm{pec}\,\mathcal{B} \to X$ that is "affine over X", i.e. the structure morphism is an affine morphism. You can think of this in two ways.

18.1.1. First, and most concretely, for any affine open set $\mathrm{Spec}\,A \subset X$, $\Gamma(\mathrm{Spec}\,A, \mathcal{B})$ is some A-algebra; call it B. Then above Spec A, $\mathcal{S}\mathrm{pec}\,\mathcal{B}$ will be Spec B.

18.1.2. Second, it will satisfy a universal property. We could define the A-scheme Spec B by the fact that maps to Spec B (from an A-scheme Y, over Spec A) correspond to maps of A-algebras $B \to \Gamma(Y, \mathcal{O}_Y)$ (this is our old friend Exercise 7.3.F). The universal property for $\beta : \mathcal{S}\mathrm{pec}\,\mathcal{B} \to X$ generalizes this. Given a morphism $\pi : Y \to X$, the X-morphisms $Y \to \mathcal{S}\mathrm{pec}\,\mathcal{B}$ are in functorial (in Y) bijection with morphisms α making

$$\begin{tikzcd} & \mathcal{O}_X \arrow[dl] \arrow[dr] & \\ \mathcal{B} \arrow[rr, "\alpha"] & & \pi_* \mathcal{O}_Y \end{tikzcd}$$

commute. Here the map $\mathcal{O}_X \to \pi_* \mathcal{O}_Y$ is that coming from the map of ringed spaces, and the map $\mathcal{O}_X \to \mathcal{B}$ comes from the $\mathcal{O}_X$-algebra structure on $\mathcal{B}$. (For experts: it needn't be true that $\pi_* \mathcal{O}_Y$ is quasicoherent, but that doesn't matter.)

By universal property nonsense, this data determines $\beta : \mathcal{S}\mathrm{pec}\, \mathcal{B} \to X$ up to unique isomorphism, assuming that it exists.

Fancy translation: in the category of X-schemes, $\beta : \mathcal{S}\mathrm{pec}\, \mathcal{B} \to X$ represents the functor

$$(\pi : Y \to X) \longmapsto \{(\alpha : \mathcal{B} \to \pi_* \mathcal{O}_Y)\}.$$

18.1.A. EXERCISE. Show that if X is affine, say $\operatorname{Spec} A$, and $\mathcal{B} = \tilde{B}$, where B is an A-algebra, then $\operatorname{Spec} B \to \operatorname{Spec} A$ satisfies this universal property. (Hint: Exercise 7.3.F.)

18.1.3. Proposition. — *Suppose* $\beta : \mathcal{S}\mathrm{pec}\, \mathcal{B} \to X$ *satisfies the universal property for* $(X, \mathcal{B})$*, and* $U \hookrightarrow X$ *is an open subset. Then* $\beta|_U : \mathcal{S}\mathrm{pec}\, \mathcal{B} \times_X U = (\operatorname{Spec} \mathcal{B})|_U \to U$ *satisfies the universal property for* $(U, \mathcal{B}|_U)$*.*

Proof. For convenience, let $V = \mathcal{S}\mathrm{pec}\, \mathcal{B} \times_X U$. A U-morphism $Y \to V$ is the same as an X-morphism $Y \to \mathcal{S}\mathrm{pec}\, \mathcal{B}$ (where by assumption $Y \to X$ factors through U). By the universal property of $\mathcal{S}\mathrm{pec}\, \mathcal{B}$, this is the same information as a map $\mathcal{B} \to \pi_* \mathcal{O}_Y$, which by the universal property definition of pullback (§ 17.3.3) is the same as $\pi^* \mathcal{B} \to \mathcal{O}_Y$, which is the same information as $(\pi|_U)^* \mathcal{B} \to \mathcal{O}_Y$. By adjointness again this is the same as $\mathcal{B}|_U \to (\pi_U)_* \mathcal{O}_Y$. $\square$

Combining the above Exercise and Proposition, we have shown the existence of $\mathcal{S}\mathrm{pec}\, \mathcal{B}$ in the case that Y is an open subscheme of an affine scheme.

18.1.B. EXERCISE. Show the existence of $\mathcal{S}\mathrm{pec}\, \mathcal{B}$ in general, following the philosophy of our construction of the fibered product, normalization, and so forth.

We make some quick observations. First $\mathcal{S}\mathrm{pec}\, \mathcal{B}$ can be "computed affine-locally on X". We also have an isomorphism $\phi : \mathcal{B} \to \beta_* \mathcal{O}_{\mathcal{S}\mathrm{pec}\, \mathcal{B}}$.

18.1.C. EXERCISE. Given an X-morphism

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathcal{S}\mathrm{pec}\, \mathcal{B} \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

show that α is the composition

$$\mathcal{B} \xrightarrow{\phi} \beta_* \mathcal{O}_{\mathcal{S}\mathrm{pec}\, \mathcal{B}} \longrightarrow \beta_* f_* \mathcal{O}_Y = \pi_* \mathcal{O}_Y.$$

The $\mathcal{S}\mathrm{pec}$ construction gives an important way to understand affine morphisms. Note that $\mathcal{S}\mathrm{pec}\, \mathcal{B} \to X$ is an affine morphism. The "converse" is also true:

18.1.D. EXERCISE. Show that if $f : Z \to X$ is an affine morphism, then we have a natural isomorphism $Z \cong \mathcal{S}\mathrm{pec}\, f_* \mathcal{O}_Z$ of X-schemes.

Hence we can recover any affine morphism in this way. More precisely, a morphism is affine if and only if it is of the form $\mathcal{S}\mathrm{pec}\, \mathcal{B} \to X$.

18.1.E. EXERCISE ($\mathcal{S}$pec BEHAVES WELL WITH RESPECT TO BASE CHANGE). Suppose $f : Z \to X$ is any morphism, and $\mathcal{B}$ is a quasicoherent sheaf of algebras on X. Show that there is a natural isomorphism $Z \times_X \mathcal{S}\mathrm{pec}\, \mathcal{A} \cong \mathcal{S}\mathrm{pec}\, f^* \mathcal{B}$.

18.1.4. *Definition.* An important example of this $\mathcal{S}$pec construction is the **total space of a finite rank locally free sheaf** $\mathcal{F}$, which we define to be $\mathcal{S}\mathrm{pec}\,\mathrm{Sym}^\bullet \mathcal{F}^\vee$.

18.1.F. EXERCISE. Show that the total space of $\mathcal{F}$ is a *vector bundle*, i.e. that given any point $p \in X$, there is a neighborhood $p \in U \subset X$ such that $\mathcal{S}\mathrm{pec}\,\mathrm{Sym}^\bullet \mathcal{F}^\vee|_U \cong \mathbb{A}^n_U$. Show that $\mathcal{F}$ is isomorphic to the sheaf of sections of the total space $\mathcal{S}\mathrm{pec}\,\mathrm{Sym}^\bullet \mathcal{F}^\vee$. (Possible hint: use transition functions.) For this reason, the total space is also called the **vector bundle associated to a locally free sheaf** $\mathcal{F}$. (Caution: some authors, e.g. [**Stacks**, tag 01M2], call $\mathcal{S}\mathrm{pec}\,\mathrm{Sym}^\bullet \mathcal{F}$, the *dual* of this vector bundle, the vector bundle associated to $\mathcal{F}$.)

In particular, if $\mathcal{F} = \mathcal{O}_X^{\oplus n}$, then $\mathcal{S}\mathrm{pec}\,\mathrm{Sym}^\bullet \mathcal{F}^\vee$ is called $\mathbb{A}^n_X$, generalizing our earlier notions of $\mathbb{A}^n_A$. As the notion of free sheaf behaves well with respect to base change, so does the notion of $\mathbb{A}^n_X$, i.e. given $X \to Y$, $\mathbb{A}^n_Y \times_Y X \cong \mathbb{A}^n_X$. (Aside: you may notice that the construction $\mathcal{S}\mathrm{pec}\,\mathrm{Sym}^\bullet$ can be applied to any coherent sheaf $\mathcal{F}$ (without dualizing, i.e. $\mathcal{S}\mathrm{pec}\,\mathrm{Sym}^\bullet \mathcal{F}$). This is sometimes called the *abelian cone* associated to $\mathcal{F}$. This concept can be useful, but we won't need it.)

18.1.G. EXERCISE. Suppose $f : \mathcal{S}\mathrm{pec}\,\mathcal{B} \to X$ is a morphism. Show that the category of quasicoherent sheaves on $\mathcal{S}\mathrm{pec}\,\mathcal{B}$ is equivalent to the category of quasicoherent sheaves on X with the structure of $\mathcal{B}$-modules (quasicoherent $\mathcal{B}$-modules on X).

This is useful if X is quite simple but $\mathcal{S}\mathrm{pec}\,\mathcal{B}$ is complicated. We will use this before long when $X \cong \mathbb{P}^1$, and $\mathcal{S}\mathrm{pec}\,\mathcal{B}$ is a more complicated curve.

18.1.H. EXERCISE (THE TAUTOLOGICAL BUNDLE ON $\mathbb{P}^n$ IS $\mathcal{O}(-1)$). Suppose k is a field. Define the subset $X \subset \mathbb{A}^{n+1}_k \times \mathbb{P}^n_k$ corresponding to "points of $\mathbb{A}^{n+1}_k$ on the corresponding line of $\mathbb{P}^n_k$", so that the fiber of the map $\pi : X \to \mathbb{P}^n$ corresponding to a point $l = [x_0; \cdots; x_n]$ is the line in $\mathbb{A}^{n+1}_k$ corresponding to l, i.e. the scalar multiples of $(x_0, \ldots, x_n)$. Show that $\pi : X \to \mathbb{P}^n_k$ is (the line bundle corresponding to) the invertible sheaf $\mathcal{O}(-1)$. (Possible hint: work first over the usual affine open sets of $\mathbb{P}^n_k$, and figure out transition functions.) For this reason, $\mathcal{O}(-1)$ is often called the **tautological bundle** of $\mathbb{P}^n_k$ (even over an arbitrary base, not just a field). (Side remark: The projection $X \to \mathbb{A}^{n+1}_k$ is the blow-up of $\mathbb{A}^{n+1}_k$ at the "origin", see Exercise 10.2.M.)

18.2 Relative Proj of a sheaf of graded algebras

In parallel with $\mathcal{S}$pec, we define a relative version of Proj, denoted $\mathcal{P}$roj (called "relative Proj" or "sheaf Proj"). To find the right universal property, we examine Exercise 17.4.A closely.

18.2.1. *Hypotheses on $\mathcal{S}_\bullet$.* We will apply this construction to a quasicoherent sheaf $\mathcal{S}_\bullet$ of graded algebras on X, so we first determine what hypotheses are necessary, by consulting the definition of Proj. (i) We require that $\mathcal{S}_0 = \mathcal{O}_X$. We require that $\mathcal{S}_\bullet$ locally satisfy the hypotheses of Exercise 17.4.A. Precisely, we require that (ii) $\mathcal{S}_1$ is finite type, and (iii) $\mathcal{S}_\bullet$ is "generated in degree 1". The cleanest way to make sense of the latter condition is to require the natural map

$$\operatorname{Sym}^\bullet_{\mathcal{O}_X} \mathcal{S}_1 \to \mathcal{S}_\bullet$$

to be surjective. Because we have checked that the $\operatorname{Sym}^\bullet$ construction may be computed affine locally (§14.5.3), we can check generation in degree 1 on any affine cover.

The X-scheme and line bundle $(\beta : \operatorname{Proj} \mathcal{S}_\bullet \to X, \mathcal{O}(1))$ is required to satisfy the following universal property. Given $\pi : Y \to X$, commuting diagrams

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

correspond to the choice of an invertible sheaf $\mathcal{L}$ on Y, and maps $\alpha : \mathcal{S}_\bullet \to \oplus_{n=0}^\infty \pi_* \mathcal{L}^{\otimes n}$, up to isomorphism of (L, α), except that two such α are identified if they locally agree in sufficiently high degree (given any point of X, there is a neighborhood of the point and an n_0, so that they agree for $n \geq n_0$). Further, $\mathcal{L}$ is required to be locally generated by $\alpha(\mathcal{S}_1)$: the composition $\pi^* \mathcal{S}_1 \to \pi^* \pi_* \mathcal{L} \to \mathcal{L}$ is surjective. (Perhaps more explicitly: given any $y \in Y$, there is a neighborhood of $\pi(y)$ so that the stalk of $\mathcal{L}$ at y is generated by the image of a section of $\mathcal{S}_1$ above that open set.)

As usual, if $(\beta : \operatorname{Proj} \mathcal{S}_\bullet \to X, \mathcal{O}(1))$ exists, it is unique up to unique isomorphism. We now show that it exists, in analogy with $\mathcal{S}$pec.

18.2.A. IMPORTANT EXERCISE. Show that if X is affine and $\mathcal{S}_\bullet$ satisfies the hypotheses of §18.2.1, then there exists some $(\beta, \mathcal{O}(1))$ satisfying the universal property. (Hint: Exercise 17.4.A. It should be clear to you what construction to use!) In doing this exercise, you will recognize each part of this tortured universal property as coming from the universal property for maps to $\operatorname{Proj} S_\bullet$.

18.2.B. EXERCISE. Show that if $(\beta, \mathcal{O}(1))$ exists for some X and $\mathcal{S}_\bullet$, and if $U \subset X$ is an open subset, then $(\beta, \mathcal{O}(1))$ exists for U and $\mathcal{S}_\bullet|_U$ (and may be obtained by taking the construction over X and restricting to U).

The previous two exercises imply that $\mathcal{P}\mathrm{roj}\, S_\bullet$, should it exist, can thus be "computed affine locally". We are left with the gluing problem.

18.2.C. IMPORTANT EXERCISE: $\mathcal{P}$roj EXISTS. Show that $(\beta : \mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet \to X, \mathcal{O}(1))$ exists.

18.2.D. EXERCISE. Describe a map of graded quasicoherent sheaves $\phi : \mathcal{S}_\bullet \to \oplus_n \beta_* \mathcal{O}(n)$, which is locally an isomorphism in high degrees (given any point of

X, there is a neighborhood of the point and an n_0, so that ϕ_n is an isomorphism for $n \geq n_0$), so that any α (in the universal property above) factors as

$$\mathcal{S}_\bullet \xrightarrow{\phi} \oplus \beta_* \mathcal{O}(n) \longrightarrow \oplus \beta_* f_* \mathcal{L}^{\otimes n} = \oplus \pi_* \mathcal{L}^{\otimes n}.$$

18.2.E. EXERCISE ($\mathcal{P}$roj BEHAVES WELL WITH RESPECT TO BASE CHANGE). Suppose $\mathcal{S}_\bullet$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet$ to exist. Let $f : Y \to X$ be any morphism. Give a natural isomorphism

$$(\mathcal{P}\mathrm{roj}\, f^* \mathcal{S}_\bullet, \mathcal{O}_{\mathcal{P}\mathrm{roj}\, f^* \mathcal{S}_\bullet}(1)) \cong (Y \times_X \mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet, g^* \mathcal{O}_{\mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet}(1))$$

where g is the "top" morphism in the base change diagram

$$\begin{tikzcd} Y \times_X \mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet \arrow[r, "g"] \arrow[d] & \mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet \arrow[d] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

18.2.2. *Definition.* If $\mathcal{F}$ is a finite type quasicoherent sheaf on X, then $\mathcal{P}\mathrm{roj}\, \mathrm{Sym}^\bullet \mathcal{F}$ is called its **projectivization**, and is denoted $\mathbb{P}\mathcal{F}$. Clearly this construction behaves well with respect to base change. Define $\mathbb{P}^n_X := \mathbb{P}(\mathcal{O}_X^{\oplus(n+1)})$. (Then $\mathbb{P}^n_{\mathrm{Spec}\, A}$ agrees with our earlier definition of $\mathbb{P}^n_A$.) More generally, if $\mathcal{F}$ is locally of free of rank $n+1$, then $\mathbb{P}\mathcal{F}$ is a **projective bundle** or **$\mathbb{P}^n$-bundle** over X. As a special case of this: if X is a nonsingular curve and $\mathcal{F}$ is locally free of rank 2, then $\mathbb{P}\mathcal{F}$ is called a **ruled surface** over C. If X is further isomorphic to $\mathbb{P}^1$, $\mathbb{P}\mathcal{F}$ is called a **Hirzebruch surface**. Grothendieck proved that all vector bundles on $\mathbb{P}^1$ split as a direct sum of line bundles (which are all of the form $\mathcal{O}(n)$, so each Hirzebruch surface is of the form $\mathbb{P}(\mathcal{O}(n_1) \oplus \mathcal{O}(n_2))$. It will follow from Exercise 18.2.G below that this depends only on $n_2 - n_1$. The Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ ($n \geq 0$) is often denoted $\mathbb{F}_n$. We will discuss the Hirzebruch surfaces in greater length in §22.2.4.

18.2.F. EXERCISE. Given the data of $(\mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet, \mathcal{O}(1))$, describe a canonical closed immersion

$$\begin{tikzcd} \mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet \arrow[rr, "i"] \arrow[dr] & & \mathbb{P}\mathcal{S}_1 \arrow[dl] \\ & X & \end{tikzcd}$$

and an isomorphism $\mathcal{O}_{\mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet}(1) \cong i^* \mathcal{O}_{\mathbb{P}\mathcal{S}_1}(1)$ arising from the surjection $\mathrm{Sym}^\bullet \mathcal{S}_1 \to \mathcal{S}_\bullet$. The importance of this exercise lies in the fact that we cannot recover $\mathcal{S}_\bullet$ from the data of $(\mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet, \mathcal{O}(1))$, but the canonical closed immersion into $\mathbb{P}\beta_* \mathcal{O}(1)$ can be recovered.

18.2.G. EXERCISE. Suppose $\mathcal{L}$ is an invertible sheaf on X, and $\mathcal{S}_\bullet$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet$ to exist. Define $\mathcal{S}'_\bullet = \oplus_{n=0} (\mathcal{S}_n \otimes \mathcal{L}^{\otimes n})$. Then $\mathcal{S}'_\bullet$ has a natural algebra structure inherited from $\mathcal{S}_\bullet$; describe it. Give a natural isomorphism of X-schemes

$$(\mathcal{P}\mathrm{roj}\, \mathcal{S}'_\bullet, \mathcal{O}_{\mathcal{P}\mathrm{roj}\, \mathcal{S}'_\bullet}(1)) \cong (\mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet, \mathcal{O}_{\mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet}(1) \otimes \pi^* \mathcal{L}),$$

where $\pi : \mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet \to X$ is the structure morphism. In other words, informally speaking, the $\mathcal{P}\mathrm{roj}$ is the same, but the $\mathcal{O}(1)$ is twisted by $\mathcal{L}$. In particular, if $\mathcal{V}$ is a finite rank locally free sheaf on X, then you will have described a canonical isomorphism $\mathbb{P}\mathcal{V} \cong \mathbb{P}(\mathcal{L} \otimes \mathcal{V})$.

18.2.H. $\star$ EXERCISE (CF. EXERCISE 9.2.Q). Show that $\mathcal{P}\mathrm{roj}(\mathcal{S}_\bullet[t]) \cong \mathcal{S}\mathrm{pec}\, \mathcal{S}_\bullet \coprod \mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet$, where $\mathcal{S}\mathrm{pec}\, \mathcal{S}_\bullet$ is an open subscheme, and $\mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet$ is a closed subscheme. Show that $\mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet$ is an effective Cartier divisor, corresponding to the invertible sheaf $\mathcal{O}_{\mathcal{P}\mathrm{roj}\, S_\bullet}(1)$. (This is the generalization of the projective and affine cone.)

18.3 Projective morphisms

In §18.1, we reinterpreted affine morphisms: $X \to Y$ is an affine morphism if there is an isomorphism $X \cong \mathcal{S}\mathrm{pec}\, \mathcal{B}$ of Y-schemes for some quasicoherent sheaf of algebras $\mathcal{B}$ on Y. We will *define* the notion of a projective morphism similarly.

You might think because projectivity is such a classical notion, there should be some obvious definition, that is reasonably behaved. But this is not the case, and there are many possible variant definitions of projective (see [**Stacks**, tag 01W8]). All are imperfect, including the accepted definition we give here. Although projective morphisms are preserved by base change, we will manage to show that they are preserved by composition only when the target is quasicompact (Exercise 18.3.E), and we will manage to show that the notion is local on the base only when we add the data of a line bundle, and even then only under locally Noetherian hypotheses (§18.3.7).

18.3.1. *Definition.* A morphism $X \to Y$ is **projective** if there is an isomorphism

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr] & & \mathcal{P}\mathrm{roj}\, \mathcal{S}_\bullet \arrow[dl] \\ & Y & \end{tikzcd}$$

for a quasicoherent sheaf of algebras $\mathcal{S}_\bullet$ on Y (satisfying the hypotheses of §18.2.1: $\mathcal{S}_\bullet$ is generated in degree 1, and $\mathcal{S}_1$ is finite type). We say X is a **projective Y-scheme**, or **projective over** Y. This generalizes the notion of a projective A-scheme.

18.3.2. *Warnings.* First, notice that $\mathcal{O}(1)$, an important part of the definition of $\mathcal{P}\mathrm{roj}$, is not mentioned. (I would prefer that it be part of the definition, but this isn't accepted practice.) As a result, the notion of affine morphism is affine-local on the target, but the notion of projectivity or a morphism is not clearly affine-local on the target. (In Noetherian circumstances, with the additional data of the invertible sheaf $\mathcal{O}(1)$, it is, as we will see in §18.3.7. We will also later see an example showing that the property of being a projective is *not* local, §25.6.9.)

Second, [**Ha**, p. 103] gives a different definition of projective morphism; we follow the more general definition of Grothendieck. These definitions turn out to be the same in nice circumstances. (An example: finite morphisms are not always projective in the sense of [**Ha**].)

18.3.A. EXERCISE (USEFUL CHARACTERIZATION OF PROJECTIVE MORPHISMS). Suppose $\mathcal{L}$ is an invertible sheaf on X, and $f : X \to Y$ is a morphism. Show that f is projective, with $\mathcal{O}(1) \cong \mathcal{L}$, if and only if there exist a finite type quasicoherent sheaf $\mathcal{S}_1$ on Y, a closed immersion $i : X \hookrightarrow \mathbb{P}\mathcal{S}_1$ (over Y, i.e. commuting with the maps to Y), and an isomorphism $i^*\mathcal{O}_{\mathbb{P}\mathcal{S}_1}(1) \cong \mathcal{L}$. Hint: Exercise 18.2.F.

18.3.3. *Definition: Quasiprojective morphisms.* In analogy with projective and quasiprojective A-schemes (§5.5.5), one may define quasiprojective morphisms. *If* Y *is quasicompact*, we say that $\pi : X \to Y$ is **quasiprojective** if π can be expressed as a quasicompact open immersion into a scheme projective over Y. (The general definition of quasiprojective is slightly delicate — see [**EGA**, II.5.3] — but we won't need it.) This isn't a great notion, as for example it isn't clear to me that it is local on the base.

18.3.4. *First properties of projective morphisms.* We start to establish a number of properties of projective morphisms. First, the property of a morphism being projective is clearly preserved by base change, as the $\mathcal{P}$roj construction behaves well with respect to base change (Exercise 18.2.E). Also, projective morphisms are proper: properness is local on the target (Theorem 11.3.4(b)), and we saw earlier that projective A-schemes are proper over A (Theorem 11.3.5). In particular (by definition of properness), projective morphisms are separated, finite type, and universally closed.

18.3.5. $\star$ **Global generation and (very) ampleness in the relative setting.**

Before establishing more properties of projective morphisms, we extend the discussion of §16.3 to the relative setting, in order to give ourselves the language of relatively base-point-freeness. With the exception of Exercise 18.3.B, we won't use this discussion later, but it comes up repeatedly in the research literature.

Suppose $\pi : X \to Y$ is a quasicompact quasiseparated morphism. In $\mathcal{F}$ is a quasicoherent sheaf on X, we say that $\mathcal{F}$ is **relatively globally generated** or **globally generated with respect to** π if the natural map of quasicoherent sheaves $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is surjective. (Quasicompactness and quasiseparatedness are needed ensure that $\pi_*\mathcal{F}$ is a quasicoherent sheaf, Exercise 14.3.I). But these hypotheses are not very restrictive. Global generation is most useful only in the quasicompact setting, and most people won't be bothered by quasiseparated hypotheses. Unimportant aside: these hypotheses can be relaxed considerably. If $\pi : X \to Y$ is a morphism of *locally ringed spaces* — not necessarily schemes — with no other hypotheses, and $\mathcal{F}$ is a quasicoherent sheaf on X, then we say that $\mathcal{F}$ is **relatively globally generated** or **globally generated with respect to** π if the natural map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ of $\mathcal{O}_X$-modules is surjective.)

Thanks to our hypotheses, as the natural map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is of quasicoherent sheaves, the condition of being relatively globally generated is affine-local on Y.

Suppose now that $\mathcal{L}$ is a locally free sheaf on X, and $\pi : X \to Y$ is a morphism. We say that $\mathcal{L}$ is **relatively base-point-free** or **base-point-free with respect to** π if it is relatively globally generated.

18.3.B. EXERCISE. Suppose $\mathcal{L}$ is a finite rank locally free sheaf on $\mathcal{L}$, $\pi : X \to Y$ is a quasicompact separated morphism, and $\pi_*\mathcal{L}$ is finite type on Y. (We will later show in Theorem 20.8.1 that this latter statement is true if π is proper and Y is

Noetherian. This is much easier if π is projective, see Theorem 20.7.1. We could work hard and prove it now, but it isn't worth the trouble.) Describe a canonical morphism $f : X \to \mathbb{P}\mathcal{L}$. (Possible hint: this generalizes the fact that base-point-free line bundles give maps to projective space, so generalize that argument, see §16.3.5.)

We say that $\mathcal{L}$ is **relatively ample** or π**-ample** or **relatively ample with respect to** π if for every affine open subset $\operatorname{Spec} B$ of Y, $\mathcal{L}|_{\pi^{-1}(\operatorname{Spec} B)}$ is ample on $\pi^{-1}(\operatorname{Spec} B)$ over B, or equivalently (by §16.3.14). $\mathcal{L}|_{\pi^{-1}(\operatorname{Spec} B)}$ is (absolutely) ample on $\pi^{-1}(\operatorname{Spec} B)$. By the discussion in §16.3.14, if $\mathcal{L}$ is ample then π is necessarily quasicompact, and (by Theorem 16.3.15) separated; if π is affine, then all invertible sheaves are ample; and if π is projective, then the corresponding $\mathcal{O}(1)$ is ample. By Exercise 16.3.N, $\mathcal{L}$ is π-ample if and only if $\mathcal{L}^{\otimes n}$ is π-ample, and if $Z \hookrightarrow X$ is a closed immersion, then $\mathcal{L}|_Z$ is ample over Y.

From Theorem 16.3.15(d) implies that we have a natural open immersion $X \to \mathcal{P}\mathrm{roj}_Y \oplus f_* \mathcal{L}^{\otimes d}$. (Do you see what this map is? Also, be careful: $\oplus f_* \mathcal{L}^{\otimes d}$ need not be a finitely generated graded sheaf of algebras, so we are using the $\mathcal{P}\mathrm{roj}$ construction where one of the usual hypotheses doesn't hold.)

The notions of relative global generation and relative ampleness are most useful in the proper setting, because of Theorem 16.3.11. Suppose $\pi : X \to Y$ is proper. If $\mathcal{L}$ is an invertible sheaf on X, then we say that $\mathcal{L}$ is **very ample (with respect to** π**)**, or (awkwardly) π**-very ample** if we can write $X = \mathcal{P}\mathrm{roj}_Y \mathcal{S}_\bullet$ where $\mathcal{S}_\bullet$ is a quasicoherent sheaf of algebras on Y satisfying the hypotheses of §18.2.1: $\mathcal{S}_1$ is finite type, and $\operatorname{Sym}^\bullet \mathcal{S}_1 \to \mathcal{S}_\bullet$ is surjective ($\mathcal{S}_\bullet$ is "generated in degree 1"). (The notion of very ampleness can be extended to more general situations, see for example [**Stacks**, tag 01VM]. But this is of interest only to people with particularly refined tastes.)

Many statements of §16.3 carry over without change. For example, we have the following. Suppose $\pi : X \to Y$ is proper, $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves on X, and $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on X. If π is affine, then $\mathcal{F}$ is relatively globally generated (from Easy Exercise 16.3.A). If $\mathcal{F}$ and $\mathcal{G}$ are relatively globally generated, so is $\mathcal{F} \otimes \mathcal{G}$ (Easy Exercise 16.3.B). If $\mathcal{L}$ is π-very ample, then it is π-base-point-free (Easy Exercise 16.3.G). If $\mathcal{L}$ is π-very ample, and $\mathcal{M}$ is π-base-point-free (if for example it is π-very ample), then $\mathcal{L} \otimes \mathcal{M}$ is π-very ample (Exercise 16.3.H).

By the nature of the statements, some of the statements of §16.3 require quasicompactness hypotheses on Y, or other patches. For example:

18.3.6. Theorem. — *Suppose $\pi : X \to Y$ is proper, $\mathcal{L}$ is an invertible sheaf on X, and Y is quasicompact. The following are equivalent.*

(a) *For some $N > 0$, $\mathcal{L}^{\otimes N}$ is π-very ample.*
(a') *For all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is π-very ample.*
(b) *For all finite type quasicoherent sheaves $\mathcal{F}$, there is an n_0 such that for $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is relatively globally generated.*
(c) *The invertible sheaf $\mathcal{L}$ is π-ample.*

18.3.C. EXERCISE. Prove Theorem 18.3.6 using Theorem 16.3.11. (Unimportant remark: The proof of Theorem 16.3.11 used Noetherian hypotheses, but as stated there, they can be removed.)

After doing the above Exercise, it will be clear how to adjust the statement of Theorem 18.3.6 if you need to remove the quasicompact assumption on Y.

18.3.D. EXERCISE (A USEFUL EQUIVALENT DEFINITION OF VERY AMPLENESS UNDER NOETHERIAN HYPOTHESES). Suppose $\pi : X \to Y$ is a proper morphism, Y is locally Noetherian (hence X is too, as f is finite type), and $\mathcal{L}$ is an invertible sheaf on X. Suppose that you know that in this situation $\pi_* \mathcal{L}$ is finite type. (We will later show this, as described in Exercise 18.3.B.) Show that $\mathcal{L}$ is very ample if and only if (i) $\mathcal{L}$ is relatively base-point-free, and (ii) the canonical Y-morphism $i : X \to \mathbb{P}\pi_* \mathcal{L}$ of Exercise 18.3.B is a closed immersion. Conclude that the notion of very ampleness is affine-local on Y (it may be checked on *any* affine cover Y), if Y is locally Noetherian and π is proper.

As a consequence, Theorem 18.3.6 implies the notion of ampleness is affine-local on Y (if π is proper and Y is locally Noetherian).

18.3.7. Properties of projective morphisms. We now give more properties of projective morphisms.

Exercise 18.3.D implies if $\pi : X \to Y$ is a proper morphism of locally Noetherian schemes, and $\mathcal{L}$ is an invertible sheaf on X, the question of whether π is a projective morphism with $\mathcal{L}$ as $\mathcal{O}(1)$ is local on Y.

18.3.E. EXERCISE (THE COMPOSITION OF PROJECTIVE MORPHISMS IS PROJECTIVE, IF THE FINAL TARGET IS QUASICOMPACT). Suppose $\pi : X \to Y$ and $\rho : Y \to Z$ are projective morphisms, and Z is quasicompact. Show that $\pi \circ \rho$ is projective. Hint: the criterion for projectivity given in Exercise 18.3.A will be useful. (i) Deal first with the case where Z is affine. Build the following commutative diagram, thereby finding a closed immersion $X \hookrightarrow \mathbb{P}\mathcal{F}^{\oplus n}$ over Z. In this diagram, all inclusions are closed immersions, and all script fonts refer to finite type quasicoherent sheaves.

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[drr, "\pi"'] & \mathbb{P}\mathcal{E} \arrow[r, hook, "(\dagger)"] \arrow[dr] & \mathbb{P}^{n-1}_Z \times_Z Y \arrow[r, hook] \arrow[d] & \mathbb{P}^{n-1}_Z \times_Z \mathbb{P}\mathcal{F} \arrow[r, hook, "\text{Segre}", "\text{cf. Ex. 10.5.D}"'] \arrow[d] & \mathbb{P}\left(\mathcal{F}^{\oplus n}\right) \arrow[ddll, bend left] \\
& & Y \arrow[r, hook] \arrow[d, "\rho"] & \mathbb{P}\mathcal{F} \arrow[dl] & \\
& & Z & &
\end{tikzcd}$$

Construct the closed immersion $(\dagger)$ as follows. Suppose $\mathcal{M}$ is the very ample line bundle on Y over Z. Then $\mathcal{M}$ is ample, and so by Theorem 16.3.11, for $m \gg 0$, $\mathcal{E} \otimes \mathcal{M}^{\otimes m}$ is generated by a finite number of global sections. Suppose $\mathcal{O}_Y^{\oplus n} \twoheadrightarrow \mathcal{E} \otimes \mathcal{M}^{\otimes m}$ is the corresponding surjection. This induces a closed immersion $\mathbb{P}(\mathcal{E} \otimes \mathcal{M}^{\otimes m}) \hookrightarrow \mathbb{P}^{n-1}_Y$. But $\mathbb{P}(\mathcal{E} \otimes \mathcal{M}^{\otimes m}) \cong \mathbb{P}\mathcal{E}$ (Exercise 18.2.G), and $\mathbb{P}^{n-1}_Y = \mathbb{P}^{n-1}_Z \times_Z Y$. (ii) Unwind this diagram to show that (for Z affine) if $m \gg 0$, if $\mathcal{L}$ is π-very ample and $\mathcal{M}$ is ρ-very ample, then for $m \gg 0$, $\mathcal{L} \otimes \mathcal{M}^{\otimes m}$ is $(\rho \circ \pi)$-very ample. Then deal with the general case by covering Z with a finite number of affines.

18.3.8. *Caution: Consequences of projectivity not being "reasonable" in the sense of §8.0.1.* Because the property of being projective is preserved by base change

(§18.3.4), and composition *to quasicompact targets* (Exercise 18.3.E), the property of being projective is "usually" preserved by products (Exercise 10.4.F): if $f : X \to Y$ and $f' : X' \to Y$ are projective, then so is $f \times f' : X \times X' \to Y \times Y'$, so long as $Y \times Y'$ is quasicompact. Also, if you follow through the proof of the Cancellation Theorem 11.1.19 for properties of morphisms, you will see that if $f : X \to Y$ is a morphisms, $g : Y \to Z$ is separated (so the diagonal δ_g is a closed immersion and hence projective), and $g \circ f$ is projective, *and* Y *is quasicompact*, then f is projective.

18.3.F. EXERCISE. Show that a morphism (over $\operatorname{Spec} k$) from a projective k-scheme to a separated k-scheme is always projective. (Hint: the Cancellation Theorem 11.1.19 for projective morphisms, cf. Caution 18.3.8.)

18.3.9. Finite morphisms are projective, and consequences.

18.3.G. IMPORTANT EXERCISE: FINITE MORPHISMS ARE PROJECTIVE (CF. EXERCISE 8.3.J). Show that finite morphisms are projective as follows. Suppose $Y \to X$ is finite, and that $Y = \mathcal{S}\mathrm{pec}\,\mathcal{B}$ where $\mathcal{B}$ is a finite type quasicoherent sheaf on X. Describe a sheaf of graded algebras $\mathcal{S}_\bullet$ where $\mathcal{S}_0 \cong \mathcal{O}_X$ and $\mathcal{S}_n \cong \mathcal{B}$ for $n > 0$. Describe an X-isomorphism $Y \cong \mathcal{P}\mathrm{roj}\,\mathcal{S}_\bullet$.

In particular, closed immersions are projective. We have the sequence of implications for morphisms

$$\text{closed immersion} \Rightarrow \text{finite} \Rightarrow \text{projective} \Rightarrow \text{proper}.$$

We know that finite morphisms are projective (Exercise 18.3.G), and have finite fibers (Exercise 8.3.K). Here is the converse.

18.3.10. Theorem (projective + finite fibers = finite). — *Suppose $\pi : X \to Y$ with Y Noetherian. Then π is projective and finite fibers if and only if it is finite. Equivalently, π is projective and quasifinite if and only it is finite.*

(Recall that quasifinite = finite fibers + finite type. But projective includes finite type.) It is true more generally that (with Noetherian hypotheses) proper + finite fibers = finite, [**EGA**, III.4.4.2].

Proof. We show π is finite near a point $y \in Y$. Fix an affine open neighborhood $\operatorname{Spec} A$ of y in Y. Pick a hypersurface H in $\mathbb{P}^n_A$ missing the preimage of y, so $H \cap X$ is closed. Let $H' = \pi_*(H \cap X)$, which is closed, and doesn't contain y. Let $U = \operatorname{Spec} A - H'$, which is an open set containing y. Then above U, π is projective and affine, so we are done by Corollary 20.1.7. $\square$

18.3.H. IMPORTANT EXERCISE. Use a similar argument to prove *semicontinuity of fiber dimension of projective morphisms*: suppose $\pi : X \to Y$ is a projective morphism where Y is locally Noetherian (or more generally $\mathcal{O}_Y$ is coherent over itself). Show that $\{y \in Y : \dim f^{-1}(y) > k\}$ is a Zariski-closed subset. In other words, the dimension of the fiber "jumps over Zariski-closed subsets". (You can interpret the case $k = -1$ as the fact that projective morphisms are closed.) This exercise is rather important for having a sense of how projective morphisms behave.

18.3.11. *Ample vector bundles.* The notion of an **ample vector bundle** is useful in some parts of the literature, so we define it, although we won't use the notion. A locally free sheaf $\mathcal{E}$ on a scheme X is **ample** if $\mathcal{O}(1)$ on its projectivization $\mathbb{P}\mathcal{E} \to X$ is ample over X.

18.3.12. $\star\star$ **Quasiaffine morphisms.**

Because we have introduced quasiprojective morphisms (Definition 18.3.3), we briefly introduce quasiaffine morphisms (and quasiaffine schemes), as some readers may have cause to use them. Many of these ideas could have been introduced long before, but because we will never use them, we deal with them all at once.

A scheme X is **quasiaffine** if it admits a quasicompact open immersion into an affine scheme. This implies that X is quasicompact and separated. Note that if X is Noetherian (the most relevant case for most people), then any open immersion is of course automatically quasicompact.

18.3.I. EXERCISE. Show that X is quasiaffine if and only if the canonical map $X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ (defined in Exercise 7.3.F and the paragraph following it) is a quasicompact open immersion. Thus a quasiaffine scheme comes with a *canonical* quasicompact open immersion into an affine scheme. Hint: Let $A = \Gamma(X, \mathcal{O}_X)$ for convenience. Suppose $X \to \operatorname{Spec} R$ is a quasicompact open immersion. We wish to show that $X \to \operatorname{Spec} A$ is a quasicompact open immersion. Factor $X \to \operatorname{Spec} R$ through $X \to \operatorname{Spec} A \to \operatorname{Spec} R$. Show that $X \to \operatorname{Spec} A$ is an open immersion in a neighborhood of any chosen point $x \in X$, as follows. Choose $r \in R$ such that $x \subset D(r) \subset X$. Notice that if $X_r = \{y \in X : r(y) \neq 0\}$, then $\Gamma(X_r, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)_r$ by Exercise 14.3.H, using the fact that X is quasicompact and quasiseparated. Use this to show that the map $X_r \to \operatorname{Spec} A_r$ is an isomorphism.

It is not hard to show that X is quasiaffine if and only if $\mathcal{O}_X$ is ample, but we won't use this fact.

A morphism $\pi : X \to Y$ is **quasiaffine** if the inverse image of every affine open subset of Y is a quasiaffine scheme. By Exercise 18.3.I, this is equivalent to π being quasicompact and separated, and the natural map $X \to \mathcal{S}\mathrm{pec}\, \pi_* \mathcal{O}_X$ being a quasicompact open immersion. This implies that the notion of quasiaffineness is local on the target (may be checked on an open cover), and also affine-local on a target (one may choose an affine cover, and check that the preimages of these open sets are quasiaffine). Quasiaffine morphisms are preserved by base change: if a morphism $X \hookrightarrow Z$ over Y is a quasicompact open immersion into an affine Y-scheme, then for any $W \to Y$, $X \times_Y W \hookrightarrow Z \times_Y W$ is a quasicompact open immersion into an affine W-scheme. (Interestingly, Exercise 18.3.I is *not* the right tool to use to show this base change property.)

One may readily check that quasiaffine morphisms are preserved by composition [**Stacks**, tag 01SN]. Thus quasicompact locally closed immersions are quasiaffine. If X is affine, then $X \to Y$ is quasiaffine if and only if it is quasicompact (as the preimage of any affine open subset of Y is an open subset of an affine scheme, namely X). In particular, from the Cancellation Theorem 11.1.19 for quasicompact morphisms, any morphism from an affine scheme to a quasiseparated scheme is quasiaffine.

18.4 Applications to curves

We now apply what we have learned to curves.

18.4.1. Theorem. — *Every integral curve* C *over a field* k *has a birational model that is a nonsingular projective curve.*

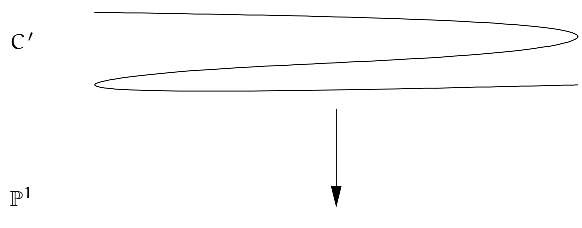


FIGURE 18.1. Constructing a projective nonsingular model of a curve C over k via a finite cover of $\mathbb{P}^1$

Proof. We can assume C is affine. By the Noether Normalization Lemma 12.2.7, we can find some $x \in K(C) \setminus k$ with $K(C)/k(x)$ a finite field extension. By identifying a standard open of $\mathbb{P}^1_k$ with $\operatorname{Spec} k[x]$, and taking the normalization of $\mathbb{P}^1$ in the function field of $K(C)$ (Definition 10.6.I), we obtain a finite morphisms $C' \to \mathbb{P}^1$, where C' is a curve ($\dim C' = \dim \mathbb{P}^1$ by Exercise 12.1.C), and nonsingular (it is reduced hence nonsingular at the generic point, and nonsingular at the closed points by the main theorem on discrete valuation rings in §13.4). Also, C' is birational to C as they have isomorphic function fields (Exercise 7.5.E).

Finally, $C' \to \mathbb{P}^1_k$ is finite (Exercise 10.6.L) hence projective (Exercise 18.3.G), and $\mathbb{P}^1_k \to \operatorname{Spec} k$ is projective, so as composition of projective morphisms (to a quasicompact target) are projective (Exercise 18.3.E), $C' \to k$ is projective. $\square$

18.4.2. Theorem. — *If* C *is an irreducible nonsingular curve over a field* k*, then there is an open immersion* $C \hookrightarrow C'$ *into some projective nonsingular curve* C' *(over* k*).*

Proof. We first prove the result in the case where C is affine. Then we have a closed immersion $C \hookrightarrow \mathbb{A}^n$, and we consider $\mathbb{A}^n$ as a standard open set of $\mathbb{P}^n$. Taking the scheme-theoretic closure of C in $\mathbb{P}^n$, we obtain a projective integral curve $\overline{C}$, containing C as an open subset. The normalization $\tilde{\overline{C}}$ of $\overline{C}$ is a finite morphism (finiteness of integral closure, Theorem 10.6.3(b)), so $\tilde{\overline{C}}$ is Noetherian, and nonsingular (as normal Noetherian dimension 1 rings are discrete valuation rings, §13.4). Moreover, by the universal property of normalization, normalization of $\overline{C}$ doesn't affect the normal open set C, so we have an open subset C, so we have an open immersion $C \hookrightarrow \tilde{\overline{C}}$. Finally, $\tilde{\overline{C}} \to \overline{C}$ is finite hence projective, and $\overline{C} \to \operatorname{Spec} k$ is projective, so (by Exercise 18.3.E) $\tilde{\overline{C}}$ is projective.

We next consider the case of general C. Let C_1 by any nonempty affine open subset of C. By the discussion in the previous paragraph, we have a nonsingular projective compactification $\overline{\widetilde{C}_1}$. The Curve-to-projective Extension Theorem 17.5.1 (applied successively to the finite number of points $C \setminus \{C_1\}$) implies that the morphism $C_1 \hookrightarrow \overline{\widetilde{C}_1}$ extends to a birational morphism $C \to \overline{\widetilde{C}_1}$. Because points of a nonsingular curve are determined by their valuation (Exercise 13.5.B, this is an inclusion of sets. Because the topology on curves is stupid (cofinite), it expresses C as an open subset of $\overline{\widetilde{C}}$. But why is it an open immersion of schemes?

We show it is an open immersion near a point $p \in C$ as follows. Let C_2 be an affine neighborhood of p in C. We repeat the construction we used on C_1, to obtain the following diagram, with open immersions marked.

$$\begin{tikzcd}
C_1 \arrow[dr, hook] \arrow[dd, hook] & & C_2 \arrow[dl, hook'] \arrow[dd, hook] \\
& C \arrow[dl] \arrow[dr] & \\
\overline{\widetilde{C}_1} & & \overline{\widetilde{C}_2}
\end{tikzcd}$$

By the Curve-to-projective Extension theorem 17.5.1, the map $C_1 \to \overline{\widetilde{C}_2}$ extends to $\pi_{12} : \overline{\widetilde{C}_1} \to \overline{\widetilde{C}_2}$, and we similarly have a morphism $\pi_{21} : \overline{\widetilde{C}_2} \to \overline{\widetilde{C}_1}$, extending $C_2 \to \overline{\widetilde{C}_1}$. The composition $\pi_{21} \circ \pi_{12}$ is the identity morphism (as it is the identity rational map, see Theorem 11.2.1). The same is true for $\pi_{12} \circ \pi_{21}$, so π_{12} and π_{21} are isomorphisms. The enhanced diagram

$$\begin{tikzcd}
C_1 \arrow[dr, hook] \arrow[dd, hook] & & C_2 \arrow[dl, hook'] \arrow[dd, hook] \\
& C \arrow[dl] \arrow[dr] & \\
\overline{\widetilde{C}_1} \arrow[rr, leftrightarrow] & & \overline{\widetilde{C}_2}
\end{tikzcd}$$

commutes (by Theorem 11.2.1 again, implying that morphisms of reduced separated schemes are determined by their behavior on dense open sets). But $C_2 \to \overline{\widetilde{C}_1}$ is an open immersion (in particular, at p), so $C \to \overline{\widetilde{C}_1}$ is an open immersion there as well. $\square$

18.4.A. EXERCISE. Show that all nonsingular proper curves over k are projective.

18.4.3. Theorem (various categories of curves are the same). — *The following categories are equivalent.*

(i) *irreducible nonsingular projective curves over* k*, and surjective* k*-morphisms.*
(ii) *irreducible nonsingular projective curves over* k*, and dominant* k*-morphisms.*
(iii) *irreducible nonsingular projective curves over* k*, and dominant rational maps over* k*.*
(iv) *irreducible reduced curves over* k*, and dominant rational maps over* k*.*

(v) *the opposite category of finitely generated fields of transcendence degree* 1 *over* k*, and* k*-homomorphisms.*

All morphisms and maps in the following discussion are assumed to be defined over k.

This Theorem has a lot of implications. For example, each quasiprojective reduced curve is birational to precisely one projective nonsingular curve. Also, thanks to §7.5.9, we know for the first time that there exist finitely generated transcendence degree 1 extensions of $\mathbb{C}$ that are not generated by a single element. We even have an example, related to Fermat's Last Theorem, from Exercise 7.5.K: the extension generated over $\mathbb{C}$ by three variables x, y, and z satisfying $x^n + y^n = z^n$, where $n > 2$.

(Aside: The interested reader can tweak the proof below to show the following variation of the theorem: in (i)–(iv), consider only geometrically irreducible curves, and in (v), consider only fields K such that $\overline{k} \cap K = k$ in $\overline{K}$. This variation allows us to exclude "weird" curves we may not want to consider. For example, if $k = \mathbb{R}$, then we are allowing curves such as $\mathbb{P}^1_{\mathbb{C}}$ which are not geometrically irreducible, as $\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C} \cong \mathbb{P}^1_{\mathbb{C}} \coprod \mathbb{P}^1_{\mathbb{C}}$.)

Proof. Any surjective morphism is a dominant morphism, and any dominant morphism is a dominant rational map, and each nonsingular projective curve is a quasiprojective curve, so we have shown (i) $\to$ (ii) $\to$ (iii) $\to$ (iv). To get from (iv) to (i), suppose we have a dominant rational map $C_1 \dashrightarrow C_2$ of irreducible reduced curves. Replace C_1 by a dense open set so the rational map is a morphism $C_1 \to C_2$. This induces a map of normalizations $\tilde{C_1} \to \tilde{C_2}$ of nonsingular irreducible curves. Let $\overline{\tilde{C_i}}$ be a nonsingular projective compactification of $\tilde{C_i}$ (for $i = 1, 2$), as in Theorem 18.4.2. Then the morphism $\tilde{C_1} \to \overline{\tilde{C_2}}$ extends to a morphism $\overline{\tilde{C_1}} \to \overline{\tilde{C_2}}$ by the Curve-to-Projective Extension Theorem 17.5.1, producing a morphism in category (i).

18.4.B. EXERCISE. Put the above pieces together to describe equivalences of categories (i) through (iv).

It remains to connect (v). This is essentially the content of Exercise 7.5.E; details are left to the reader. □

18.4.4. Degree of a projective morphism from a curve to a nonsingular curve.

You might already have a reasonable sense that a map of compact Riemann surfaces has a well-behaved degree, that the number of preimages of a point of C' is constant, so long as the preimages are counted with appropriate multiplicity. For example, if f locally looks like $z \mapsto z^m = y$, then near $y = 0$ and $z = 0$ (but not *at* $z = 0$), each point has precisely m preimages, but as y goes to 0, the m preimages coalesce. Enlightening Example 10.3.3 showed this phenomenon in a more complicated context.

We now show the algebraic version of this fact. Suppose $f : C \to C'$ is a surjective (or equivalently, dominant) map of nonsingular projective curves. We will show that f has a well-behaved degree, in a sense that we will now make precise.

Then f is finite, as f is a projective morphism with finite fibers (Theorem 18.3.10). Alternatively, we can see the finiteness of f as follows. Let C'' be the normalization of C' in the function field of C. Then we have an isomorphism $K(C) \cong K(C'')$ which leads to birational maps $C \dashleftarrow\dashrightarrow C''$ which extend to morphisms as both C and C'' are nonsingular and projective (by the Curve-to-projective Extension Theorem 17.5.1). Thus this yields an isomorphism of C and C''. But $C'' \to C$ is a finite morphism by the finiteness of integral closure (Theorem 10.6.3).

18.4.5. Proposition. — *Suppose that $\pi : C \to C'$ is a finite morphism, where C is a (pure dimension 1) curve, and C' is a nonsingular curve. Then $\pi_* \mathcal{O}_C$ is locally free of finite rank.*

The nonsingularity hypothesis on C' is necessary: the normalization of a nodal curve (Figure 8.4) is an example where most points have one preimage, and one point (the node) has two.

18.4.6. *Definition.* If C' is irreducible, the rank of this locally free sheaf is the **degree** of π.

18.4.C. EXERCISE. Recall that the degree of a rational map from one irreducible curve to another is defined as the degree of the function field extension (Definition 7.5.6). Show that (with the notation of Proposition 18.4.5) if C and C' are irreducible, the degree of π as a rational map is the same as the rank of $\pi_* \mathcal{O}_C$.

18.4.7. *Remark for those with complex-analytic background (algebraic degree = analytic degree).* If $C \to C'$ is a finite map of nonsingular complex algebraic curves, Proposition 18.4.5 establishes that algebraic degree as defined above is the same as analytic degree (counting preimages, with multiplicity).

18.4.D. EXERCISE. We use the notation of Proposition 18.4.5. Suppose p is a point of C'. The scheme-theoretic preimage $\pi^* p$ of p is a dimension 0 scheme over k.

(a) Suppose C' is finite type over a field k, and n is the dimension of the structure sheaf of $\pi^* p$ as k-vector space. Show that $n = (\deg \pi)(\deg p)$. (The degree of a point was defined in §6.3.8.)
(b) Suppose that C is nonsingular, and $\pi^{-1} p = \{p_1, \dots, p_m\}$. Suppose t is a uniformizer of the discrete valuation ring $\mathcal{O}_{C', p}$. Show that

$$\deg \pi = \sum_{i=1}^{m} (\mathrm{val}_{p_i} \pi^* t) \deg(\kappa(p_i)/\kappa(p)),$$

where $\deg(\kappa(p_i)/\kappa(p))$ denotes the degree of the field extension of the residue fields.

(Can you extend (a) to remove the hypotheses of working over a field? If you are a number theorist, can you recognize (b) in terms of splitting primes in extensions of rings of integers in number fields?)

18.4.E. EXERCISE. Suppose that C is an irreducible nonsingular curve, and s is a nonzero rational function on C. Show that the number of zeros of s (counted with appropriate multiplicity) equals the number of poles. Hint: recognize this as the degree of a morphism $s : C \to \mathbb{P}^1$. (In the complex category, this is an important consequence of the Residue Theorem. Another approach is given in Exercise 20.4.D.)

18.4.8. *Revisiting Example 10.3.3.* Proposition 18.4.5 and Exercise 18.4.D make precise what general behavior we observed in Example 10.3.3. Suppose C' is irreducible, and that d is the rank of this allegedly locally free sheaf. Then the fiber over any point of C with residue field K is the Spec of an algebra of dimension d over K. This means that the number of points in the fiber, counted with appropriate multiplicity, is always d.

As a motivating example, we revisit Example 10.3.3, the map $\mathbb{Q}[y] \to \mathbb{Q}[x]$ given by $x \mapsto y^2$, the projection of the parabola $x = y^2$ to the x-axis. We observed the following.

(i) The fiber over $x = 1$ is $\mathbb{Q}[y]/(y^2 - 1)$, so we get 2 points.
(ii) The fiber over $x = 0$ is $\mathbb{Q}[y]/(y^2)$ — we get one point, with multiplicity 2, arising because of the nonreducedness.
(iii) The fiber over $x = -1$ is $\mathbb{Q}[y]/(y^2 + 1) \cong \mathbb{Q}(i)$ — we get one point, with multiplicity 2, arising because of the field extension.
(iv) Finally, the fiber over the generic point $\operatorname{Spec} \mathbb{Q}(x)$ is $\operatorname{Spec} \mathbb{Q}(y)$, which is one point, with multiplicity 2, arising again because of the field extension (as $\mathbb{Q}(y)/\mathbb{Q}(x)$ is a degree 2 extension).

We thus see three sorts of behaviors ((iii) and (iv) are really the same). Note that even if you only work with algebraically closed fields, you will still be forced to this third type of behavior, because residue fields at generic points are usually not algebraically closed (witness case (iv) above).

18.4.9. *Proof of Proposition 18.4.5 in the case C is integral.* To emphasize the main idea in the proof, we prove it in the case where C is integral. You can remove this hypothesis in Exercise 18.4.F. (We will later see that what matters here is that the morphism is finite and *flat*.) A key idea, useful in other circumstances, is to reduce to the case of a discrete valuation ring (when C' is the Spec of a discrete valuation ring).

The question is local on the target, so we may assume that C' is affine. We may also assume C' is integral (by Exercise 6.4.B).

Our plan is as follows: by Important Exercise 14.7.J, if the rank of the finite type quasicoherent sheaf $\pi_*\mathcal{O}_C$ is constant, then (as C' is reduced) $\pi_*\mathcal{O}_C$ is locally free. We will show this by showing the rank at any closed point p of C' is the same as the rank at the generic point.

If $\mathcal{F}$ is a quasicoherent sheaf on $\operatorname{Spec} A$, and $\mathfrak{p} \subset A$ is a prime ideal, then the rank of $\mathcal{F}$ at $[\mathfrak{p}]$ is (by definition) the dimension (as a vector space) of the pullback of $\mathcal{F}$ under $\operatorname{Spec} \kappa([\mathfrak{p}]) = \operatorname{Spec} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to \operatorname{Spec} A$. Thus on an integral scheme C', if we wish to compare the rank at a point p and the generic point η of C', we can pull back to $\operatorname{Spec} \mathcal{O}_{C',p}$, and compute there, as the inclusions of the spectra of both residue fields factor through this intermediate space:

$$\begin{tikzcd}
\operatorname{Spec} \kappa(p) \arrow[dr] & & \\
& \operatorname{Spec} \mathcal{O}_{C',p} \arrow[r] & C' \\
\operatorname{Spec} \kappa(\eta) \arrow[ur] & &
\end{tikzcd}$$

Thus we may assume C' is the spectrum of a discrete valuation ring.

Now $\pi_* \mathcal{O}_C$ is finite type (Exercise 17.2.C — Noetherianness is implicit in our hypothesis of nonsingularity) and $\pi_* \mathcal{O}_C$ is torsion-free (as $\Gamma(C, \mathcal{O}_C)$ is an integral domain). By Remark 13.4.16, any finitely generated torsion free module over a discrete valuation ring is free, so we are done. $\square$

18.4.F. EXERCISE (REMOVING THE INTEGRALITY HYPOTHESIS). Prove Proposition 18.4.5 without the "integral" hypothesis added in the proof. (Hint: the key fact used in the last paragraph was that the uniformizer t pulled back from C' was not a zerodivisor. But if it was, then $V(\pi^* t)$ would be dimension 1, whereas the pullback of a point $\pi^{-1}(V(t))$ must be dimension 0, by finiteness.)

18.4.10. *Remark: Flatness.* Everything we have discussed since the start of §18.4.4 is secretly about flatness, as you will see in §25.4.5.

CHAPTER 19

$\star$ Blowing up a scheme along a closed subscheme

We next discuss an important construction in algebraic geometry, the blow-up of a scheme along a closed subscheme (cut out by a finite type ideal sheaf). We won't use this much in later chapters, so feel free to skip this topic for now. But it is an important tool. For example, one can use it to resolve singularities, and more generally, indeterminacy of rational maps. In particular, blow-ups can be used to relate birational varieties to each other.

We will start with a motivational example that will give you a picture of the construction in a particularly important (and the historically earliest) case, in §19.1. We will then see a formal definition, in terms of a universal property, §19.2. The definition won't immediately have a clear connection to the motivational example. We will deduce some consequences of the definition (assuming that the blow-up actually exists). We then prove that the blow-up exists, by describing it quite explicitly, in §19.3. As a consequence, we will find that the blow-up morphism is projective, and we will deduce more consequences from this. In §19.4, we will do a number of explicit computations, to see various sorts of applications, and to see that many things can be computed by hand.

19.1 Motivating example: blowing up the origin in the plane

We will to generalize the following notion, which will correspond to "blowing up" the origin of $\mathbb{A}^2_k$ (Exercise 10.2.M). We will be informal. Consider the subset of $\mathbb{A}^2 \times \mathbb{P}^1$ corresponding to the following. We interpret $\mathbb{P}^1$ as parametrizing the lines through the origin. Consider the subvariety $\mathrm{Bl}_{(0,0)} \mathbb{A}^2 := \{(p \in \mathbb{A}^2, [\ell] \in \mathbb{P}^1) : p \in \ell)\}$, which is the data of a point p in the plane, and a line ℓ containing both p and the origin. Algebraically: let x and y be coordinates on $\mathbb{A}^2$, and X and Y be projective coordinates on $\mathbb{P}^1$ ("corresponding" to x and y); we will consider the subset $\mathrm{Bl}_{(0,0)} \mathbb{A}^2$ of $\mathbb{A}^2 \times \mathbb{P}^1$ corresponding to $xY - yX = 0$. We have the useful diagram

$$\begin{tikzcd} \mathrm{Bl}_{(0,0)} \mathbb{A}^2 \arrow[r, hook] \arrow[dr, "\beta"'] & \mathbb{A}^2 \times \mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{P}^1 \\ & \mathbb{A}^2 & \end{tikzcd}$$

You can verify that it is smooth over k (§13.2.4) directly (you can now make the paragraph after Exercise 10.2.M precise), but here is a informal argument, using the projection $\mathrm{Bl}_{(0,0)} \mathbb{A}^2 \to \mathbb{P}^1$. The projective line $\mathbb{P}^1$ is smooth, and for each point

$[\ell]$ in $\mathbb{P}^1$, we have a smooth choice of points on the line ℓ. Thus we are verifying smoothness by way of a fibration over $\mathbb{P}^1$.

We next consider the projection to $\mathbb{A}^2$, $\beta : \mathrm{Bl}_{(0,0)} \mathbb{A}^2 \to \mathbb{A}^2$. This is an isomorphism away from the origin. Loosely speaking, if p is not the origin, there is precisely one line containing p and the origin. On the other hand, if p *is* the origin, then there is a full $\mathbb{P}^1$ of lines containing p and the origin. Thus the preimage of $(0,0)$ is a curve, and hence a divisor (an effective Cartier divisor, as the blown-up surface is nonsingular). This is called the *exceptional divisor* of the blow-up.

If we have some curve $C \subset \mathbb{A}^2$ singular at the origin, it can be potentially partially desingularized, using the blow-up, by taking the closure of $C \setminus \{(0,0)\}$ in $\mathrm{Bl}_{(0,0)} \mathbb{A}^2$. (A **desingularization** or a **resolution of singularities** of a variety X is a proper birational morphism $\tilde{X} \to X$ from a nonsingular scheme.) For example, the curve $y^2 = x^3 + x^2$, which is nonsingular except for a node at the origin, then we can take the preimage of the curve minus the origin, and take the closure of this locus in the blow-up, and we will obtain a nonsingular curve; the two branches of the node downstairs are separated upstairs. (You can check this in Exercise 19.4.B once we have defined things properly. The result will be called the *proper transform* (or *strict transform*) of the curve.) We are interested in desingularizations for many reasons. For example, we will soon understand nonsingular curves quite well (Chapter 21), and we could hope to understand other curves through their desingularizations. This philosophy holds true in higher dimension as well.

More generally, we can blow up $\mathbb{A}^n$ at the origin (or more informally, "blow up the origin"), getting a subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$. Algebraically, If $x_1, \dots, x_n$ are coordinates on $\mathbb{A}^n$, and $X_1, \dots, X_n$ are projective coordinates on $\mathbb{P}^{n-1}$, then the blow-up $\mathrm{Bl}_{\vec{0}} \mathbb{A}^n$ is given by the equations $x_i X_j - x_j X_i = 0$. Once again, this is smooth: $\mathbb{P}^{n-1}$ is smooth, and for each point $[\ell] \in \mathbb{P}^{n-1}$, we have a smooth choice of $p \in \ell$.

We can extend this further, by blowing up $\mathbb{A}^{n+m}$ along a coordinate m-plane $\mathbb{A}^n$ by adding m more variables $x_{n+1}, \dots, x_{n+m}$ to the previous example; we get a subset of $\mathbb{A}^{n+m} \times \mathbb{P}^{n-1}$.

Because in complex geometry, smooth submanifolds of smooth manifolds locally "look like" coordinate m-planes in n-space, you might imagine that we could extend this to blowing up a nonsingular subvariety of a nonsingular variety. In the course of making this precise, we will accidentally generalize this notion greatly, defining the blow-up of any finite type sheaf of ideals in a scheme. In general, blowing up may not have such an intuitive description as in the case of blowing up something nonsingular inside something nonsingular — it can do great violence to the scheme — but even then, it is very useful. The result will be very powerful, and will touch on many other useful notions in algebra (such as the Rees algebra).

Our description will depend only the closed subscheme being blown up, and not on coordinates. That remedies a defect was already present in the first example, of blowing up the plane at the origin. It is not obvious that if we picked different coordinates for the plane (preserving the origin as a closed subscheme) that we wouldn't have two different resulting blow-ups.

As is often the case, there are two ways of understanding this notion, and each is useful in different circumstances. The first is by universal property, which lets you show some things without any work. The second is an explicit construction, which lets you get your hands dirty and compute things (and implies for example that the blow-up morphism is projective).

The motivating example here may seem like a very special case, but if you understand the blow-up of the origin in n-space well enough, you will understand blowing up in general.

19.2 Blowing up, by universal property

We now define the blow-up by a universal property. The disadvantage of starting here is that this definition won't obviously be the same as (or even related to) the examples of §19.1.

Suppose $X \hookrightarrow Y$ is a closed subscheme corresponding to a finite type sheaf of ideals. (If Y is locally Noetherian, the "finite type" hypothesis is automatic, so Noetherian readers can ignore it.)

The blow-up of $X \hookrightarrow Y$ is a fiber diagram

(19.2.0.1)
$$\begin{tikzcd} E_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_X Y \arrow[d, "\beta"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

such that $E_X Y$ (the scheme-theoretical pullback of X on Y) is an effective Cartier divisor (defined in §9.1.2) on $\mathrm{Bl}_X Y$, such any other such fiber diagram

(19.2.0.2)
$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & W \arrow[d] \\ X \arrow[r, hook] & Y, \end{tikzcd}$$

where D is an effective Cartier divisor on W, factors uniquely through it:

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & W \arrow[d] \\ E_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_X Y \arrow[d] \\ X \arrow[r, hook] & Y. \end{tikzcd}$$

We call $\mathrm{Bl}_X Y$ the **blow-up** (of Y along X, or of Y with center X). (A somewhat archaic term for this is **monoidal transformation**; we won't use this.) We call $E_X Y$ the **exceptional divisor** of the blow-up. (Bl and β stand for "blow-up", and E stands for "exceptional".)

By a typical universal property argument, if the blow-up exists, it is unique up to unique isomorphism. (We can even recast this more explicitly in the language of Yoneda's lemma: consider the category of diagrams of the form (19.2.0.2), where morphisms are diagrams of the form

$$\begin{tikzcd}
D \arrow[r, hook] \arrow[ddr] \arrow[drr] & W \arrow[drr] \arrow[ddr] & & \\
& & D' \arrow[r, hook] \arrow[dl] & W' \arrow[dl] \\
& X \arrow[r, hook] & Y. &
\end{tikzcd}$$

Then the blow-up is a final object in this category, if one exists.)

If $Z \hookrightarrow Y$ is any closed subscheme of Y, then the (scheme-theoretic) pullback $\beta^{-1}Z$ is called the **total transform** of Z. We will soon see that β is an isomorphism away from X (Observation 19.2.2). $\overline{\beta^{-1}(Z-X)}$ is called the **proper transform** or **strict transform** of Z. (We will use the first terminology. We will also define it in a more general situation.) We will soon see (in the Blow-up closure lemma 19.2.6) that the proper transform is naturally isomorphic to $\mathrm{Bl}_{Z\cap X} Z$, where $Z \cap X$ is the scheme-theoretic intersection.

We will soon show that the blow-up always exists, and describe it explicitly. We first make a series of observations, *assuming that the blow up exists*.

19.2.1. *Observation.* If X is the empty set, then $\mathrm{Bl}_X Y = Y$. More generally, if X is an effective Cartier divisor, then the blow-up is an isomorphism. (Reason: $\mathrm{id}_Y : Y \to Y$ satisfies the universal property.)

19.2.A. EXERCISE. If U is an open subset of Y, then $\mathrm{Bl}_{U\cap X} U \cong \beta^{-1}(U)$, where $\beta : \mathrm{Bl}_X Y \to Y$ is the blow-up.

Thus "we can compute the blow-up locally."

19.2.B. EXERCISE. Show that if Y_α is an open cover of Y (as α runs over some index set), and the blow-up of Y_α along $X \cap Y_\alpha$ exists, then the blow-up of Y along X exists.

19.2.2. *Observation.* Combining Observation 19.2.1 and Exercise 19.2.A, we see that the blow-up is an isomorphism away from the locus you are blowing up:

$$\beta|_{\mathrm{Bl}_X Y - E_X Y} : \mathrm{Bl}_X Y - E_X Y \to Y - X$$

is an isomorphism.

19.2.3. *Observation.* If $X = Y$, then the blow-up is the empty set: the only map $W \to Y$ such that the pullback of X is a Cartier divisor is $\varnothing \hookrightarrow Y$. In this case we have "blown Y out of existence"!

19.2.C. EXERCISE (BLOW-UP PRESERVES IRREDUCIBILITY AND REDUCEDNESS). Show that if Y is irreducible, and X doesn't contain the generic point of Y, then $\mathrm{Bl}_X Y$ is irreducible. Show that if Y is reduced, then $\mathrm{Bl}_X Y$ is reduced.

19.2.4. Existence in a first nontrivial case: blowing up a locally principal closed subscheme.

We next see why $\mathrm{Bl}_X Y$ exists if $X \hookrightarrow Y$ is locally cut out by one equation. As the question is local on Y (Exercise 19.2.B), we reduce to the affine case $\operatorname{Spec} A/(t) \hookrightarrow$

Spec A. (A good example to think through is $A = k[x,y]/(xy)$ and $t = x$.) Let

$$I = \ker(A \to A_t) = \{a \in A \ : \ t^n a = 0 \text{ for some } n > 0\},$$

and let $\phi : A \to A/I$ be the projection.

19.2.D. EXERCISE. Show that $\phi(t)$ is not a zerodivisor in A/I.

19.2.E. EXERCISE. Show that $\beta : \operatorname{Spec} A/I \to \operatorname{Spec} A$ is the blow up of $\operatorname{Spec} A$ along $\operatorname{Spec} A/t$. In other words, show that

$$\begin{tikzcd} \operatorname{Spec} A/(t,I) \arrow[r] \arrow[d] & \operatorname{Spec} A/I \arrow[d, "\beta"] \\ \operatorname{Spec} A/t \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

is a "blow up diagram" (19.2.0.1). Hint: In checking the universal property reduce to the case where W (in (19.2.0.2)) is affine. Then solve the resulting problem about rings. Depending on how you proceed, you might find Exercise 11.2.C, about the uniqueness of extension of maps over effective Cartier divisors, helpful.

19.2.F. EXERCISE. Show that $\operatorname{Spec} A/I$ is the scheme-theoretic closure of $D(t)$ in $\operatorname{Spec} A$.

Thus you might geometrically interpret $\operatorname{Spec} A/I \to \operatorname{Spec} A$ as "shaving off any fuzz supported in $V(t)$". In the Noetherian case, this can be interpreted as removing those associated points in $V(t)$. This is intended to be vague, and you should think about how to make it precise only if you want to.

19.2.5. The Blow-up closure lemma.
Suppose we have a fibered diagram

$$\begin{tikzcd} W \arrow[r, hook, "\text{cl. imm.}"] \arrow[d] & Z \arrow[d] \\ X \arrow[r, hook, "\text{cl. imm.}"] & Y \end{tikzcd}$$

where the bottom closed immersion corresponds to a finite type ideal sheaf (and hence the upper closed immersion does too). The first time you read this, it may be helpful to consider only the special case where $Z \to Y$ is a closed immersion.

Then take the fibered product of this square by the blow-up $\beta : \operatorname{Bl}_X Y \to Y$, to obtain

$$\begin{tikzcd} Z \times_Y E_X Y \arrow[r, hook] \arrow[d] & Z \times_Y \operatorname{Bl}_X Y \arrow[d] \\ E_X Y \arrow[r, hook, "\text{Cartier}"] & \operatorname{Bl}_X Y. \end{tikzcd}$$

The bottom closed immersion is locally cut out by one equation, and thus the same is true of the top closed immersion as well. However, the local equation on $Z \times_Y \operatorname{Bl}_X Y$ need not be a non-zerodivisor, and thus the top closed immersion is not necessarily an effective Cartier divisor.

Let $\overline{Z}$ be the scheme-theoretic closure of $Z \times_Y \operatorname{Bl}_X Y \setminus W \times_Y \operatorname{Bl}_X Y$ in $Z \times_Y \operatorname{Bl}_X Y$. (As $W \times_Y \operatorname{Bl}_X Y$ is locally principal, we are in precisely the situation of §19.2.4, so the scheme-theoretic closure is not mysterious.) Note that in the special case where $Z \to Y$ is a closed immersion, $\overline{Z}$ is the proper transform, as defined in §19.2. For this reason, it is reasonable to call $\overline{Z}$ the *proper transform of* Z even if Z *isn't* a closed immersion. Similarly, it is reasonable to call $Z \times_Z \operatorname{Bl}_X Y$ the *total transform of* Z even if Z isn't a closed immersion.

Define $E_{\overline{Z}} \hookrightarrow \overline{Z}$ as the pullback of $E_X Y$ to $\overline{Z}$, i.e. by the fibered diagram

$$\begin{tikzcd} E_{\overline{Z}} \arrow[r, hook] \arrow[d, hook, "\text{cl. imm.}"] & \overline{Z} \arrow[d, hook, "\text{cl. imm.}"] & \text{proper transform} \\ Z \times_Y E_X Y \arrow[r, hook, "\text{loc. prin.}"] \arrow[d] & Z \times_Y \operatorname{Bl}_X Y \arrow[d] & \text{total transform} \\ E_X Y \arrow[r, hook, "\text{Cartier}"] & \operatorname{Bl}_X Y. & \end{tikzcd}$$

Note that $E_{\overline{Z}}$ is an effective Cartier divisor on $\overline{Z}$. (It is locally cut out by one equation, pulled back from a local equation of $E_X Y$ on $\operatorname{Bl}_X Y$. Can you see why this is not locally a zerodivisor?)

19.2.6. Blow-up closure lemma. — $(\operatorname{Bl}_Z W, E_Z W)$ *is canonically isomorphic to* $(\overline{Z}, E_{\overline{Z}})$. *More precisely: if the blow-up* $\operatorname{Bl}_X Y$ *exists, then* $(\overline{Z}, E_{\overline{Z}})$ *is the blow-up of* W *along* Z.

This will be very useful. We make a few initial comments. The first three apply to the special case where $Z \to W$ is a closed immersion, and the fourth comment basically tells us we shouldn't have concentrated on this special case.

(1) First, note that if $Z \to Y$ is a closed immersion, then this states that the proper transform (as defined in §19.2) is the blow-up of Z along the scheme-theoretic intersection $W = X \cap Z$.

(2) In particular, it lets you actually compute blow-ups, and we will do lots of examples soon. For example, suppose C is a plane curve, singular at a point p, and we want to blow up C at p. Then we could instead blow up the plane at p (which we have already described how to do, even if we haven't yet proved that it satisfies the universal property of blowing up), and then take the scheme-theoretic closure of $C \setminus \{p\}$ in the blow-up.

(3) More generally, if W is some nasty subscheme of Z that we wanted to blow-up, and Z were a finite type k-scheme, then the same trick would work. We could work locally (Exercise 19.2.A), so we may assume that Z is affine. If W is cut out by r equations $f_1, \dots, f_r \in \Gamma(\mathcal{O}_Z)$, then complete the f's to a generating set $f_1, \dots, f_n$ of $\Gamma(\mathcal{O}_Z)$. This gives a closed immersion $Y \hookrightarrow \mathbb{A}^n$ such that W is the scheme-theoretic intersection of Y with a coordinate linear space $\mathbb{A}^r$.

19.2.7. (4) Most generally still, this reduces the existence of the blow-up to a specific special case. (If you prefer to work over a fixed field k, feel free to replace $\mathbb{Z}$ by k in this discussion.) Suppose that for each n, $\operatorname{Bl}_{(x_1,\dots,x_n)} \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$ exists. Then I claim that the blow-up always exists. Here's why. We may assume that Y is affine, say $\operatorname{Spec} B$, and $X = \operatorname{Spec} B/(f_1, \dots, f_n)$. Then we have a morphism $Y \to \mathbb{A}^n_{\mathbb{Z}}$ given by $x_i \mapsto f_i$, such that X is the scheme-theoretic pullback of the origin. Hence by the blow-up closure lemma, $\operatorname{Bl}_X Y$ exists.

19.2.G. ⋆ TRICKY EXERCISE. Prove the Blow-up Closure Lemma 19.2.6. Hint: obviously, construct maps in both directions, using the universal property. Constructing the following diagram may or may not help.

$$\begin{tikzcd}
 & & E_{\overline{Z}} \arrow[rr, hook, "\text{Cartier}"] \arrow[dll] \arrow[ddr, hook] & & \overline{Z} \arrow[dll] \arrow[ddr, hook] & & \\
E_W Z \arrow[rr, hook, "\text{Cartier}"] \arrow[ddr] \arrow[drrr] & & \mathrm{Bl}_W Z \arrow[ddr] \arrow[drrr] & & & & \\
 & & & Z \times_Y E_X Y \arrow[rr, hook, "\text{loc. prin.}"] \arrow[dll] \arrow[ddr] & & Z \times_Y \mathrm{Bl}_X Y \arrow[dll] \arrow[ddr] & \\
 & W \arrow[rr, hook] \arrow[ddr] & & Z \arrow[ddr] & & & \\
 & & & & E_X Y \arrow[rr, hook, "\text{Cartier}"] \arrow[dll] & & \mathrm{Bl}_X Y \arrow[dll] \\
 & & X \arrow[rr, hook] & & Y & &
\end{tikzcd}$$

Hooked arrows indicate closed immersions; and when morphisms are furthermore locally principal or even effective Cartier, they are so indicated. Exercise 11.2.C, on the uniqueness of extension of maps over effective Cartier divisors, may or may not help as well. Note that if $Z \to Y$ is actually a closed immersion, then so is $Z \times_Y \mathrm{Bl}_X Y \to \mathrm{Bl}_X Y$ and hence $\overline{Z} \to \mathrm{Bl}_X Y$.

19.3 The blow-up exists, and is projective

19.3.1. It is now time to show that the blow up always exists. We will see two arguments, which are enlightening in different ways. Both will imply that the blow-up morphism is projective, and hence quasicompact, proper, finite type, and separated. In particular, if $Y \to Z$ is quasicompact (resp. proper, finite type, separated), so is $\mathrm{Bl}_X Y \to Z$. (And if $Y \to Z$ is projective, and Z is quasicompact, then $\mathrm{Bl}_X Y \to Z$ is projective. See the solution to Exercise 18.3.E for the reason for this annoying extra hypothesis.) The blow-up of a k-variety is a k-variety (using the fact that reducedness is preserved, Exercise 19.2.C), and the blow-up of a irreducible k-variety is a irreducible k-variety (using the fact that irreducibility is preserved, also Exercise 19.2.C),

Approach 1. As explained in §19.2.7, it suffices to show that $\mathrm{Bl}_{V(x_1,\dots,x_n)} \operatorname{Spec} \mathbb{Z}[x_1,\dots,x_n]$ exists. But we know what it is supposed to be: the locus in $\operatorname{Spec} \mathbb{Z}[x_1,\dots,x_n] \times \operatorname{Proj} \mathbb{Z}[X_1,\dots,X_n]$ cut out by the equations $x_i X_j - x_j X_i = 0$. We will show this by the end of the section.

Approach 2. We can describe the blow-up all at once as a $\mathcal{P}$roj.

19.3.2. Theorem ($\mathcal{P}$roj description of the blow-up). — *Suppose $X \hookrightarrow Y$ is a closed subscheme cut out by a finite type quasicoherent sheaf of ideals $\mathcal{I} \hookrightarrow \mathcal{O}_Y$. Then*

$$\mathcal{P}\mathrm{roj}\left(\mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots\right) \to Y$$

satisfies the universal property of blowing up.

(We made sense of products of ideal sheaves, and hence $\mathcal{I}^n$, in Exercise 15.3.D.)

We will prove Theorem 19.3.2 soon (§19.3.3), after seeing what it tells us. Because I is finite type, the graded sheaf of algebras has degree 1 piece that is finite type. The graded sheaf of algebras is also clearly generated in degree 1. Thus the sheaf of algebras satisfy the hypotheses of §18.2.1.

But first, we should make sure that the preimage of X is indeed an effective Cartier divisor. We can work affine-locally (Exercise 19.2.A), so we may assume that $Y = \operatorname{Spec} B$, and X is cut out by the finitely generated ideal I. Then

$$\mathrm{Bl}_X Y = \operatorname{Proj}\left(B \oplus I \oplus I^2 \oplus \cdots\right).$$

(The ring $B \oplus I \oplus \cdots$ is called the **Rees algebra** of the ideal I in B, although we will not need this terminology.) We are slightly abusing notation by using the notation $\mathrm{Bl}_X Y$, as we haven't yet shown that this satisfies the universal property.

The preimage of X isn't just any effective Cartier divisor; it corresponds to the invertible sheaf $\mathcal{O}(1)$ on this $\mathcal{P}$roj. Indeed, $\mathcal{O}(1)$ corresponds to taking our graded ring, chopping off the bottom piece, and sliding all the graded pieces to the left by 1 (§16.2); it is the invertible sheaf corresponding to the graded module

$$I \oplus I^2 \oplus I^3 \oplus \cdots$$

(where that first summand I has grading 0). But this can be interpreted as the scheme-theoretic pullback of X, which corresponds to the ideal I of B:

$$I\left(B \oplus I \oplus I^2 \oplus \cdots\right) \hookrightarrow B \oplus I \oplus I^2 \oplus \cdots.$$

Thus the scheme-theoretic pullback of $X \hookrightarrow Y$ to $\mathcal{P}\mathrm{roj}(\mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots)$, the invertible sheaf corresponding to $\mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots$, is an effective Cartier divisor in class $\mathcal{O}(1)$. Once we have verified that this construction is indeed the blow-up, this divisor will be our exceptional divisor $E_X Y$.

Moreover, we see that the exceptional divisor can be described beautifully as a $\mathcal{P}$roj over X:

$$E_X Y = \mathcal{P}\mathrm{roj}_X\left(B/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots\right). \tag{19.3.2.1}$$

We will later see (§19.4.12) that in good circumstances (if X is a local complete intersection in something nonsingular, or more generally a local complete intersection in a Cohen-Macaulay scheme) this is a projectivization of a vector bundle (the "projectivized normal bundle").

19.3.3. Proof of the universal property, Theorem 19.3.2. Let's prove that this $\mathcal{P}$roj construction satisfies the universal property. Then Approach 1 will also follow, as a special case of Approach 2.

19.3.4. *Aside: why approach 1?.* Before we begin, you may be wondering why we bothered with Approach 1. One reason is that you may find it more comfortable to work with this one nice ring, and the picture may be geometrically clearer to you (in the same way that thinking about the Blow-up Closure Lemma 19.2.6

in the case where $Z \to Y$ is a closed immersion is more intuitive). Another reason is that, as you will find in the exercises, you will see some facts more easily in this explicit example, and you can then pull them back to more general examples. Perhaps most important, Approach 1 lets you actually compute blow-ups by working affine locally: if $f_1, \dots, f_n$ are elements of a ring A, cutting a subscheme $X = \operatorname{Spec} A/(f_1, \dots, f_n)$ of $Y = \operatorname{Spec} A$, then $\operatorname{Bl}_X Y$ can be interpreted as a closed subscheme of $\mathbb{P}^{n-1}_A$, by pulling back from $\operatorname{Bl}_{V(x_1,\dots,x_n)} \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$, and taking the closure of the locus "above X" as dictated by the Blow-up Closure Lemma 19.2.6.

Proof. Reduce to the case of affine target $\operatorname{Spec} R$ with ideal $I \subset R$. Reduce to the case of affine source, with principal effective Cartier divisor t. (A principal effective Cartier divisor is locally cut out by a single non-zerodivisor.) Thus we have reduced to the case $\operatorname{Spec} S \to \operatorname{Spec} R$, corresponding to $f : R \to S$. Say $(x_1, \dots, x_n) = I$, with $(f(x_1), \dots, f(x_n)) = (t)$. We will describe *one* map $\operatorname{Spec} S \to \operatorname{Proj} R[I]$ that will extend the map on the open set $\operatorname{Spec} S_t \to \operatorname{Spec} R$. It is then unique, by Exercise 11.2.C. We map $R[I]$ to S as follows: the degree one part is $f : R \to S$, and $f(X_i)$ (where X_i corresponds to x_i, except it is in degree 1) goes to $f(x_i)/t$. Hence an element X of degree d goes to $X/(t^d)$. On the open set $D_+(X_1)$, we get the map $R[X_2/X_1, \dots, X_n/X_1]/(x_2 - X_2/X_1 x_1, \dots, x_i X_j - x_j X_i, \dots) \to S$ (where there may be many relations) which agrees with f away from $D(t)$. Thus this map does extend away from $V(I)$. $\square$

Here are some applications and observations arising from this construction of the blow-up. First, we can verify that our initial motivational examples are indeed blow-ups. For example, blowing up $\mathbb{A}^2$ (with coordinates x and y) at the origin yields: $B = k[x, y]$, $I = (x, y)$, and $\operatorname{Proj}(B \oplus I \oplus I^2 \oplus \cdots) = \operatorname{Proj} B[X, Y]$ where the elements of B have degree 0, and X and Y are degree 1 and "correspond to" x and y respectively.

19.3.5. *Normal bundles to exceptional divisors.* We will soon see that the normal bundle to a Cartier divisor D is the (space associated to the) invertible sheaf $\mathcal{O}(D)|_D$, the invertible sheaf corresponding to the D on the total space, then restricted to D (Exercise 23.2.H). Thus in the case of the blow-up of a point in the plane, the exceptional divisor has normal bundle $\mathcal{O}(-1)$. (As an aside: Castelnuovo's criterion states that conversely given a smooth surface containing $E \cong \mathbb{P}^1$ with normal bundle $\mathcal{O}(-1)$, E can be blown-down to a point on another smooth surface.) In the case of the blow-up of a nonsingular subvariety of a nonsingular variety, the blow up turns out to be nonsingular (a fact discussed soon in §19.4.12), and the exceptional divisor is a projective bundle over X, and the normal bundle to the exceptional divisor restricts to $\mathcal{O}(-1)$.

19.3.A. HARDER BUT ENLIGHTENING EXERCISE. If $X \hookrightarrow \mathbb{P}^n$ is a projective scheme, show that the exceptional divisor of the blow up the affine cone over X (§9.2.11) at the origin is isomorphic to X, and that its normal bundle (§19.3.5) is isomorphic to $\mathcal{O}_X(-1)$. (In the case $X = \mathbb{P}^1$, we recover the blow-up of the plane at a point. In particular, we recover the important fact that the normal bundle to the exceptional divisor is $\mathcal{O}(-1)$.)

19.3.6. *The normal cone.* Partially motivated by (19.3.2.1), we make the following definition. If X is a closed subscheme of Y cut out by $\mathcal{I}$, then the **normal cone** $N_X Y$ of X in Y is defined as

$$N_X Y := \mathcal{S}\mathrm{pec}_X \left(\mathcal{O}_Y/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots \right).$$

This can profitably be thought of as an algebro-geometric version of a "tubular neighborhood". But some cautions are in order. If Y is smooth, $N_X Y$ may not be smooth. (You can work out the example of $Y = \mathbb{A}^2_k$ and $X = V(xy)$.) And even if X and Y is smooth, then although $N_X Y$ is smooth (as we will see shortly, §19.4.12), it doesn't "embed" in any way in Y.

If X is a closed point p, then the normal cone is called the **tangent cone** to Y at p. The **projectivized tangent cone** is the exceptional divisor $E_X Y$ (the $\mathcal{P}\mathrm{roj}$ of the same graded sheaf of algebras). Following §9.2.12, the tangent cone and the projectivized tangent cone can be put together in the projective completion of the tangent cone, which contains the tangent cone as an open subset, and the projectivized tangent cone as a complementary effective Cartier divisor.

19.3.B. EXERCISE. Suppose $Y = \operatorname{Spec} k[x,y]/(y^2 - x^2 - x^3)$ (the bottom of Figure 8.4). Assume (to avoid distraction) that $\operatorname{char} k \neq 2$. Show that the tangent cone to Y at the origin is isomorphic to $\operatorname{Spec} k[x,y]/(y^2 - x^2)$. Thus, informally, the tangent cone "looks like" the original variety "infinitely magnified".

We will later see that at a smooth point of Y, the tangent cone may be identified with the tangent space, and the normal cone may often be identified with the total space of the normal bundle (see §19.4.12).

19.3.C. EXERCISE. Suppose $S_\bullet$ is a finitely generated graded algebra over a field k. Exercise 19.3.A gives an isomorphism of $\operatorname{Proj} S_\bullet$ with the exceptional divisor to the blow-up of $\operatorname{Spec} S_\bullet$ at the origin. Show that the tangent cone to $\operatorname{Spec} S_\bullet$ at the origin is isomorphic $\operatorname{Spec} S_\bullet$ itself. (Your geometric intuition should lead you to find these facts believable.)

The following construction is key to the modern understanding of intersection theory in algebraic geometry, as developed by Fulton and MacPherson, **[F]**.

19.3.D. $\star$ EXERCISE: DEFORMATION TO THE NORMAL CONE. Suppose Y is a k-variety, and $X \hookrightarrow Y$ is a closed subscheme.
(a) Show that the exceptional divisor of $\beta : \mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1) \to Y \times \mathbb{P}^1$ is isomorphic to the projective completion of the normal cone to X in Y.
(b) Let $\pi : \mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1) \to \mathbb{P}^1$ be the composition of β with the projection to $\mathbb{P}^1$. Show that $\pi^*(0)$ is the scheme-theoretic union of $\mathrm{Bl}_X Y$ with the projective completion of the normal cone to X and Y, and the intersection of these two subschemes may be identified with $E_X Y$, which is a closed subscheme of $\mathrm{Bl}_X Y$ in the usual way (as the exceptional divisor of the blow-up $\mathrm{Bl}_X Y \to Y$), and a closed subscheme of the projective completion of the normal cone as described in Exercise 9.2.Q.

The map

$$\mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1) \setminus \mathrm{Bl}_X Y \to \mathbb{P}^1$$

is called the **deformation to the normal cone** (short for *deformation of* Y *to the normal cone of* X *in* Y). Notice that the fiber above every k-point away from $0 \in \mathbb{P}^1$ is canonically isomorphic to Y, and the fiber over 0 is the normal cone. Because this family is "nice" (more precisely, *flat*, the topic of Chapter 25), we can prove things about general Y (near X) by way of this degeneration.

19.4 Examples and computations

In this section we will do a number of explicit of examples, to get a sense of how blow-ups behave, how they are useful, and how one can work with them explicitly. To avoid distraction, **all of the following discussion takes place over an algebraically closed field k of characteristic 0,** although these hypotheses are often not necessary. The examples and exercises are loosely arranged in a number of topics, but the topics are not in order of importance.

19.4.1. Example: Blowing up the plane along the origin. Let's first blow up the plane $\mathbb{A}^2_k$ along the origin, and see that the result agrees with our discussion in §19.1. Let x and y be the coordinates on $\mathbb{A}^2_k$. The blow-up is $\operatorname{Proj} k[x, y, X, Y]$ where $xY - yX = 0$. (Here x and y have degree 0 and X and Y have degree 1.) This is naturally a closed subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$, cut out (in terms of the projective coordinates X and Y on $\mathbb{P}^1_k$) by $xY - yX = 0$. We consider the two usual patches on $\mathbb{P}^1_k$: $[X; Y] = [s; 1]$ and $[1; t]$. The first patch yields $\operatorname{Spec} k[x, y, s]/(sy - x)$, and the second gives $\operatorname{Spec} k[x, y, t]/(y - xt)$. Notice that both are nonsingular: the first is naturally $\operatorname{Spec} k[y, s] \cong \mathbb{A}^2_k$, the second is $\operatorname{Spec} k[x, t] \cong \mathbb{A}^2_k$.

Let's describe the exceptional divisor. We first consider the first (s) patch. The ideal is generated by (x, y), which in our ys-coordinates is $(ys, y) = (y)$, which is indeed principal. Thus on this patch the exceptional divisor is generated by y. Similarly, in the second patch, the exceptional divisor is cut out by x. (This can be a little confusing, but there is no contradiction!) This explicit description will be useful in working through some of the examples below.

19.4.A. EXERCISE. Let p be a k-valued point of $\mathbb{P}^2_k$. Exhibit an isomorphism between $\operatorname{Bl}_p \mathbb{P}^2_k$ and the Hirzebruch surface $\mathbb{F}_1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ (Definition 18.2.2). (The map $\operatorname{Bl}_p \mathbb{P}^2_k \to \mathbb{P}^1$ informally corresponds to taking a point to the line connecting it to the origin. Do not be afraid: You can do this by explicitly working with coordinates.)

19.4.2. Resolving singularities.

19.4.3. *The proper transform of a nodal curve (Figure 19.1).* (You may wish to flip to Figure 8.4 while thinking through this exercise.) Consider next the curve $y^2 = x^3 + x^2$ inside the plane $\mathbb{A}^2_k$. Let's blow up the origin, and compute the total and proper transform of the curve. (By the Blow-up Closure Lemma 19.2.6, the latter is the blow-up of the nodal curve at the origin.) In the first patch, we get $y^2 - s^2y^2 - s^3y^3 = 0$. This factors: we get the exceptional divisor y with multiplicity two, and the curve $1 - s^2 - y^3 = 0$. You can easily check that the proper transform is nonsingular. Also, notice that the proper transform $\tilde{C}$ meets the exceptional divisor at two points, $s = \pm 1$. This corresponds to the two tangent directions at the origin (as $s = x/y$).

19.4.B. EXERCISE (FIGURE 19.1). Describe both the total and proper transform of the curve C given by $y = x^2 - x$ in $\mathrm{Bl}_{(0,0)} \mathbb{A}^2$. Show that the proper transform of C is isomorphic to C. Interpret the intersection of the proper transform of C with the exceptional divisor E as the slope of C at the origin.

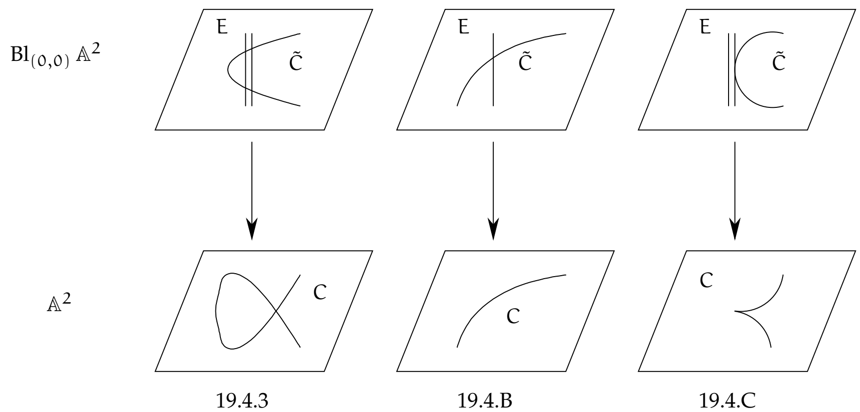


FIGURE 19.1. Resolving curve singularities (§19.4.3, Exercise 19.4.B, and Exercise 19.4.C)

19.4.C. EXERCISE: BLOWING UP A CUSPIDAL PLANE CURVE (CF. EXERCISE 10.6.F). Describe the proper transform of the cuspidal curve C given by $y^2 = x^3$ in the plane $\mathbb{A}^2_k$. Show that it is nonsingular. Show that the proper transform of C meets the exceptional divisor E at one point, and is tangent to E there.

The previous two exercises are the first in an important sequence of singularities, which we now discuss.

19.4.D. EXERCISE: RESOLVING A_n CURVE SINGULARITIES. Resolve the singularity $y^2 = x^{n+1}$ in $\mathbb{A}^2$, by first blowing up its singular point, then considering its proper transform and deciding what to do next.

19.4.4. *Definition: A_n curve singularities.* You will notice that your solution to Exercise 19.4.D depends only on the "power series expansion" of the singularity at the origin, and not on the precise equation. For example, if you compare your solution to Exercise 19.4.B with the $n = 1$ case of Exercise 19.4.D, you will see that they are "basically the same". A k-curve singularity analytically isomorphic (in the sense of Definition 13.6.2) to that of Exercise 19.4.D is called an A_n **curve singularity**. Thus by Definition 13.6.2, an A_1-singularity (resp. A_2-singularity, A_3-singularity) is a node (resp. cusp, tacnode).

19.4.E. EXERCISE (WARM-UP TO EXERCISE 19.4.F). Blow up the cone point $z^2 = x^2 + y^2$ (Figure 4.4) at the origin. Show that the resulting surface is nonsingular. Show that the exceptional divisor is isomorphic to $\mathbb{P}^1$. (Remark: you can check that the normal bundle to this $\mathbb{P}^1$ is not $\mathcal{O}(-1)$, as is the case when you blow up a point on a smooth surface, see §19.3.5; it is $\mathcal{O}(-2)$.)

19.4.F. EXERCISE (RESOLVING A_n SURFACE SINGULARITIES). Resolve the singularity $z^2 = y^2 + x^{n+1}$ in $\mathbb{A}^3$ by first blowing up its singular point, then considering its proper transform, and deciding what to do next. (A k-surface singularity analytically isomorphic this is called an A_n **surface singularity**. This exercise is a bit time consuming, but is rewarding in that it shows that you can really resolve singularities by hand.)

19.4.5. *Remark:* ADE*-surface singularities and Dynkin diagrams (see Figure 19.2).* A k-singularity analytically isomorphic to $z^2 = x^2 + y^{n+1}$ (resp. $z^2 = x^3 + y^4$, $z^2 = x^3 + xy^3$, $z^2 = x^3 + y^5$) is called a D_n surface singularity (resp. E_6, E_7, E_8 surface singularity). You can guess the definition of the corresponding curve singularity. If you (minimally) desingularize each of these surfaces by sequentially blowing up singular points as in Exercise 19.4.F, and look at the arrangement of exceptional divisors (the various exceptional divisors and how they meet), you will discover the corresponding Dynkin diagram. More precisely, if you create a graph, where the vertices correspond to exceptional divisors, and two vertices are joined by an edge if the two divisors meet, you will find the underlying graph of the corresponding Dynkin diagram. This is the start of several very beautiful stories.

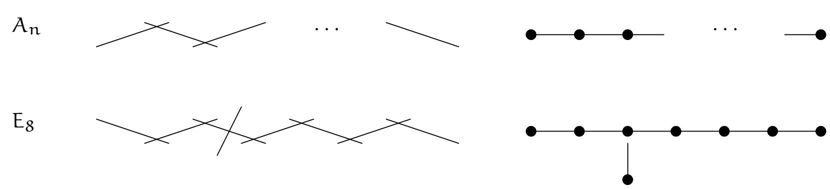


FIGURE 19.2. The exceptional divisors for resolutions of some ADE surface singularities, and their corresponding dual graphs (see Remark 19.4.5)

19.4.6. *Remark: Resolution of singularities.* Hironaka's theorem on resolution of singularities implies that this idea of trying to resolve singularities by blowing up singular loci in general can succeed in characteristic 0. More precisely, if X is a variety over a field of characteristic 0, then X can be resolved by a sequence of blow-ups, where the nth blow-up is along a nonsingular subvariety that lies in the singular locus of the variety produced after the $(n-1)$st stage (see **[Hir]**, and **[Ko]**). As of this writing, it is not known if an analogous statement is true in positive characteristic, but de Jong's Alteration Theorem **[dJ]** gives a result which is good enough for most applications. Rather than producing a birational proper map $\tilde{X} \to X$ from something nonsingular, it produces a proper map from something nonsingular that is generically finite (and the corresponding extension of function fields is separable).

Here are some other exercises related to resolution of singularities.

19.4.G. EXERCISE. Blowing up a nonreduced subscheme of a nonsingular scheme can give you something singular, as shown in this example. Describe the blow up of the ideal (y, x^2) in $\mathbb{A}^2_k$. Show that you get an A_1 surface singularity (basically, the cone point).

19.4.H. EXERCISE. Desingularize the tacnode $y^2 = x^4$, not in two steps (as in Exercise 19.4.D), but in a single step by blowing up (y, x^2).

19.4.I. EXERCISE (RESOLVING A SINGULARITY BY AN UNEXPECTED BLOW-UP). Suppose Y is the cone $x^2 + y^2 = z^2$, and X is the ruling of the cone $x = 0$, $y = z$. Show that $\mathrm{Bl}_X Y$ is nonsingular. (In this case we are blowing up a codimension 1 locus that is not an effective Cartier divisor (Problem 13.1.3). But it *is* an effective Cartier divisor away from the cone point, so you should expect your answer to be an isomorphism away from the cone point.)

19.4.J. EXERCISE. Show that the multiplicity of the exceptional divisor in the total transform of a subscheme Z of $\mathbb{A}^n$ when you blow up the origin is the lowest degree that appears in a defining equation of Z. (For example, in the case of the nodal and cuspidal curves above, Example 19.4.3 and Exercise 19.4.C respectively, the exceptional divisor appears with multiplicity 2.) This is called the **multiplicity** of the singularity of Z at the origin. It actually depends only on Z, and not on $\mathbb{A}^n$. This can be shown by reinterpreting it as the smallest m such that $\mathrm{Sym}^m \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^m/\mathfrak{m}^{m+1}$ is not an isomorphism, if Z is singular, and 1 otherwise. In this guise, it makes sense in more generality, such as for a closed point of a k-smooth variety. The multiplicity of a subscheme Z at a point p is denoted $\mathrm{mult}_p Z$.

19.4.7. Resolving rational maps.

19.4.K. EXERCISE (UNDERSTANDING THE BIRATIONAL MAP $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ VIA BLOW-UPS). Let p and q be two distinct k-points of $\mathbb{P}^2_k$, and let r be a k-point of $\mathbb{P}^1_k \times \mathbb{P}^1_k$. Describe an isomorphism $\mathrm{Bl}_{\{p,q\}} \mathbb{P}^2_k \leftrightarrow \mathrm{Bl}_r \mathbb{P}^1_k \times \mathbb{P}^1_k$. (Possible hint: Consider lines ℓ through p and m through q; the choice of such a pair corresponds to the parametrized by $\mathbb{P}^1_k \times \mathbb{P}^1_k$. A point s of $\mathbb{P}^2$ not on line pq yields a pair of lines $(\overline{ps}, \overline{qs})$ of $\mathbb{P}^1_k \times \mathbb{P}^1_k$. Conversely, a choice of lines (ℓ, m) such that neither ℓ and m is line $\overline{pq}$ yields a point $s = \ell \cap m \in \mathbb{P}^2_k$. This describes a birational map $\mathbb{P}^2_k \dashrightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k$. Exercise 19.4.A is related.)

This exercise is an example of the general phenomenon explored in the next two exercises.

19.4.L. HARDER BUT USEFUL EXERCISE (BLOW-UPS RESOLVE BASE LOCI OF RATIONAL MAPS TO PROJECTIVE SPACE). Suppose we have a scheme Y, an invertible sheaf $\mathcal{L}$, and a number of sections $s_0, \dots, s_n$ of $\mathcal{L}$ (a *linear series*, Definition 16.3.6). Then away from the closed subscheme X cut out by $s_0 = \cdots = s_n = 0$ (the base locus of the linear series), these sections give a morphism to $\mathbb{P}^n$. Show that this morphism extends uniquely to a morphism $\mathrm{Bl}_X Y \to \mathbb{P}^n$, where this morphism corresponds to the invertible sheaf $(\beta^*\mathcal{L})(-E_X Y)$, where $\beta : \mathrm{Bl}_X Y \to Y$ is the blow-up morphism. In other words, "blowing up the base scheme resolves this rational map". Hint: it suffices to consider an affine open subset of Y where $\mathcal{L}$ is trivial. Uniqueness might use Exercise 11.2.C.

19.4.8. *Remarks.* (i) This exercise immediately implies that blow-ups can be used to resolve rational maps to projective schemes $Y \dashrightarrow Z \hookrightarrow \mathbb{P}^n$.

(ii) The following interpretation is enlightening. The linear series on Y pulls back to a linear series on $\mathrm{Bl}_X Y$, and the base locus of the linear series on Y pulls back to the base locus on $\mathrm{Bl}_X Y$. The base locus on $\mathrm{Bl}_X Y$ is $E_X Y$, an effective Cartier divisor. Because $E_X Y$ is not just locally principal, but also locally a non-zerodivisor, it can be "divided out" from the $\beta^* s_i$ (yielding a section of $(\beta^*\mathcal{L})(-E_X Y)$, thereby removing the base locus, and leaving a base-point-free linear series. (In a sense that can be made precise through the universal property, this is the smallest "modification" of Y that can remove the base locus.) If X is already Cartier (as for example happens with any nontrivial linear system if Y is a nonsingular pure-dimensional curve), then we can remove a base locus by just "dividing out X".

19.4.9. *Examples.* (i) The rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ given by $[x_0;\cdots;x_n] \dashrightarrow [x_1;\cdots;x_n]$, defined away from $p = [1;0;\cdots;0]$, is resolved by blowing up p. Then by the Blow-up Closure Lemma 19.2.6, if Y is any locally closed subscheme of $\mathbb{P}^n$, we can project to $\mathbb{P}^{n-1}$ once we blow up p in Y, and the invertible sheaf giving the map to $\mathbb{P}^{n-1}$ is (somewhat informally speaking) $\beta^*(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}(-E_p Y)$.

(ii) Consider two general cubic equations C_1 and C_2 in three variables, yielding two cubic curves in $\mathbb{P}^2$. We shall see that they are smooth, and meet in 9 points $p_1, \ldots, p_9$ (using our standing assumption that we work over an algebraically closed field). Then $[C_1; C_2]$ gives a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. To resolve the rational map, we blow up $p_1, \ldots, p_9$. The result is (generically) an *elliptic fibration* $\mathrm{Bl}_{p_1,\ldots,p_9} \mathbb{P}^2 \to \mathbb{P}^1$. (This is by no means a complete argument.)

(iii) Fix six general points $p_1, \ldots, p_6$ in $\mathbb{P}^2$. There is a four-dimensional vector space of cubics vanishing at these points, and they vanish scheme-theoretically precisely at these points. This yields a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$, which is resolved by blowing up the six points. The resulting morphism turns out to be a closed immersion, and the image in $\mathbb{P}^3$ is a (smooth) cubic surface. This is the famous fact that the blow up of the plane at six general points may be represented as a (smooth) cubic in $\mathbb{P}^3$. (Again, this argument is not intended to be complete.)

In reasonable circumstances, Exercise 19.4.L has an interpretation in terms of graphs of rational maps.

19.4.M. EXERCISE. Suppose $s_0, \ldots, s_n$ are sections of an invertible sheaf $\mathcal{L}$ on an integral scheme X, not all 0. By Remark 17.4.3, this data gives a rational map $\phi : X \dashrightarrow \mathbb{P}^n$. Give an isomorphism between the graph of ϕ (defined in §10.2.6) and $\mathrm{Bl}_{V(s_0,\ldots,s_n)} X$.

You may enjoy exploring the previous idea by working out how the Cremona transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ (Exercise 7.5.J) can be interpreted in terms of the graph of the rational map $[x;y;z] \dashrightarrow [1/x; 1/y; 1/z]$.

19.4.N. ⋆ EXERCISE. Resolve the rational map

$$\operatorname{Spec} k[w,x,y,z]/(wz-xy) \overset{[w;x]}{\dashrightarrow} \mathbb{P}^1_k$$

from the cone over the quadric surface to the projective line. Let X be the resulting variety, and $\pi : X \to \operatorname{Spec} k[w,x,y,z]/(wz-xy)$. the projection to the cone over the quadric surface. Show that π is an isomorphism away from the cone point, and that the preimage of the cone point is isomorphic to $\mathbb{P}^1$ (and thus has codimension 2, and thus is different from the resolution obtained by simply blowing up the cone point). This is an example of a small resolution. (A **small resolution** $X \to Y$ is a resolution where the space of points of Y where the fiber has dimension r is of codimension greater than $2r$. We will not use this notion again in any essential way.) Notice that this resolution of the morphism involves blowing up the base locus $w = x = 0$, which is a cone over one of the lines on the quadric surface $wz = xy$. We are blowing up an effective Weil divisor, which is necessarily not Cartier as the blow-up is not an isomorphism. In Exercise 13.1.D, we saw that (w, x) was not principal, while here we see that (w, x) is not even locally principal.

19.4.10. *Remark: non-isomorphic small resolutions.* If you instead resolved the map $[w; y]$, you would obtain a similar looking small resolution $\pi' : X' \to \operatorname{Spec} k[w, x, y, z]/(wz - xy)$ (it is an isomorphism away from the origin, and the fiber over the origin is $\mathbb{P}^1$). But it is different! More precisely, there is no morphism $X \to X'$ making the following the diagram commute.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & \operatorname{Spec} k[w,x,y,z]/(wz-xy) & \end{tikzcd}$$

19.4.11. *Factorization of birational maps.* We end our discussion of resolution of rational maps by noting that just as Hironaka's theorem states that one may resolve all singularities of varieties in characteristic by a sequence of blow-ups along smooth centers, the weak factorization theorem (first proved by Włoldarczyk) states that any two birational varieties X and Y in characteristic 0 may be related by blow-ups and blow-downs along smooth centers. More precisely, there are varieties X_0, $\dots, X_n, X_{01}, \dots, X_{(n-1)n}$, with $X_0 = X$ and $X_n = Y$, with morphisms $X_{i(i+1)} \to X_i$ and $X_{i(i+1)} \to X_{i+1}$ ($0 \leq i < n$) which are blow-ups of smooth subvarieties.

19.4.12. The blow-up of a local complete intersection in a $\overline{k}$-smooth variety.

We now examine the case of a reduced local complete intersection in a $\overline{k}$-smooth variety. Suppose A is a finitely generated algebra over a field k, such that $\operatorname{Spec} A$ is nonsingular of pure dimension n. Suppose further that $f_1, \dots, f_m$ cut out an *integral* complete intersection $Z := \operatorname{Spec} A/I$ in $\operatorname{Spec} A$ ($I = (f_1, \dots, f_m)$) of codimension m (§13.3.4). Then we have a commutative diagram

$$\begin{tikzcd} \operatorname{Bl}_Z \operatorname{Spec} A \arrow[rr, hook, "\text{cl. imm.}"] \arrow[dr] & & \mathbb{P}^{m-1}_A \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

(cf. §19.3.4). Pulling back by the closed immersion $Z \hookrightarrow A$, we have

$$\begin{tikzcd} E_Z \operatorname{Spec} A \arrow[rr, hook, "\alpha", "\text{cl. imm.}"'] \arrow[dr] & & \mathbb{P}_Z^{m-1} \arrow[dl] \\ & Z & \end{tikzcd}$$

Now $E_Z \operatorname{Spec} A$ is an effective Cartier divisor, hence of pure dimension $n-1$. But $\mathbb{P}_Z^{m-1}$ is of dimension $m-1+\dim Z = n-1$, and is integral. Hence the closed immersion $E_Z \operatorname{Spec} A \hookrightarrow \mathbb{P}_Z^{m-1}$ is an isomorphism.

19.4.O. EXERCISE. Remove the hypothesis "Z irreducible" from the above discussion.

We now extract a couple of results from this.

19.4.13. Theorem. — *Suppose $X \hookrightarrow Y$ is a closed immersion of k-smooth varieties. Then $\mathrm{Bl}_X Y$ is k-smooth.*

Proof. By Theorem 13.3.5, $X \hookrightarrow Y$ is a local complete intersection, so the above discussion applies. We need only check the points of $E_X Y$, as $\mathrm{Bl}_Y \setminus E_X Y \cong Y \setminus X$ is $\overline{k}$-smooth. But $E_X Y \cong \mathbb{P}_Z^{m-1}$ is an effective Cartier divisor, and is nonsingular of dimension $n-1$. By the slicing criterion for nonsingularity (Exercise 13.2.B), it follows that Y is nonsingular along $E_X Y$. $\square$

Furthermore, we also proved that for any reduced complete intersection Z in a nonsingular scheme Y, $E_Z Y$ is a $\mathbb{P}^{n-1}$-bundle over Z. We will later identify this as the projectivized normal bundle of Z in Y, and will remove the reducedness hypothesis.

CHAPTER 20

Čech cohomology of quasicoherent sheaves

This topic is surprisingly simple and elegant. You may think cohomology must be complicated, and that this is why it appears so late in these notes. But you will see that we need very little background. After defining schemes, we could have immediately defined quasicoherent sheaves, and then defined cohomology, and verified that it had many useful properties.

20.1 (Desired) properties of cohomology

Rather than immediately defining cohomology of quasicoherent sheaves, we first discuss why we care, and what properties it should have.

As $\Gamma(X, \cdot)$ is a left-exact functor, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves on X, then

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X)$$

is exact. We dream that this sequence continues to the right, giving a long exact sequence. More explicitly, there should be some covariant functors H^i ($i \geq 0$) from quasicoherent sheaves on X to groups such that H^0 is the global section functor Γ, and so that there is a "long exact sequence in cohomology".

(20.1.0.1)
$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H})$$
$$\longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \longrightarrow \cdots$$

(In general, whenever we see a left-exact or right-exact functor, we should hope for this, and in good cases our dreams will come true. The machinery behind this usually involves *derived functors*, which we will discuss in Chapter 24.)

Before defining cohomology groups of quasicoherent sheaves explicitly, we first describe their important properties, which are in some ways more important than the formal definition. The boxed properties will be the important ones.

Suppose X is a separated and quasicompact A-scheme. For each quasicoherent sheaf $\mathcal{F}$ on X, we will define A-modules $H^i(X, \mathcal{F})$. In particular, if $A = k$, they are k-vector spaces. In this case, we define $h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})$ (where k is left implicit on the left side).

(i) Each $\boxed{H^i \text{ is a covariant functor in the sheaf } \mathcal{F}}$ extending the usual covariance for $H^0(X, \cdot)$: $\mathcal{F} \to \mathcal{G}$ induces $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G})$.

(ii) The functor H^0 is identified with functor Γ: $\boxed{H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}).}$

(iii) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of quasicoherent sheaves on X, then we have a $\boxed{\text{long exact sequence}}$ (20.1.0.1). The maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$ come from covariance, and similarly for $H^i(X, \mathcal{G}) \to H^i(X, \mathcal{H})$. The *connecting homomorphisms* $H^i(X, \mathcal{H}) \to H^{i+1}(X, \mathcal{F})$ will have to be defined.

(iv) If $f : X \to Y$ is any morphism of quasicompact separated schemes, and $\mathcal{F}$ is a quasicoherent sheaf on X, then there is a natural morphism $\boxed{H^i(Y, f_*\mathcal{F}) \to H^i(X, \mathcal{F})}$ extending $\Gamma(Y, f_*\mathcal{F}) \to \Gamma(X, \mathcal{F})$. (Note that f is quasicompact and separated by the Cancellation Theorem 11.1.19 for quasicompact and separated morphisms, taking $Z = \operatorname{Spec} k$ in the statement of the Cancellation Theorem, so $f_*\mathcal{F}$ is indeed a quasicoherent sheaf by Exercise 14.3.I.) We will later see this as part of a larger story, the *Leray spectral sequence* (Exercise 24.4.E). If $\mathcal{G}$ is a quasicoherent sheaf on Y, then setting $\mathcal{F} := f^*\mathcal{G}$ and using the adjunction map $\mathcal{G} \to f_*f^*\mathcal{G}$ and covariance of (ii) gives a natural **pullback map** $H^i(Y, \mathcal{G}) \to H^i(X, f^*\mathcal{G})$ (via $H^i(Y, \mathcal{G}) \to H^i(Y, f_*f^*\mathcal{G}) \to H^i(X, f^*\mathcal{G})$) extending $\Gamma(Y, \mathcal{G}) \to \Gamma(X, f^*\mathcal{G})$. In this way, H^i is a "contravariant functor in the space".

(v) If $f : X \to Y$ is an affine morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X, the natural map of (iv) is an isomorphism: $\boxed{H^i(Y, f_*\mathcal{F}) \xrightarrow{\sim} H^i(X, \mathcal{F})\,.}$ When f is a closed immersion and $Y = \mathbb{P}^N_A$, this isomorphism translates calculations on arbitrary projective A-schemes to calculations on $\mathbb{P}^N_A$.

(vi) If X can be covered by n affines, then $\boxed{H^i(X, \mathcal{F}) = 0}$ for $i \geq n$ for all $\mathcal{F}$. In particular, on affine schemes, all higher ($i > 0$) quasicoherent cohomology groups vanish. The vanishing of H^1 in this case, along with the long exact sequence **(iii)** implies that Γ is an exact functor for quasicoherent sheaves on affine schemes, something we already knew (Exercise 14.4.A). It is also true that if $\dim X = n$, then $H^i(X, \mathcal{F}) = 0$ for all $i > n$ and for all $\mathcal{F}$ (**dimensional vanishing**). We will prove this for projective A-schemes (Theorem 20.2.6) and even quasiprojective A-schemes (Exercise 20.2.I). See §20.2.8 for discussion of the general case.

20.1.1. *Side remark: the cohomological criterion for affineness.* The converse to **(vi)** in the case when $n = 1$ is Serre's *cohomological criterion for affineness:* in reasonable circumstances, a scheme, all of whose higher cohomology groups vanish for all quasicoherent sheaves, must be affine.

(vii) The functor H^i behaves well under direct sums, and more generally under colimits: $\boxed{H^i(X, \varinjlim \mathcal{F}_j) = \varinjlim H^i(X, \mathcal{F}_j).}$

(viii) We will also identify the cohomology of all $\mathcal{O}(m)$ on $\mathbb{P}^n_A$:

20.1.2. Theorem. —

- $H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ *is a free* A*-module of rank* $\binom{n+m}{n}$ *if* $i = 0$ *and* $m \geq 0$*, and* 0 *otherwise.*
- $H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ *is a free* A*-module of rank* $\binom{-m-1}{-n-m-1}$ *if* $m \leq -n-1$*, and* 0 *otherwise.*
- $H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = 0$ *if* $0 < i < n$.

We already have shown the first statement in Essential Exercise 15.1.C.

Theorem 20.1.2 has a number of features that will be the first appearances of facts that we will prove later.

- The cohomology of these bundles vanish above n (**(vi)** above)
- These cohomology groups are always *finitely-generated* A-modules. This will be true for all coherent sheaves on projective A-schemes (Theorem 20.1.3(i)), and indeed (with more work) on proper A-schemes (Theorem 20.8.1).
- The top cohomology group vanishes for $m > -n-1$. (This is a first appearance of *Kodaira vanishing*.)
- The top cohomology group is one-dimensional for $m = -n-1$ if $A = k$. This is the first appearance of the *dualizing sheaf*.
- There is a natural duality

$$H^i(X, \mathcal{O}(m)) \times H^{n-i}(X, \mathcal{O}(-n-1-m)) \to H^n(X, \mathcal{O}(-n-1))$$

This is the first appearance of *Serre duality*.

Before proving these facts, let's first use them to prove interesting things, as motivation. We begin with a single exercise to see get practice in computing cohomology groups.

20.1.A. EXERCISE. Compute the cohomology groups $H^i(\mathbb{A}^2_k \setminus \{(0,0)\}, \mathcal{O})$. (Hint: the case $i = 0$ was done in Example 5.4.1. The case $i > 1$ is clear from property **(vi)** above.) In particular, show that $H^1(\mathbb{A}^2_k \setminus \{(0,0)\}, \mathcal{O}) \neq 0$, and thus give another proof (see §5.4.3) of the fact that $\mathbb{A}^2_k \setminus \{(0,0)\}$ is not affine. (Cf. Serre's cohomological criterion for affineness, Remark 20.1.1.)

We now develop a string of interesting results out of the properties of cohomology described above. By Theorem 16.3.1, for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_A$ we can find a surjection $\mathcal{O}(m)^{\oplus j} \to \mathcal{F}$, which yields the exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}(m)^{\oplus j} \to \mathcal{F} \to 0 \tag{20.1.2.1}$$

for some coherent sheaf $\mathcal{G}$. We can use this to prove the following.

20.1.3. Theorem. — *(i) For any coherent sheaf $\mathcal{F}$ on a projective A-scheme X where A is Noetherian, $H^i(X, \mathcal{F})$ is a coherent (finitely generated) A-module.*
(ii) (Serre vanishing) Furthermore, for $m \gg 0$, $H^i(X, \mathcal{F}(m)) = 0$ for all $i > 0$ (even without Noetherian hypotheses).

A slightly fancier version of Serre vanishing will be given later.

Proof. Because cohomology of a closed scheme can be computed on the ambient space (**(v)** above), we may immediately reduce to the case $X = \mathbb{P}^n_A$.

(i) Consider the long exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathbb{P}^n_A, \mathcal{G}) \arrow[r] & H^0(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \arrow[r] & H^0(\mathbb{P}^n_A, \mathcal{F}) \arrow[r] & {} \\ & H^1(\mathbb{P}^n_A, \mathcal{G}) \arrow[r] & H^1(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \arrow[r] & H^1(\mathbb{P}^n_A, \mathcal{F}) \arrow[r] & \cdots \\ \cdots \arrow[r] & H^{n-1}(\mathbb{P}^n_A, \mathcal{G}) \arrow[r] & H^{n-1}(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \arrow[r] & H^{n-1}(\mathbb{P}^n_A, \mathcal{F}) \arrow[r] & {} \\ & H^n(\mathbb{P}^n_A, \mathcal{G}) \arrow[r] & H^n(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \arrow[r] & H^n(\mathbb{P}^n_A, \mathcal{F}) \arrow[r] & 0 \end{tikzcd}$$

The exact sequence ends here because $\mathbb{P}^n_A$ is covered by $n+1$ affines (**(vi)** above). Then $H^n(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j})$ is finitely generated by Theorem 20.1.2, hence $H^n(\mathbb{P}^n_A, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. Hence in particular, $H^n(\mathbb{P}^n_A, \mathcal{G})$ is finitely generated. As $H^{n-1}(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j})$ is finitely generated, and $H^n(\mathbb{P}^n_A, \mathcal{G})$ is too, we have that $H^{n-1}(\mathbb{P}^n_A, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. We continue inductively downwards.

(ii) Twist (20.1.2.1) by $\mathcal{O}(N)$ for $N \gg 0$. Then

$$H^n(\mathbb{P}^n_A, \mathcal{O}(m+N)^{\oplus j}) = \oplus_j H^n(\mathbb{P}^n_A, \mathcal{O}(m+N)) = 0$$

(by **(vii)** above), so $H^n(\mathbb{P}^n_A, \mathcal{F}(N)) = 0$. Translation: for any coherent sheaf, its top cohomology vanishes once you twist by $\mathcal{O}(N)$ for N sufficiently large. Hence this is true for $\mathcal{G}$ as well. Hence from the long exact sequence, $H^{n-1}(\mathbb{P}^n_A, \mathcal{F}(N)) = 0$ for $N \gg 0$. As in (i), we induct downwards, until we get that $H^1(\mathbb{P}^n_A, \mathcal{F}(N)) = 0$. (The induction stops here, as it is *not* true that $H^0(\mathbb{P}^n_A, \mathcal{O}(m+N)^{\oplus j}) = 0$ for large N — quite the opposite.) □

20.1.B. ⋆⋆ EXERCISE FOR THOSE WHO LIKE NON-NOETHERIAN RINGS. Prove part (i) in the above result without the Noetherian hypotheses, assuming only that A is a coherent A-module (A is "coherent over itself"). (Hint: induct downwards as before. Show the following in order: $H^n(\mathbb{P}^n_A, \mathcal{F})$ finitely generated, $H^n(\mathbb{P}^n_A, \mathcal{G})$ finitely generated, $H^n(\mathbb{P}^n_A, \mathcal{F})$ coherent, $H^n(\mathbb{P}^n_A, \mathcal{G})$ coherent, $H^{n-1}(\mathbb{P}^n_A, \mathcal{F})$ finitely generated, $H^{n-1}(\mathbb{P}^n_A, \mathcal{G})$ finitely generated, etc.)

In particular, we have proved the following, that we would have cared about even before we knew about cohomology.

20.1.4. Corollary. — *Any projective* k*-scheme has a finite-dimensional space of global sections. More generally, if* A *is Noetherian and* $\mathcal{F}$ *is a coherent sheaf on a projective* A*-scheme, then* $H^0(X, \mathcal{F})$ *is a coherent* A*-module.*

(We will generalize this in Theorem 20.7.1.) I want to emphasize how remarkable this proof is. It is a question about global sections, i.e. H^0, which we think of as the most down to earth cohomology group, yet the proof is by downward induction for H^n, starting with n large.

Corollary 20.1.4 is true more generally for proper k-schemes, not just projective k-schemes (see Theorem 20.8.1).

Here are some important consequences. They can also be shown directly, without the use of cohomology, but with much more elbow grease. We begin with the analogue of the following fact in complex analysis: the only holomorphic functions on a compact complex manifold are locally constant (because of the maximum principle).

20.1.C. EXERCISE (THE ONLY FUNCTIONS ON PROJECTIVE INTEGRAL SCHEMES ARE CONSTANTS). Suppose X is a projective integral scheme over an algebraically closed field. Show that $h^0(X, \mathcal{O}_X) = 1$. Hint: show that $H^0(X, \mathcal{O}_X)$ is a finite-dimensional k-algebra, and a domain. Hence show it is a field. (For experts: the same argument holds with the weaker hypotheses where X is proper, geometrically connected and geometrically reduced (§10.4.2), over an arbitrary field. The key facts needed are the extension of Corollary 20.1.4 to proper morphisms mentioned above, given in Theorem 20.8.1, and Exercise 20.2.G.)

20.1.5. As a partial converse, if $h^0(X, \mathcal{O}_X) = 1$, then X is connected (why?), but need not be reduced: witness the subscheme in $\mathbb{P}^2$ cut out by $x^2 = 0$. (For experts: the geometrically connected hypothesis is necessary, as $X = \operatorname{Spec} \mathbb{C}$ is a projective integral $\mathbb{R}$-scheme, with $h^0(X, \mathcal{O}_X) = 2$. Similarly, a nontrivial purely inseparable field extension can be used to show that the geometrically reduced hypothesis is also necessary.)

20.1.D. EXERCISE (THE $S_\bullet$-MODULE ASSOCIATED TO A COHERENT SHEAF ON $\operatorname{Proj} S_\bullet$ IS COHERENT, PROMISED IN REMARK 16.4.3). Suppose $S_\bullet$ is a finitely generated graded ring generated in degree 1 over a Noetherian ring A, and $\mathcal{F}$ is a coherent sheaf on $\operatorname{Proj} S_\bullet$. Show that $\Gamma_\bullet \mathcal{F}$ is a coherent $S_\bullet$-module. (Feel free to remove the generation in degree 1 hypothesis.)

20.1.E. CRUCIAL EXERCISE (PUSHFORWARDS OF COHERENTS ARE COHERENT). Suppose $f : X \to Y$ is a projective morphism of Noetherian schemes. Show that the pushforward of a coherent sheaf on X is a coherent sheaf on Y. (See Grothendieck's Coherence Theorems 20.7.1 and 20.8.1 for generalizations.)

20.1.6. *Unimportant remark, promised in Exercise 17.2.C.* As a consequence, if $f : X \to Y$ is a finite morphism, and $\mathcal{O}_Y$ is coherent over itself, then f_* sends coherent sheaves on X to coherent sheaves on Y.

Finite morphisms are affine (from the definition) and projective (18.3.G). We can now show that this is a characterization of finiteness.

20.1.7. Corollary. — *If $\pi : X \to Y$ is projective and affine and Y is locally Noetherian, then π is finite.*

We will see in Exercise 20.8.A that the projective hypotheses can be relaxed to proper.

Proof. By Exercise 20.1.E, $\pi_* \mathcal{O}_X$ is coherent and hence finite type. □

20.1.F. EXERCISE. Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on projective X with $\mathcal{F}$ coherent. Show that for $n \gg 0$,

$$0 \to H^0(X, \mathcal{F}(n)) \to H^0(X, \mathcal{G}(n)) \to H^0(X, \mathcal{H}(n)) \to 0$$

is also exact. (Hint: for $n \gg 0$, $H^1(X, \mathcal{F}(n)) = 0$.)

20.2 Definitions and proofs of key properties

This section could be read much later; the facts we will use are all stated in the previous section. However, the arguments are not complicated, so you want to read this right away. As you read this, you should go back and check off all the facts in the previous section, to assure yourself that you understand everything promised.

20.2.1. Čech cohomology. Čech cohomology in general settings is often defined using a limit over finer and finer covers of a space. In our algebro-geometric setting, the situation is much cleaner, and we can use a single cover.

Suppose X is quasicompact and separated, for example if X is quasiprojective over A. In particular, X may be covered by a finite number of affine open sets, and the intersection of any two affine open sets is also an affine open set (by separatedness, Proposition 11.1.8). We will use quasicompactness and separatedness only in order to ensure these two nice properties.

Suppose $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{U} = \{U_i\}_{i=1}^n$ is a *finite* collection of affine open sets covering X. For $I \subset \{1, \dots, n\}$ define $U_I = \cap_{i \in I} U_i$, which is affine by the separated hypothesis. (The strong analogy for those who have seen cohomology in other contexts: cover a topological space X with a finite number of open sets U_i, such that all intersections $\cap_{i \in I} U_i$ are contractible.) Consider the **Čech complex**

$$0 \to \prod_{\substack{|I| = 1 \\ I \subset \{1, \dots, n\}}} \mathcal{F}(U_I) \to \cdots \to \prod_{\substack{|I| = i \\ I \subset \{1, \dots, n\}}} \mathcal{F}(U_I) \to \prod_{\substack{|I| = i+1 \\ I \subset \{1, \dots, n\}}} \mathcal{F}(U_I) \to \cdots . \tag{20.2.1.1}$$

The maps are defined as follows. The map from $\mathcal{F}(U_I) \to \mathcal{F}(U_J)$ is 0 unless $I \subset J$, i.e. $J = I \cup \{j\}$. If j is the kth element of J, then the map is $(-1)^{k-1}$ times the restriction map res_{U_I, U_J}.

20.2.A. EASY EXERCISE (FOR THOSE WHO HAVEN'T SEEN ANYTHING LIKE THE ČECH COMPLEX BEFORE). Show that the Čech complex is indeed a complex, i.e. that the composition of two consecutive arrows is 0.

Define $H^i_{\mathcal{U}}(X, \mathcal{F})$ to be the ith cohomology group of the complex (20.2.1.1). Note that if X is an A-scheme, then $H^i_{\mathcal{U}}(X, \mathcal{F})$ is an A-module. We have almost succeeded in defining the Čech cohomology group H^i, except our definition seems to depend on a choice of a cover $\mathcal{U}$.

20.2.B. EASY EXERCISE. Show that $H^0_{\mathcal{U}}(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$. (Hint: use the sheaf axioms for $\mathcal{F}$.)

20.2.C. EXERCISE. Suppose $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves on a topological space, and $\mathcal{U}$ is an open cover such that on any intersection of open subsets in $\mathcal{U}$, the sections of $\mathcal{F}_2$ surject onto $\mathcal{F}_3$. (Note that this applies in our case!) Show that we get a "long exact sequence of cohomology for $H^i_{\mathcal{U}}$".

20.2.2. Theorem/Definition. — *Our standing assumption is that* X *is quasicompact and separated.* $H^i_{\mathcal{U}}(X, \mathcal{F})$ *is independent of the choice of (finite) cover* $\{U_i\}$*. More precisely, for any two covers* $\{U_i\} \subset \{V_i\}$*, the maps* $H^i_{\{V_i\}}(X, \mathcal{F}) \to H^i_{\{U_i\}}(X, \mathcal{F})$ *induced by the natural maps of Čech complexes* (20.2.1.1) *are isomorphisms. Define the Čech cohomology group* $H^i(X, \mathcal{F})$ *to be this group.*

If you are unsure of what the "natural maps of Čech complexes" is, by (20.2.3.1) it should become clear.

20.2.3. For experts: maps of complexes inducing isomorphisms on cohomology groups are called *quasiisomorphisms*. We are actually getting a finer invariant than cohomology out of this construction; we are getting an element of the *derived category of* A*-modules*.

Proof. We need only prove the result when $|\{V_i\}| = |\{U_i\}| + 1$. We will show that if $\{U_i\}_{1 \leq i \leq n}$ is a cover of X, and U_0 is any other open set, then the map $H^i_{\{U_i\}_{0 \leq i \leq n}}(X, \mathcal{F}) \to H^i_{\{U_i\}_{1 \leq i \leq n}}(X, \mathcal{F})$ is an isomorphism. Consider the exact sequence of complexes

(20.2.3.1)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \prod_{\substack{|I| = i-1 \\ 0 \in I}} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{\substack{|I| = i \\ 0 \in I}} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{\substack{|I| = i+1 \\ 0 \in I}} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \prod_{|I| = i-1} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I| = i} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I| = i+1} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \prod_{\substack{|I| = i-1 \\ 0 \notin I}} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{\substack{|I| = i \\ 0 \notin I}} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{\substack{|I| = i+1 \\ 0 \notin I}} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Throughout, $I \subset \{0, \ldots, n\}$. The bottom two rows are Čech complexes with respect to two covers, and the map between them induces the desired map on cohomology. We get a long exact sequence of cohomology from this short exact sequence of complexes (Exercise 2.6.C). Thus we wish to show that the top row is exact and thus has vanishing cohomology. (Note that $U_0 \cap U_j$ is affine by our separatedness hypothesis, Proposition 11.1.8.) But the ith cohomology of the top row is precisely $H^i_{\{U_i \cap U_0\}_{i>0}}(U_i, \mathcal{F})$ except at step 0, where we get 0 (because the complex starts off

$0 \to \mathcal{F}(U_0) \to \prod_{j=1}^n \mathcal{F}(U_0 \cap U_j))$. So it suffices to show that higher Čech groups of affine schemes are 0. Hence we are done by the following result. $\square$

20.2.4. Theorem. — *The higher Čech cohomology $H^i_{\mathcal{U}}(X, \mathcal{F})$ of an affine A-scheme X vanishes (for any affine cover $\mathcal{U}$, $i > 0$, and quasicoherent $\mathcal{F}$).*

Serre describes this as a partition of unity argument.

Proof. (The following argument can be made shorter using spectral sequences, but we avoid this for the sake of clarity.) We want to show that the "extended" complex

$$0 \to \mathcal{F}(X) \to \prod_{|I|=1} \mathcal{F}(U_I) \to \prod_{|I|=2} \mathcal{F}(U_I) \to \cdots \tag{20.2.4.1}$$

(where the global sections $\mathcal{F}(X)$ have been appended to the start) has no cohomology, i.e. is exact. We do this with a trick.

Suppose first that some U_i, say U_0, is X. Then the complex is the middle row of the following short exact sequence of complexes

(20.2.4.2)

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \prod_{|I|=1, 0 \in I} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I|=2, 0 \in I} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{F}(X) \arrow[r] \arrow[d] & \prod_{|I|=1} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I|=2} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{F}(X) \arrow[r] & \prod_{|I|=1, 0 \notin I} \mathcal{F}(U_I) \arrow[r] & \prod_{|I|=2, 0 \notin I} \mathcal{F}(U_I) \arrow[r] & \cdots
\end{tikzcd}$$

The top row is the same as the bottom row, slid over by 1. The corresponding long exact sequence of cohomology shows that the central row has vanishing cohomology. (You should show that the "connecting homomorphism" on cohomology is indeed an isomorphism.) This might remind you of the *mapping cone* construction (Exercise 2.7.E).

We next prove the general case by sleight of hand. Say $X = \operatorname{Spec} R$. We wish to show that the complex of A-modules (20.2.4.1) is exact. It is also a complex of R-modules, so we wish to show that the complex of R-modules (20.2.4.1) is exact. To show that it is exact, it suffices to show that for a cover of $\operatorname{Spec} R$ by distinguished open sets $D(f_i)$ ($1 \leq i \leq r$) (i.e. $(f_1, \dots, f_r) = 1$ in R) the complex is exact. (Translation: exactness of a sequence of sheaves may be checked locally.) We choose a cover so that each $D(f_i)$ is contained in some $U_j = \operatorname{Spec} A_j$. Consider the complex localized at f_i. As

$$\Gamma(\operatorname{Spec} A, \mathcal{F})_f = \Gamma(\operatorname{Spec}(A_j)_f, \mathcal{F})$$

(by quasicoherence of $\mathcal{F}$, Exercise 14.3.D), as $U_j \cap D(f_i) = D(f_i)$, we are in the situation where one of the U_i's is X, so we are done. $\square$

We have now proved properties **(i)–(iii)** of the previous section.

20.2.D. EXERCISE (PROPERTY **(v)**). Suppose $f : X \to Y$ is an affine morphism, and Y is a quasicompact and separated A-scheme (and hence X is too, as affine morphisms are both quasicompact and separated). If $\mathcal{F}$ is a quasicoherent sheaf on X, describe a natural isomorphism $H^i(Y, f_*\mathcal{F}) \cong H^i(X, \mathcal{F})$. (Hint: if $\mathcal{U}$ is an affine cover of Y, "$f^{-1}(\mathcal{U})$" is an affine cover X. Use these covers to compute the cohomology of $\mathcal{F}$.)

20.2.E. EXERCISE (PROPERTY **(iv)**). Suppose $f : X \to Y$ is any quasicompact separated morphism, $\mathcal{F}$ is a quasicoherent sheaf on X, and Y is a quasicompact separated A-scheme. The hypotheses on f ensure that $f_*\mathcal{F}$ is a quasicoherent sheaf on Y. Describe a natural morphism $H^i(Y, f_*\mathcal{F}) \to H^i(X, \mathcal{F})$ extending $\Gamma(Y, f_*\mathcal{F}) \to \Gamma(X, \mathcal{F})$.

20.2.F. UNIMPORTANT EXERCISE. Prove Property **(vii)** of the previous section. (This can be done by hand. Hint: in the category of modules over a ring, taking the colimit over a directed sets is an exact functor, §2.6.11.)

20.2.5. Useful facts about cohomology for k-schemes.

20.2.G. EXERCISE (COHOMOLOGY AND CHANGE OF BASE FIELD). Suppose X is a quasicompact separated k-scheme, and $\mathcal{F}$ is a coherent sheaf on X. Give an isomorphism

$$H^i(X, \mathcal{F}) \otimes_k K \cong\!\!= H^i(X \times_{\operatorname{Spec} k} \operatorname{Spec} K, \mathcal{F} \otimes_k K)$$

for all i, where K/k is any field extension. Here $\mathcal{F} \otimes_k k$ means the pullback of $\mathcal{F}$ to $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. Hence $h^i(X, \mathcal{F}) = h^i(X \times_{\operatorname{Spec} k} \operatorname{Spec} K, \mathcal{F} \otimes_k K)$. If $i = 0$ (taking $H^0 = \Gamma$), show the result without the quasicompact and separated hypotheses. (This is useful for relating facts about k-schemes to facts about schemes over algebraically closed fields. Your proof might use vector spaces — i.e. linear algebra — in a fundamental way. If it doesn't, you may prove something more general, if $k \to K$ is replaced by a flat ring map $B \to A$. Recall that $B \to A$ is flat if $\otimes_B A$ is an exact functor $Mod_B \to Mod_A$. A hint for this harder exercise: the FHHF theorem, Exercise 2.6.H. See Exercise 20.7.B(b) for the next generalization of this.)

20.2.H. EXERCISE (BASE-POINT-FREENESS IS INDEPENDENT OF EXTENSION OF BASE FIELD). Suppose X is a scheme over a field k, $\mathcal{L}$ is an invertible sheaf on X, and K/k is a field extension. Show that $\mathcal{L}$ is base-point-free if and only if its pullback to $X \otimes_{\operatorname{Spec} k} \operatorname{Spec} K$ is base-point-free. (Hint: Exercise 20.2.G with $i = 0$ implies that a basis of sections of $\mathcal{L}$ over k becomes, after tensoring with K, a basis of sections of $\mathcal{L} \otimes_k K$.)

20.2.6. Theorem (dimensional vanishing for quasicoherent sheaves on projective k-schemes). — *Suppose X is a projective k-scheme, and $\mathcal{F}$ is a quasicoherent sheaf on X. Then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.*

In other words, cohomology vanishes above the dimension of X. It turns out that n affine open sets are necessary. (One way of proving this is by showing that the complement of an affine set is always pure codimension 1.)

Proof. Suppose $X \hookrightarrow \mathbb{P}^N$, and let $n = \dim X$. We show that X may be covered by n affine open sets. Exercise 12.3.C shows that there are n effective Cartier divisors on $\mathbb{P}^N$ such that their complements $U_0, \ldots, U_n$ cover X. Then U_i is affine, so $U_i \cap X$ is affine, and thus we have covered X with n affine open sets. $\square$

20.2.7. $\star$ *Dimensional vanishing more generally.* Using the theory of blowing up (Chapter 19), Theorem 20.2.6 can be extended to quasiprojective k-schemes. Suppose X is a quasiprojective k-variety of dimension n. We show that X may be covered by $n+1$ affine open subsets. As X is quasiprojective, there is some projective variety Y with an open immersion $X \hookrightarrow Y$. By replacing Y with the closure of X in Y, we may assume that $\dim Y = n$. Put any subscheme structure Z on the complement of X in Y (for example the reduced subscheme structure, §9.3.8). Let $Y' = \mathrm{Bl}_Z Y$. Then Y' is a projective variety (§19.3.1), which can be covered by $n+1$ affine open subsets. The complement of X in Y' is an affective Cartier divisor ($E_Z Y$), so the restriction to X of each of these affine open subsets of Y is also affine, by Exercise 8.3.F. (You might then hope that *any* dimension n variety can be covered by $n+1$ affine open subsets. This is not true. For each integer m, there is a threefold that requires at least m affine open sets to cover it, see [**RV**, Ex. 4.9].)

(Here is a fact useful in invariant theory, which can be proved in the same way. Suppose $p_1, \dots, \dots, p_n$ are closed points on a quasiprojective k-variety X. Then there is an affine open subset of X containing all of them.)

20.2.I. EXERCISE (DIMENSIONAL VANISHING FOR QUASIPROJECTIVE VARIETIES). Suppose X is a quasiprojective k-scheme of dimension d. Show that for any quasicoherent sheaf $\mathcal{F}$ on X, $H^i(X, \mathcal{F}) = 0$ for $i > d$.

20.2.8. *Dimensional vanishing most generally.* Dimensional vanishing is even true in much greater generality. To state it, we need to define cohomology with the more general machinery of derived functors (Chapter 24). If X is a Noetherian topological space (§4.6.3) and $\mathcal{F}$ is any sheaf of abelian groups on X, we have $H^i(X, \mathcal{F}) = 0$ for all $i > \dim X$. (See [**Ha**, Theorem III.2.7] for Grothendieck's elegant proof.) In particular, if X is a k-variety of dimension n, we *always* have dimensional vanishing, even for crazy varieties that can't be covered with $n+1$ affine open subsets (§20.2.7).

20.3 Cohomology of line bundles on projective space

We now finally prove the last promised basic fact about cohomology, property **(viii)** of §20.1, Theorem 20.1.2, on the cohomology of line bundles on projective space. More correctly, we will do one case and you will do the rest.

20.3.1. *Remark.* Essential Exercise 15.1.C and the ensuing discussion showed that $H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ should be interpreted as the homogeneous degree m polynomials in $x_0, \dots, x_n$ (with A-coefficients). Similarly, $H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ should be interpreted as the homogeneous degree m Laurent polynomials in $x_0, \dots, x_n$, where in each monomial, each x_i appears with degree at most -1.

20.3.2. *Proof of Theorem 20.1.2 for $n = 2$.* We take the standard cover $U_0 = D(x_0)$, $\dots$, $U_n = D(x_n)$ of $\mathbb{P}^n_A$.

20.3.A. EXERCISE. If $I \subset \{1, \dots, n\}$, then give an isomorphism (of A-modules) of $\Gamma(\mathcal{O}(m), U_I)$ with the Laurent monomials (in x_0, $\dots$, x_n, with coefficients in A) where each x_i for $i \notin I$ appears with non-negative degree. Your construction should be such that the restriction map $\Gamma(\mathcal{O}(m), U_I) \to \Gamma(\mathcal{O}(m), U_J)$ $(I \subset J)$ corresponds to the natural inclusion: a Laurent polynomial in $\Gamma(\mathcal{O}(m), U_I)$ maps to the *same* Laurent polynomial in $\Gamma(\mathcal{O}(m), U_J)$.

The Čech complex for $\mathcal{O}(m)$ is the degree m part of

(20.3.2.1)
$$0 \longrightarrow A[x_0, x_1, x_2, x_0^{-1}] \times A[x_0, x_1, x_2, x_1^{-1}] \times A[x_0, x_1, x_2, x_2^{-1}] \longrightarrow$$
$$A[x_0, x_1, x_2, x_0^{-1}, x_1^{-1}] \times A[x_0, x_1, x_2, x_1^{-1}, x_2^{-1}] \times A[x_0, x_1, x_2, x_0^{-1}, x_2^{-1}]$$
$$\longrightarrow A[x_0, x_1, x_2, x_0^{-1}, x_1^{-1}, x_2^{-1}] \longrightarrow 0.$$

Rather than consider $\mathcal{O}(m)$ for each m independently, it is notationally simpler to consider them all at once, by considering $\mathcal{F} = \oplus_{m \in \mathbb{Z}} \mathcal{O}(m)$: the Čech complex for $\mathcal{F}$ is (20.3.2.1). It is useful to write which U_I corresponds to which factor (see (20.3.2.2) below). The maps (from one factor of one term to one factor of the next) are all natural inclusions, or negative of natural inclusions, and in particular preserve degree.

We extend (20.3.2.1) by replacing the $0 \to$ on the left by $0 \to A[x_0, x_1, x_2] \to$:

(20.3.2.2)
$$\begin{array}{ccccccccccc} & & H^0 & & U_0\ U_1\ U_2 & & & & U_{012} & & \\ 0 & \longrightarrow & A[x_0, x_1, x_2] & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & A[x_0, x_1, x_2, x_0^{-1}, x_1^{-1} x_2^{-1}] & \longrightarrow & 0. \end{array}$$

20.3.B. EXERCISE. Show that if (20.3.2.2) is exact, except that at U_{012} the cohomology/cokernel is $A[x_0^{-1}, x_1^{-1}, x_2^{-1}]$, then Theorem 20.1.2 holds for $n = 2$. (Hint: Remark 20.3.1.)

Because the maps in (20.3.2.2) preserve multidegree (degrees of each x_i independently), we can study exactness of (20.3.2.2) monomial by monomial.

The "0-positive" case. Consider first the monomial $x_0^{a_0} x_1^{a_1} x_2^{a_2}$, where the exponents a_i are all negative. Then (20.3.2.2) in this multidegree is:

$$0 \longrightarrow 0_{H^0} \longrightarrow 0_0 \times 0_1 \times 0_2 \longrightarrow 0_{01} \times 0_{12} \times 0_{02} \longrightarrow A_{012} \longrightarrow 0.$$

Here the subscripts serve only to remind us which "Čech" terms the factors correspond to. (For example, A_{012} corresponds to the coefficient of $x_0^{a_0} x_1^{a_1} x_2^{a_2}$ in $A[x_0, x_1, x_2, x_0^{-1}, x_1^{-1}, x_2^{-1}]$.) Clearly this complex only has (co)homology at the U_{012} spot, as desired.

The "1-positive" case. Consider next the case where *two* of the exponents, say a_0 and a_1, are negative. Then the complex in this multidegree is

$$0 \longrightarrow 0_{H^0} \longrightarrow 0_0 \times 0_1 \times 0_2 \longrightarrow A_{01} \times 0_{12} \times 0_{02} \longrightarrow A_{012} \longrightarrow 0,$$

which is clearly exact.

The "2-positive" case. We next consider the case where *one* of the exponents, say a_0, is negative. Then the complex in this multidegree is

$$0 \longrightarrow 0_{H^0} \longrightarrow A_0 \times 0_1 \times 0_2 \longrightarrow A_{01} \times 0_{12} \times A_{02} \longrightarrow A_{012} \longrightarrow 0$$

With a little thought (paying attention to the signs on the arrows $A \to A$), you will see that it is exact. (The subscripts, by reminding us of the subscripts in the original Čech complex, remind us what signs to take in the maps.)

The "3-positive" case. Finally, consider the case where *none* of the exponents are negative. Then the complex in this multidegree is

$$0 \longrightarrow A_{H^0} \longrightarrow A_0 \times A_1 \times A_2 \longrightarrow A_{01} \times A_{12} \times A_{02} \longrightarrow A_{012} \longrightarrow 0$$

We wish to show that this is exact. We write this complex as the middle of a short exact sequence of complexes:

(20.3.2.3)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_{02} \times A_{12} \arrow[r] \arrow[d] & A_{012} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & A_{H^0} \arrow[r] \arrow[d] & A_0 \times A_1 \times A_2 \arrow[r] \arrow[d] & A_{01} \times A_{12} \times A_{02} \arrow[r] \arrow[d] & A_{012} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A_{H^0} \arrow[r] & A_0 \times A_1 \arrow[r] & A_{01} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Thus we get a long exact sequence in cohomology (Theorem 2.6.5). But the top and bottom rows are exact (basically from the 2-positive case), i.e. cohomology-free, so the middle row must be exact too.

20.3.C. EXERCISE. Prove Theorem 20.1.2 for general n. (I could of course just have given you the proof for general n, but seeing the argument in action may be enlightening. In particular, your argument may be much shorter. For example, the 1-positive case could be done in the same way as the 2-positive case, so you will not need $n + 1$ separate cases if you set things up carefully.)

20.3.3. *Remarks. (i)* In fact we don't really need the exactness of the top and bottom rows of (20.3.2.3); we just need that they are the same, just as with (20.2.4.2).

(ii) This argument is basically the proof that the reduced homology of the boundary of a simplex S (known in some circles as a "sphere") is 0, unless S is the empty set, in which case it is one-dimensional. The "empty set" case corresponds to the "0-positive" case.

20.3.D. EXERCISE. Show that $H^i(\mathbb{P}^m_k \times_k \mathbb{P}^n_k, \mathcal{O}(a,b)) = \sum_{j=0}^i H^j(\mathbb{P}^m_k, \mathcal{O}(a)) \otimes_k H^{i-j}(\mathbb{P}^n_k, \mathcal{O}(b))$. (Can you generalize this Kunneth-type formula further?)

20.4 Riemann-Roch, degrees of coherent sheaves, arithmetic genus, and Serre duality

We have seen some powerful uses of Čech cohomology, to prove things about spaces of global sections, and to prove Serre vanishing. We will now see some classical constructions come out very quickly and cheaply.

In this section, we will work over a field k. Suppose $\mathcal{F}$ is a coherent sheaf on a projective k-scheme X. Recall the notation (§20.1) $h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$. By Theorem 20.1.3, $h^i(X, \mathcal{F})$ is finite. (The arguments in this section will extend without change to proper X once we have this finiteness for proper morphisms, by Grothendieck's Coherence Theorem 20.8.1.) Define the **Euler characteristic**

$$\chi(X,\mathcal{F}) := \sum_{i=0}^{\dim X} (-1)^i h^i(X,\mathcal{F}).$$

We will see repeatedly here and later that Euler characteristics behave better than individual cohomology groups. As one sign, notice that for fixed n, and $m \geq 0$,

$$h^0(\mathbb{P}_k^n, \mathcal{O}(m)) = \binom{n+m}{m} = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

Notice that the expression on the right is a polynomial in m of degree n. (For later reference, notice also that the leading coefficient is $m^n/n!$.) But it is not true that

$$h^0(\mathbb{P}_k^n, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}$$

for *all* m — it breaks down for $m \leq -n-1$. Still, you can check (using Theorem 20.1.2) that

$$\chi(\mathbb{P}_k^n, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

So one lesson is this: if one cohomology group (usual the top or bottom) behaves well in a certain range, and then messes up, likely it is because (i) it is actually the Euler characteristic which behaves well *always*, and (ii) the other cohomology groups vanish in that cetain range.

In fact, we will see that it is often hard to calculate cohomology groups (even h^0), but it can be easier calculating Euler characteristics. So one important way of getting a hold of cohomology groups is by computing the Euler characteristics, and then showing that all the *other* cohomology groups vanish. Hence the ubiquity and importance of *vanishing theorems*. (A vanishing theorem usually states that a certain cohomology group vanishes under certain conditions.) We will see this in action when discussing curves. (One of the first applications will be (21.2.4.1).)

The following exercise shows another way in which Euler characteristic behaves well: it is *additive in exact sequences*.

20.4.A. EXERCISE. Show that if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on a projective k-scheme X, then $\chi(X,\mathcal{G}) = \chi(X,\mathcal{F}) + \chi(X,\mathcal{H})$. (Hint: consider the long exact sequence in cohomology.) More generally, if

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

is an exact sequence of sheaves, show that

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

20.4.1. The Riemann-Roch Theorem for line bundles on a nonsingular projective curve. Suppose $D := \sum_{p \in C} a_p[p]$ is a divisor on a nonsingular projective curve C over a field k (where $a_p \in \mathbb{Z}$, and all but finitely many a_p are 0). Define the **degree of** D by

$$\deg D = \sum a_p \deg p.$$

(The degree of a point p was defined in §6.3.8, as the degree of the field extension of the residue field over k.)

20.4.B. ESSENTIAL EXERCISE: THE RIEMANN-ROCH THEOREM FOR LINE BUNDLES ON A NONSINGULAR PROJECTIVE CURVE. Show that

$$\chi(C, \mathcal{O}_C(D)) = \deg D + \chi(C, \mathcal{O}_C)$$

by induction on $\sum |a_p|$ (where $D = \sum a_p[p]$ as above). Hint: to show that $\chi(C, \mathcal{O}_C(D)) = \deg p + \chi(C, \mathcal{O}_C(D-p))$, tensor the closed subscheme exact sequence

$$0 \to \mathcal{O}_C(-p) \to \mathcal{O}_C \to \mathcal{O}|_p \to 0$$

(where $\mathcal{O}|_p$ is the structure sheaf of the scheme p, not the stalk $\mathcal{O}_{C,p}$) by $\mathcal{O}_C(D)$, and use additivity of Euler characteristics in exact sequences (Exercise 20.4.A).

As every invertible sheaf $\mathcal{L}$ is of the form $\mathcal{O}_C(D)$ for some D (see §15.2), this exercise is very powerful.

20.4.C. IMPORTANT EXERCISE. Suppose $\mathcal{L}$ is an invertible sheaf on a nonsingular projective curve C over k. Define the **degree** of $\mathcal{L}$ as $\chi(C, \mathcal{L}) - \chi(C, \mathcal{O}_C)$. Let s be a non-zero rational section on C. Let D be the divisor of zeros and poles of s:

$$D := \sum_{p \in C} \nu_p(s)[p]$$

Show that $\deg \mathcal{L} = \deg D$. In particular, the degree can be computed by counting zeros and poles of *any* section not vanishing on a component of C.

20.4.D. EXERCISE. Give a new solution to Exercise 18.4.E (roughly, a nonzero rational function on a projective curve has the same number of zeros and poles, counted appropriately) using the ideas above.

20.4.E. EXERCISE. If $\mathcal{L}$ and $\mathcal{M}$ are two line bundles on a nonsingular projective curve C, show that $\deg \mathcal{L} \otimes \mathcal{M} = \deg \mathcal{L} + \deg \mathcal{M}$. (Hint: choose rational sections of $\mathcal{L}$ and $\mathcal{M}$.)

20.4.F. EXERCISE. Suppose $f : C \to C'$ is a degree d morphism of integral projective nonsingular curves, and $\mathcal{L}$ is an invertible sheaf on C'. Show that $\deg_C f^*\mathcal{L} = d \deg_{C'} \mathcal{L}$. Hint: compute $\deg_{\mathcal{L}}$ using any non-zero rational section s of $\mathcal{L}$, and compute $\deg f^*\mathcal{L}$ using the rational section f^*s of $f^*\mathcal{L}$. Note that zeros pull back to zeros, and poles pull back to poles. Reduce to the case where $\mathcal{L} = \mathcal{O}(p)$ for a single point p. Use Exercise 18.4.D.

20.4.G. $\star\star$ EXERCISE (COMPLEX-ANALYTIC INTERPRETATION OF DEGREE; ONLY FOR THOSE WITH SUFFICIENT ANALYTIC BACKGROUND). Suppose X is a connected nonsingular projective complex curve. Show that the degree map is the composition of group homomorphisms

$$\operatorname{Pic} X \longrightarrow \operatorname{Pic} X_{an} \xrightarrow{c_1} H^2(X_{an}, \mathbb{Z}) \xrightarrow{\cap [X_{an}]} H_0(X_{an}, \mathbb{Z}) \cong \mathbb{Z}.$$

Hint: show it for a generator $\mathcal{O}(p)$ of the group $\operatorname{Pic} X$, using explicit transition functions. (The first map was discussed in Exercise 14.1.J. The second map is takes a line bundle to its first Chern class, and can be interpreted as follows. The transition functions for a line bundle yield a Čech 1-cycle for $\mathcal{O}^*_{X_{an}}$; this yields a map $\operatorname{Pic} X_{an} \to H^1(X_{an}, \mathcal{O}^*_{X_{an}})$. Combining this with the map $H^1(X_{an}, \mathcal{O}^*_{X_{an}}) \to H^2(X_{an}, \mathbb{Z})$ from the long exact sequence in cohomology corresponding to the exponential exact sequence (3.4.9.1) yields the first Chern class map.)

20.4.2. Arithmetic genus.

Motivated by geometry, we define the **arithmetic genus** of a scheme X as $1 - \chi(X, \mathcal{O}_X)$. This is sometimes denoted $p_a(X)$. For irreducible reduced curves over an algebraically closed field, as $h^0(X, \mathcal{O}_X) = 1$ (Exercise 20.1.C), $p_a(X) = h^1(X, \mathcal{O}_X)$. (In higher dimension, this is a less natural notion.)

We can restate the Riemann-Roch formula for curves (Exercise 20.4.B) as:

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} - p_a(C) + 1.$$

This is the most common formulation of the Riemann-Roch formula.

20.4.3. Miracle. If C is a nonsingular irreducible projective complex curve, then the corresponding complex-analytic object, a compact *Riemann surface*, has a notion called the *genus* g, which is the number of holes (see Figure 20.1). Miraculously, $g = p_a$ in this case (see Exercise 23.5.H), and for this reason, we will often write g for p_a when discussing nonsingular (projective irreducible) curves, over any field. We will discuss genus further in §20.5.3, when we will be able to compute it in many interesting cases. (Warning: the arithmetic genus of $\mathbb{P}^1_{\mathbb{C}}$ as an $\mathbb{R}$-variety is -1!)

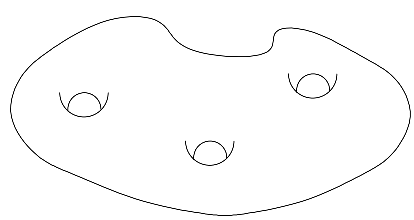


FIGURE 20.1. A genus 3 Riemann surface

20.4.4. Serre duality.

Another common version of Riemann-Roch involves Serre duality, which unlike Riemann-Roch is *hard*.

20.4.5. Theorem (Serre duality for smooth projective varieties). — *Suppose X is a geometrically irreducible smooth k-variety, of dimension n. Then there is an invertible sheaf $\mathcal{K}$ on X such that*

$$h^i(X, \mathcal{F}) = h^{n-i}(X, \mathcal{K} \otimes \mathcal{F}^\vee)$$

for all $i \in \mathbb{Z}$ and all coherent sheaves $\mathcal{F}$.

20.4.6. This is a simpler version of a better statement, which we will prove later ((27.1.1.1) and Important Exercise 27.6.E. The *dualizing sheaf* $\mathcal{K}$ is the determinant of the cotangent bundle $\Omega_{X/k}$ of X, but we haven't yet defined the cotangent bundle. (We will discuss differentials, and the cotangent bundle, in Chapter 23.) This equality is a consequence of a perfect pairing

$$H^i(X, \mathcal{F}) \times H^{n-i}(X, \mathcal{K} \otimes \mathcal{F}^\vee) \to H^n(X, \mathcal{K}) \cong k.$$

We remark that smoothness can be relaxed, to the condition of being *Cohen-Macaulay.*

For our purposes, it suffices to note that $h^1(C, \mathcal{L}) = h^0(C, \mathcal{K} \otimes \mathcal{L}^\vee)$, where $\mathcal{K}$ is the (invertible) sheaf of differentials $\Omega_{X/k}$. Then the Riemann-Roch formula can be rewritten as

$$h^0(C, \mathcal{L}) - h^0(\mathcal{K} \otimes \mathcal{L}^\vee) = \deg \mathcal{L} - p_a(C) + 1.$$

If $\mathcal{L} = \mathcal{O}(D)$, just as it is convenient to interpret $h^0(C, \mathcal{L})$ as rational functions with zeros and poles constrained by D, it is convenient to interpret $h^0(\mathcal{K} \otimes \mathcal{L}^\vee) = h^0(\mathcal{K}(-D))$ as rational *differentials* with zeros and poles constrained by D (in the opposite way).

20.4.H. EXERCISE (ASSUMING SERRE DUALITY). Suppose C is a geometrically integral smooth curve over k.

(a) Show that $h^0(C, \mathcal{K}_C)$ is the genus g of C.
(b) Show that $\deg \mathcal{K} = 2g - 2$. (Hint: Riemann-Roch for $\mathcal{L} = \mathcal{K}$.)

20.4.7. *Aside: a special case.* If $C = \mathbb{P}^1_k$, Exercise 20.4.H implies that $\mathcal{K}_C \cong \mathcal{O}(-2)$. And indeed, $h^1(\mathbb{P}^1, \mathcal{O}(-2)) = 1$. Moreover, we also have a natural perfect pairing

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) \times H^1(\mathbb{P}^1, \mathcal{O}(-2-n)) \to k.$$

We can interpret this pairing as follows. If $n < 0$, both factors on the left are 0, so we assume $n > 0$. Then $H^0(\mathbb{P}^1, \mathcal{O}(n))$ corresponds to homogeneous degree n polynomials in x and y, and $H^1(\mathbb{P}^1, \mathcal{O}(-2-n))$ corresponds to homogeneous degree $-2-n$ Laurent polynomials in x and y so that the degrees of x and y are both at most $n-1$ (see Remark 20.3.1). You can quickly check that the dimension of both vector spaces are $n+1$. The pairing is given as follows: multiply the polynomial by the Laurent polynomial, to obtain a Laurent polynomial of degree -2. Read off the co-efficient of $x^{-1}y^{-1}$. (This works more generally for $\mathbb{P}^n_k$; see the discussion after the statement of Theorem 20.1.2.)

20.4.I. EXERCISE (AMPLE DIVISORS ON A CONNECTED SMOOTH PROJECTIVE VARIETY ARE CONNECTED). Suppose X is a connected smooth projective $\overline{k}$-variety, and D is an ample divisor. Show that D is connected. (Hint: Suppose $D = V(s)$, where s is a section of an ample invertible sheaf. Then $V(s^n) = V(s)$ for all $n > 0$, so we may replace $\mathcal{L}$ with a high power of our choosing. Use the long exact sequence for $0 \to \mathcal{O}_X(-nD) \to \mathcal{O}_X \to \mathcal{O}_{V(s^n)} \to 0$ to show that for $n \gg 0$, $h^0(\mathcal{O}_{V_{s^n}}) = 1$.

Once we know that Serre duality holds for Cohen-Macaulay projective schemes, this result will automatically extend to these schemes when s is an effective Cartier divisor (and with a little thought will extend to show that all ample divisors on such schemes). On the other hand, the result is false if X is the union of two randomly chosen 2-planes in $\mathbb{P}^4$ (why?), so this will imply that X is not Cohen-Macaulay.

20.4.8. Degree of a line bundle, and degree and rank of a coherent sheaf.

Suppose C is an irreducible reduced projective curve (pure dimension 1, over a field k). If $\mathcal{F}$ is a coherent sheaf on C, define the **rank** of $\mathcal{F}$, denoted $\operatorname{rank} \mathcal{F}$, to be its rank at the generic point of C (see §14.7.4 for the definition of rank at a point).

20.4.J. EASY EXERCISE. Show that the rank is additive in exact sequences: if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves, show that $\operatorname{rank} \mathcal{F} - \operatorname{rank} \mathcal{G} + \operatorname{rank} \mathcal{H} = 0$.

Define the **degree of** $\mathcal{F}$ by

$$\deg \mathcal{F} = \chi(C, \mathcal{F}) - (\operatorname{rank} \mathcal{F}) \cdot \chi(C, \mathcal{O}_C). \tag{20.4.8.1}$$

If $\mathcal{F}$ is an invertible sheaf (or if more generally the rank is the same on each irreducible component), we can drop the irreducibility hypothesis.

This generalizes the notion of the degree of a line bundle on a nonsingular curve (Important Exercise 20.4.C).

20.4.K. EASY EXERCISE. Show that degree (as a function of coherent sheaves on a fixed curve C) is additive in exact sequences.

20.4.L. EXERCISE. Show that the degree of a vector bundle is the degree of its determinant bundle (cf. Exercise 14.5.H).

The statement (20.4.8.1) is often called Riemann-Roch for coherent sheaves (or vector bundles) on a projective curve.

20.4.9. Extending this to proper curves.

20.4.M. EXERCISE. Suppose X is a projective curve over a field k, and $\mathcal{F}$ is a coherent sheaf on C. Show that $\chi(\mathcal{L} \otimes \mathcal{F}) - \chi(\mathcal{F})$ is the sum over the irreducible components C_i of C of the degree $\mathcal{L}$ on C_i^{red} times the length of $\mathcal{F}$ at the generic point η_i of C_i (the length of $\mathcal{F}_{\eta_i}$ as an $\mathcal{O}_{\eta_i}$-module). Hints: (1) First reduce to the case where $\mathcal{F}$ is scheme-theoretically supported on C^{red}, by showing that both sides of the alleged equality are additive in short exact sequences, and using the filtration

$$0 = \mathcal{I}^r \mathcal{F} \subset \mathcal{I}^{r-1} \mathcal{F} \subset \cdots \subset \mathcal{I} \mathcal{F} \subset \mathcal{F}$$

of $\mathcal{F}$, where $\mathcal{I}$ is the ideal sheaf cutting out C^{red} in C. Thus we need only consider the case where C is reduced. (2) As $\mathcal{L}$ is projective, we can write $\mathcal{L} \cong \mathcal{O}(\sum n_i p_i)$ where the p_i are nonsingular points distinct from the associated points of $\mathcal{F}_i$. Use this avatar of $\mathcal{L}$, and perhaps induction on the number of p_i.

In Exercise 20.6.E, we will see that all proper curves over k are projective, so "projective" can be replaced by "proper" in this exercise. In this guise, we will use it when discussing intersection theory in Chapter 22.

20.4.10. $\star$ **Numerical equivalence, the Néron-Severi group, nef line bundles, and the nef and ample cones.**

The notion of a degree on a line bundle leads to important and useful notions. Suppose X is a proper k-variety, and $\mathcal{L}$ is an invertible sheaf on X. If $i : C \hookrightarrow X$ is a one-dimensional closed subscheme of X, define the degree of $\mathcal{L}$ on C by $\deg_C \mathcal{L} := \deg_C i^* \mathcal{L}$. If $\deg_C \mathcal{L} = 0$ for all C, we say that $\mathcal{L}$ is **numerically trivial**.

20.4.N. EASY EXERCISE.

(a) Show that $\mathcal{L}$ is numerically trivial if and only if $\deg_C \mathcal{L} = 0$ for all *integral* curves C in X.
(b) Show that if $\pi : X \to Y$ is a proper morphism, and $\mathcal{L}$ is a numerically trivial invertible sheaf on Y, then $\pi^* \mathcal{L}$ is numerically trivial on X.
(c) Show that $\mathcal{L}$ is numerically trivial if and only if $\mathcal{L}$ is numerically trivial on each of the irreducible components of X.
(d) Show that if $\mathcal{L}$ and $\mathcal{L}'$ are numerically trivial, then $\mathcal{L} \otimes \mathcal{L}'$ is numerically trivial. Show that if $\mathcal{L}$ and $\mathcal{L}'$ are numerically trivial, then $\mathcal{L} \otimes \mathcal{L}'$ and $\mathcal{L}^\vee$ are both numerically trivial.

20.4.11. *Numerical equivalence.* By part (d), the numerically trivial invertible sheaves form a subgroup of $\operatorname{Pic} X$, denoted $\operatorname{Pic}^\tau X$. The resulting equivalence on line bundles is called **numerical equivalence**. Two lines bundles equivalent modulo the subgroup of numerically trivial line bundles are called **numerically equivalent**. A property of invertible sheaves stable under numerical equivalence is said to be a *numerical property*. We will see that "nefness" and ampleness are numerical properties (Definition 20.4.12 and Remark 22.3.2 respectively).

We will later define the *Néron-Severi group* $NS(X)$ of X as $\operatorname{Pic} X$ modulo algebraic equivalence (Exercise 25.6.D). (We will define algebraic equivalence once we have discussed flatness.) The highly nontrivial **Néron-Severi Theorem** (or **Theorem of the Base**) states that $NS(X)$ is a finitely generated group. The group $\operatorname{Pic} X / \operatorname{Pic}^\tau X$ is denoted $N^1(X)$. We will see (in the chapter on flatness) that it is a quotient of $NS(X)$, so it is also finitely generated. As the group $N^1(X)$ is clearly abelian and torsion-free, it is finite free $\mathbb{Z}$-module (by the classification of finitely generated modules over a principal ideal domain, see §1.2). The rank of $N^1(X)$ is called the **Picard number**, and is denoted $\rho(X)$ (although we won't have need of this notion). For example, $\rho(\mathbb{P}^n) = 1$ and $\rho((\mathbb{P}^1)^n) = n$. We let define $N^1_{\mathbb{Q}}(X) := N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (so $\rho(X) = \dim_{\mathbb{Q}} N^1_{\mathbb{Q}}(X)$), and call the elements of this group **$\mathbb{Q}$-line bundles**, for lack of any common term in the literature.

20.4.O. ⋆⋆ EXERCISE (FINITENESS OF PICARD NUMBER IN THE COMPLEX CASE, ONLY FOR THOSE WITH SUFFICIENT BACKGROUND). Show (without the Néron-Severi Theorem) that if X is a complex proper variety, then $\rho(X)$ is finite, by interpreting it as a subquotient of $H^2(X, \mathbb{Z})$. Hint: show that the image of $(\mathcal{L}, C)$ under the map $H^2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \to H_0(X, \mathbb{Z}) \to \mathbb{Z}$ is $\deg_C \mathcal{L}$.

20.4.12. *Definition.* We say that an invertible sheaf $\mathcal{L}$ is **numerically effective**, or **nef** if for all such C, $\deg_C \mathcal{L} \geq 0$. Clearly nefness is a numerical property.

20.4.P. EASY EXERCISE.

(a) Show that $\mathcal{L}$ is nef if and only if $\deg_C \mathcal{L} \geq 0$ for all *integral* curves C in X.
(b) Show that if $\pi : X \to Y$ is a proper morphism, and $\mathcal{L}$ is a nef invertible sheaf on Y, then $\pi^* \mathcal{L}$ is nef on X.
(c) Show that $\mathcal{L}$ is nef if and only if $\mathcal{L}$ is nef on each of the irreducible components of X.
(d) Show that if $\mathcal{L}$ and $\mathcal{L}'$ are nef, then $\mathcal{L} \otimes \mathcal{L}'$ is nef. Thus the nef elements of $\operatorname{Pic} X$ form a semigroup.
(e) Show that ample invertible sheavesare nef.
(f) Suppose $n \in \mathbb{Z}^+$. Show that $\mathcal{L}$ is nef if and only if $\mathcal{L}^{\otimes n}$ is nef.

20.4.Q. EXERCISE. Define what it means for a $\mathbb{Q}$-line bundle to be nef. Show that the nef $\mathbb{Q}$-line bundles form a closed cone in $N^1_{\mathbb{Q}}(X)$. This is called the **nef cone**.

It is a surprising fact that whether an invertible sheaf $\mathcal{L}$ on X is ample depends only on its class in $N^1_{\mathbb{Q}}(X)$, i.e. on how it intersects the curves in X. Because of this (as for any $n \in \mathbb{Z}^+$, $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ is ample, see Theorem 16.3.11), it makes sense to define when a $\mathbb{Q}$-line bundle is ample. Then by Exercise 16.3.L, the ample divisors form a cone in $N^1_{\mathbb{Q}}(X)$, necessarily contained in the nef cone by Exercise 20.4.P(e). It turns out that if X is projective, the ample divisors are precisely the interior of the nef cone. The new facts in this paragraph are a consequence of Kleiman's numerical criterion for ampleness, Theorem 22.3.6.

20.4.R. EXERCISE. Describe the nef cones of $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. (Notice in the latter case that the two boundaries of the cone correspond to linear series contracting one of the $\mathbb{P}^1$'s. This is true in general: informally speaking, linear series corresponding to the boundaries of the cone give interesting contractions. Another example will be given in Exercise 22.2.F.)

20.5 Hilbert polynomials, genus, and Hilbert functions

If $\mathcal{F}$ is a coherent sheaf on X, define the **Hilbert function of** $\mathcal{F}$ by

$$h_{\mathcal{F}}(n) := h^0(X, \mathcal{F}(n)).$$

The **Hilbert function of** X is the Hilbert function of the structure sheaf.

20.5.A. EXERCISE. Suppose $p_1, \dots, p_m$ are m distinct closed points of $\mathbb{P}^n_k$. Find the Hilbert function of the structure sheaf of the union of the p_i in the following two cases:
(a) $p_1, \dots, p_m$ span a projective space of dimension $m-1$ (the maximum possible).
(b) $p_1, \dots, p_m$ are collinear (lie on a $\mathbb{P}^1$).
In particular, show that the Hilbert function of 3 distinct points in $\mathbb{P}^2$ depends on whether they are collinear or not, but in both cases the Hilbert function is "eventually always 3".

The ancients were aware that the Hilbert function is "eventually polynomial", i.e. for large enough n, it agrees with some polynomial, called the **Hilbert polynomial** (and denoted $p_{\mathcal{F}}(n)$ or $p_X(n)$). This polynomial contains lots of interesting geometric information, as we will soon see. In modern language, we expect that this "eventual polynomiality" arises because the Euler characteristic should be a polynomial, and that for $n \gg 0$, the higher cohomology vanishes. This is indeed the case, as we now verify.

20.5.1. Theorem. — *If $\mathcal{F}$ is a coherent sheaf on a projective k-scheme $X \hookrightarrow \mathbb{P}^n_k$, $\chi(X, \mathcal{F}(m))$ is a polynomial of degree equal to $\dim \operatorname{Supp} \mathcal{F}$. Hence by Serre vanishing (Theorem 20.1.3 (ii)), for $m \gg 0$, $h^0(X, \mathcal{F}(m))$ is a polynomial of degree $\dim \operatorname{Supp} \mathcal{F}$. In particular, for $m \gg 0$, $h^0(X, \mathcal{O}_X(m))$ is polynomial with degree = $\dim X$.*

Here $\mathcal{O}_X(m)$ is the restriction or pullback of $\mathcal{O}_{\mathbb{P}^n_k}(1)$. Both the degree of the 0 polynomial and the dimension of the empty set is defined to be -1. In particular, the only coherent sheaf with Hilbert polynomial 0 is the zero-sheaf.

This argument uses the notion of associated primes of (finitely generated) modules (over a Noetherian ring); see Theorem 6.5.4. (The resolution given by the Hilbert Syzygy Theorem, §16.3.2, can give a shorter proof; but we haven't proved the Hilbert Syzygy Theorem.)

20.5.B. EASY EXERCISE. Using the results of §6.5, define the notion of associated points of a coherent sheaf on a locally Noetherian scheme.

Proof. Define $p_{\mathcal{F}}(m) = \chi(X, \mathcal{F}(m))$. We will show that $p_{\mathcal{F}}(m)$ is a polynomial of the desired degree.

We first use Exercise 20.2.G to reduce to the case where k is algebraically closed, and in particular infinite. (This is one of those cases where even if you are concerned with potentially arithmetic questions over some non-algebraically closed field like $\mathbb{F}_p$, you are forced to consider the "geometric" situation where the base field is algebraically closed.)

The coherent sheaf $\mathcal{F}$ has a finite number of associated points. We show a useful fact that we will use again.

20.5.C. EXERCISE. Suppose X is a projective k-scheme with k infinite, and $\mathcal{F}$ is a coherent sheaf on X. Show that if $\mathcal{L}$ is a very ample invertible sheaf on X, then there is an effective divisor D on X with $\mathcal{L} \cong \mathcal{O}(D)$, and where D does not meet the associated points of $\mathcal{F}$. (Hint: show that given any finite set of points of $\mathbb{P}^n_k$, there is a hyperplane not containing any of them.)

Thus there is a hyperplane $x = 0$ ($x \in \Gamma(X, \mathcal{O}(1))$) missing this finite number of points. (This is where we use the infinitude of k.)

Then the map $\mathcal{F}(-1) \xrightarrow{\times x} \mathcal{F}$ is injective (on any affine open subset, $\mathcal{F}$ corresponds to a module, and x is not a zerodivisor on that module, as it doesn't vanish at any associated point of that module, see Theorem 6.5.4(c)). Thus we have a short exact sequence

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{20.5.1.1}$$

where $\mathcal{G}$ is a coherent sheaf.

20.5.D. EXERCISE. Show that $\operatorname{Supp} \mathcal{G} = \operatorname{Supp} \mathcal{F} \cap V(x)$. (Hint: show that $\mathcal{F}(-1) \to \mathcal{F}$ is an isomorphism away from $V(x)$, and hence $\mathcal{G} = 0$ on this locus. If $p \in V(x)$, show that the $\mathcal{F}(-1)|_x \to \mathcal{F}|_x$ is the 0 map, and hence $\mathcal{F}|_x \to \mathcal{G}|_x$ is an isomorphism.)

Hence $\dim \operatorname{Supp} \mathcal{G} = \dim \operatorname{Supp} \mathcal{F} - 1$ by Krull's Principal Ideal Theorem 12.3.3 unless $\mathcal{F} = 0$ (in which case we already know the result, so assume this is not the case).

Twisting (20.5.1.1) by $\mathcal{O}(m)$ yields

$$0 \longrightarrow \mathcal{F}(m-1) \longrightarrow \mathcal{F}(m) \longrightarrow \mathcal{G}(m) \longrightarrow 0$$

Euler characteristics are additive in exact sequences, from which $p_{\mathcal{F}}(m) - p_{\mathcal{F}}(m-1) = p_{\mathcal{G}}(m)$. Now $p_{\mathcal{G}}(m)$ is a polynomial of degree $\dim \operatorname{Supp} \mathcal{F} - 1$.

The result is then a consequence from the following elementary fact about polynomials in one variable.

20.5.E. EXERCISE. Suppose f and g are functions on the integers, $f(m+1) - f(m) = g(m)$ for all m, and $g(m)$ is a polynomial of degree $d \geq 0$. Show that f is a polynomial of degree $d+1$.

$\square$

Definition. The **Hilbert polynomial** $p_{\mathcal{F}}(m)$ was defined in the above proof. If $X \subset \mathbb{P}^n$ is a projective k-scheme, define $p_X(m) := p_{\mathcal{O}_X}(m)$.

Example 1. $p_{\mathbb{P}^n}(m) = \binom{m+n}{n}$, where we interpret this as the polynomial $(m+1)\cdots(m+n)/n!$.

Example 2. Suppose H is a degree d hypersurface in $\mathbb{P}^n$. Then from the closed subscheme exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_H \longrightarrow 0,$$

we have

$$p_H(m) = p_{\mathbb{P}^n}(m) - p_{\mathbb{P}^n}(m-d) = \binom{m+n}{n} - \binom{m+n-d}{n}.$$

(Note: implicit in this argument is the fact that if $i : H \hookrightarrow \mathbb{P}^n$ is the closed immersion, then $(i_* \mathcal{O}_H) \otimes \mathcal{O}_{\mathbb{P}^n}(m) \cong i_*(\mathcal{O}_H \otimes i^* \mathcal{O}_{\mathbb{P}^n}(m))$. This follows from the projection formula, Exercise 17.3.H(b).)

20.5.F. EXERCISE. Show that the twisted cubic (in $\mathbb{P}^3$) has Hilbert polynomial $3m+1$. (The twisted cubic was defined in Exercise 9.2.A.)

20.5.G. EXERCISE. More generally, find the Hilbert polynomial for the dth Veronese embedding of $\mathbb{P}^n$ (i.e. the closed immersion of $\mathbb{P}^n$ in a bigger projective space by way of the line bundle $\mathcal{O}(d)$, §9.2.6).

20.5.H. EXERCISE. Suppose $X \subset Y \subset \mathbb{P}^n_k$ are a sequence of closed subschemes.

(a) Show that $p_X(m) \leq p_Y(m)$ for $m \gg 0$. Hint: let $\mathcal{I}_{X/Y}$ be the ideal sheaf of X in Y. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_{X/Y}(m) \longrightarrow \mathcal{O}_Y(m) \longrightarrow \mathcal{O}_X(m) \longrightarrow 0.$$

(b) If $p_X(m) = p_Y(m)$ for $m \gg 0$, show that $X = Y$. Hint: Show that if the Hilbert polynomial of $\mathcal{I}_{X/Y}$ is 0, then $\mathcal{I}_{X/Y}$ must be the 0 sheaf. (Handy trick: For $m \gg 0$, $\mathcal{I}_{X/Y}(m)$ is generated by global sections and is also 0. This of course applies with $\mathcal{I}$ replaced by *any* coherent sheaf.)

This fact will be used several times in Chapter 21.

From the Hilbert polynomial, we can extract many invariants, of which two are particularly important. The first is the *degree*. The **degree of a projective k-scheme of dimension n** to be leading coefficient of the Hilbert polynomial (the coefficient of m^n) times $n!$.

Using the examples above, we see that the degree of $\mathbb{P}^n$ in itself is 1. The degree of the twisted cubic is 3.

20.5.I. EXERCISE. Show that the degree is always an integer. Hint: by induction, show that any polynomial in m of degree k taking on only integer values must have coefficient of m^k an integral multiple of $1/k!$. Hint for this: if $f(x)$ takes on only integral values and is of degree k, then $f(x+1) - f(x)$ takes on only integral values and is of degree $k-1$.

20.5.J. EXERCISE. Show that the degree of a degree d hypersurface (Definition 9.2.2) is d (preventing a notational crisis).

20.5.K. EXERCISE. Suppose a curve C is embedded in projective space via an invertible sheaf of degree d (as defined in §20.4.8). In other words, this line bundle determines a closed immersion. Show that the degree of C under this embedding is d, preventing another notational crisis. (Hint: Riemann-Roch, Exercise 20.4.B.)

20.5.L. EXERCISE. Show that the degree of the dth Veronese embedding of $\mathbb{P}^n$ is d^n.

20.5.M. EXERCISE (BÉZOUT'S THEOREM, GENERALIZING EXERCISES 9.2.E AND 17.4.G). Suppose X is a projective scheme of dimension at least 1, and H is a degree d hypersurface not containing any associated points of X. (For example, if X is a projective variety, then we are just requiring H not to contain any irreducible components of X.) Show that $\deg H \cap X = d \deg X$. (As an example, we have Bézout's theorem for plane curves: if C and D are plane curves of degrees m and n respectively, with no common components, then C and D meet at mn points, counted with appropriate multiplicity.)

This is a very handy theorem! For example: if two projective plane curves of degree m and degree n share no irreducible components, then they intersect in mn points, counted with appropriate multiplicity. The notion of multiplicity of intersection is just the degree of the intersection as a k-scheme.

20.5.N. EXERCISE. Classically, the degree of a complex projective variety of dimension n was defined as follows. We slice the variety with n generally chosen hyperplanes. Then the intersection will be a finite number of points. The degree is this number of points. Use Bézout's theorem to make sense of this in a way that agrees with our definition of degree. You will need to assume that k is infinite.

Thus the classical definition of the degree, which involved making a choice and then showing that the result is independent of choice, has been replaced by making a cohomological definition involving Euler characteristics. This is analogous to how the degree of a line bundle was initially defined (as the degree of a divisor, Important Exercise 20.4.C) is better defined in terms of Euler characteristics (§20.4.8).

20.5.2. *Revisiting an earlier example.* We revisit the enlightening example of Example 10.3.3 and §18.4.8: let $k = \mathbb{Q}$, and consider the parabola $x = y^2$. We intersect it with the four lines, $x = 1$, $x = 0$, $x = -1$, and $x = 2$, and see that we get 2 each time (counted with the same convention as with the last time we saw this example).

If we intersect it with $y = 2$, we only get one point — but that's because this isn't a projective curve, and we really should be doing this intersection on $\mathbb{P}^2_k$, and in this case, the conic meets the line in two points, one of which is "at ∞".

20.5.O. EXERCISE. Show that the degree of the d-fold Veronese embedding of $\mathbb{P}^n$ is d^n in a different way from Exercise 20.5.L as follows. Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding. To find the degree of the image, we intersect it with n hyperplanes in $\mathbb{P}^N$ (scheme-theoretically), and find the number of intersection points (counted with multiplicity). But the pullback of a hyperplane in $\mathbb{P}^N$ to $\mathbb{P}^n$ is a degree d hypersurface. Perform this intersection in $\mathbb{P}^n$, and use Bézout's theorem (Exercise 20.5.M).

20.5.3. Genus.

There is another central piece of information residing in the Hilbert polynomial. Notice that $p_X(0)$ is the arithmetic genus $\chi(X, \mathcal{O}_X)$, an *intrinsic* invariant of the scheme X, independent of the projective embedding.

Imagine how amazing this must have seemed to the ancients: they defined the Hilbert function by counting how many "functions of various degrees" there are; then they noticed that when the degree gets large, it agrees with a polynomial; and then when they plugged 0 into the polynomial — extrapolating backwards, to where the Hilbert function and Hilbert polynomials didn't agree — they found a magic invariant! Furthermore, in the case when X is a complex curve, this invariant was basically the topological genus!

We can now see a large family of curves over an algebraically closed field that is provably not $\mathbb{P}^1$! Note that the Hilbert polynomial of $\mathbb{P}^1$ is $(m+1)/1 = m+1$, so $\chi(\mathcal{O}_{\mathbb{P}^1}) = 1$. Suppose C is a degree d curve in $\mathbb{P}^2$. Then the Hilbert polynomial of C is

$$p_{\mathbb{P}^2}(m) - p_{\mathbb{P}^2}(m-d) = (m+1)(m+2)/2 - (m-d+1)(m-d+2)/2.$$

Plugging in $m = 0$ gives us $-(d^2 - 3d)/2$. Thus when $d > 2$, we have a curve that cannot be isomorphic to $\mathbb{P}^1$! (And it is not hard to show that there exists a *nonsingular* degree d curve, Exercise 13.2.J.)

Now from $0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$, using $h^1(\mathcal{O}_{\mathbb{P}^2}(d)) = 0$, we have that $h^0(C, \mathcal{O}_C) = 1$. As $h^0 - h^1 = \chi$, we have

$$h^1(C, \mathcal{O}_C) = (d-1)(d-2)/2. \tag{20.5.3.1}$$

We now revisit an interesting question we first saw in §7.5.9. If k is an algebraically closed field, is every finitely generated transcendence degree 1 extension of k isomorphic to $k(x)$? In that section, we found ad hoc (but admittedly beautiful) examples showing that the answer is "no". But we now have a better answer. The question initially looks like an algebraic question, but we now recognize it as a fundamentally geometric one. There is an integer-valued cohomological invariant of such field extensions that is has good geometric meaning: the genus.

Equation (20.5.3.1) yields examples of curves of genus $0, 1, 3, 6, 10, \dots$ (corresponding to degree 1 or $2, 3, 4, 5, \dots$). This begs some questions, such as: are there curves of other genera? (We will see soon, in §21.4.5, that the answer is yes.) Are there other genus 0 curves? (Not if k is algebraically closed, but sometimes yes otherwise — consider $x^2+y^2+z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$, which has no $\mathbb{R}$-points and hence is not isomorphic to $\mathbb{P}^1_{\mathbb{R}}$ — we will discuss this more in §21.3.) Do we have all the curves of genus 3?

(Almost all, but not quite. We will see more in §21.6.) Do we have all the curves of genus 6? (We are missing "most of them".)

Caution: The Euler characteristic of the structure sheaf doesn't distinguish between isomorphism classes of projective schemes, nonsingular, over algebraically closed fields. For example, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ both have Euler characteristic 1 (see Theorem 20.1.2 and Exercise 20.3.D), but are not isomorphic — $\operatorname{Pic} \mathbb{P}^2 \cong \mathbb{Z}$ (§15.2.6) while $\operatorname{Pic} \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{Z} \oplus \mathbb{Z}$ (Exercise 15.2.N).

20.5.4. Complete intersections.

We define a **complete intersection** in $\mathbb{P}^n$ inductively as follows. $\mathbb{P}^n$ is a complete intersection in itself. A closed subscheme $X_r \hookrightarrow \mathbb{P}^n$ of dimension r (with $r < n$) is a complete intersection if there is a complete intersection X_{r+1}, and X_r is an effective Cartier divisor in class $\mathcal{O}_{X_{r+1}}(d)$.

20.5.P. EXERCISE. Show that if X is a complete intersection of dimension r in $\mathbb{P}^n$, then $H^i(X, \mathcal{O}_X(m)) = 0$ for all $0 < i < r$ and all m. Show that if $r > 0$, then $H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$ is surjective. (Hint: long exact sequences.)

Now X_r is the divisor of a section of $\mathcal{O}_{X_{r+1}}(m)$ for some m. But this section is the restriction of a section of $\mathcal{O}(m)$ on $\mathbb{P}^n$. Hence X_r is the scheme-theoretic intersection of X_{r+1} with a hypersurface. Thus inductively X_r is the scheme-theoretic intersection of $n - r$ hypersurfaces. (By Bézout's theorem, Exercise 20.5.M, $\deg X_r$ is the product of the degree of the defining hypersurfaces.)

20.5.Q. EXERCISE (POSITIVE-DIMENSIONAL COMPLETE INTERSECTIONS ARE CONNECTED). Show that complete intersections of *positive* dimension are connected. (Hint: show that $h^0(X, \mathcal{O}_X) = 1$.) For experts: this argument will even show that they are geometrically connected (§10.4.2), using Exercise 20.1.C.

20.5.R. EXERCISE. Find the genus of the complete intersection of 2 quadrics in $\mathbb{P}^3_k$.

20.5.S. EXERCISE. More generally, find the genus of the complete intersection of a degree m surface with a degree n surface in $\mathbb{P}^3_k$. (If $m = 2$ and $n = 3$, you should get genus 4. We will see in §21.7 that in some sense most genus 4 curves arise in this way. You might worry about whether there are any nonsingular curves of this form. You can check this by hand, but Bertini's Theorem 26.5.2 will save us this trouble.)

20.5.T. EXERCISE. Show that the rational normal curve of degree d in $\mathbb{P}^d$ is *not* a complete intersection if $d > 2$. (Hint: If it *were* the complete intersection of $d - 1$ hypersurfaces, what would the degree of the hypersurfaces be? Why could none of the degrees be 1?)

20.5.U. EXERCISE. Show that the union of two distinct planes in $\mathbb{P}^4$ is not a complete intersection. Hint: it is connected, but you can slice with another plane and get something not connected (see Exercise 20.5.Q).

This is another important scheme in algebraic geometry that is an example of many sorts of behavior. We will see it again!

20.6 ⋆ Serre's cohomological characterization of ampleness

Theorem 16.3.11 gave a number of characterizations of ampleness, in terms of projective geometry, global generation, and the Zariski topology. Here is another characterization, this time cohomological, under Noetherian hypotheses. Because (somewhat surprisingly) we won't use this result much (and mainly the fact that all proper curves over k are projective, Exercise 20.6.E), this section is starred.

20.6.1. Theorem (Serre's cohomological criterion for ampleness). — *Suppose A is a Noetherian ring, X is a proper A-scheme, and $\mathcal{L}$ is an invertible sheaf on X. Then the following are equivalent.*

(a-c) *The invertible sheaf $\mathcal{L}$ is ample on X (over A).*

(e) *For all coherent sheaves $\mathcal{F}$ on X, there is an n_0 such that for $n \geq n_0$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $i > 0$.*

The label (a-c) is in intended to reflect the statement of Theorem 16.3.11. We avoid the label (d) because it appeared in Theorem 16.3.15. Before getting to the proof, we motivate this result by giving some applications.

20.6.A. IMPORTANT EXERCISE (USED REPEATEDLY). Suppose $f : X \to Y$ is a finite morphism of proper Noetherian A-schemes, and $\mathcal{L}$ is an ample line bundle on Y. Show that $f^*\mathcal{L}$ is ample on X.

20.6.B. EXERCISE. Suppose $\mathcal{L}$ is basepoint free, and hence induces some morphism $\phi : X \to \mathbb{P}^n$. Then $\mathcal{L}$ is ample if and only if ϕ is finite. (Hint: if ϕ is finite, use Exercise 20.6.A. If ϕ is not finite, show that there is a curve C contracted by π, using the fact that quasifinite + projective = finite, Theorem 18.3.10. Show that $\mathcal{L}$ has degree 0 on C.)

20.6.C. EXERCISE. Suppose X is a proper A-scheme, and $\mathcal{L}$ is an invertible sheaf on X. Show that $\mathcal{L}$ is ample on X if and only if $\mathcal{L}|_{X^{\text{red}}}$ is ample on X^{red}. Hint: for the "only if" direction, use Exercise 20.6.A. For the "if" direction, let $\mathcal{I}$ be the ideal sheaf cutting out the closed subscheme X^{red} in X. Filter $\mathcal{F}$ by powers of $\mathcal{I}$:

$$0 = \mathcal{I}^r\mathcal{F} \subset \mathcal{I}^{r-1}\mathcal{F} \subset \cdots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}.$$

(Essentially the same filtration appeared in Exercise 20.4.M, for similar reasons.) Show that each quotient $\mathcal{I}^n\mathcal{F}/\mathcal{I}^{n-1}\mathcal{F}$, twisted by a high enough power of $\mathcal{L}$, has no higher cohomology. Use descending induction on n to show each part $\mathcal{I}^n\mathcal{F}$ of the filtration (and hence in particular $\mathcal{F}$) has this property as well.

20.6.D. EXERCISE. Suppose X is a proper A-scheme, and $\mathcal{L}$ is an invertible sheaf on X. Show that $\mathcal{L}$ is ample on X if and only if $\mathcal{L}$ is ample on each component. Hint: follow the outline of the solution to the previous exercise, taking instead $\mathcal{I}$ as the ideal sheaf of one component. Perhaps first reduce to the case where $X = X^{\text{red}}$.

20.6.E. EXERCISE. Show that every proper curve over a field k is projective as follows. Recall that every nonsingular integral proper curve is projective (Exercise 18.4.A). Show that every reduced integral proper curve is projective. (Hint: Exercise 20.6.A.) Show that on any reduced integral proper curve C, you can find a very ample divisor supported only of nonsingular points of C. Show that every reduced proper curve is projective. (Hint: Exercise 20.6.D.) Show that every proper curve C is projective. (Hint: Exercise 20.6.C. To apply it, you will have to find a line bundle on C that you will show is ample.)

20.6.F. EXERCISE. (In Exercise 21.2.E, we will show that on a projective nonsingular integral curve, an invertible sheaf is ample if and only if it has positive degree. Use this fact in this exercise. There will be no logical circularity.) Show that a line bundle on a projective curve is ample if and only if it has positive degree on each component.

20.6.2. *Very ample versus ample.* The previous exercises don't work with "ample" replaced by "very ample", which shows again how the notion of ampleness is better-behaved than very ampleness.

20.6.3. *Proof of Theorem 20.6.1.* For the fact that (a-c) implies (e), use the fact that $\mathcal{L}^{\otimes N}$ is very ample for some N (Theorem 16.3.11(a)), and apply Serre vanishing (Theorem 20.1.3(ii)) to $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}, \dots$, and $\mathcal{F} \otimes \mathcal{L}^{\otimes (N-1)}$.

So we now assume (e), and show that $\mathcal{L}$ is ample by criterion (b) of Theorem 16.3.11: we will show that for any coherent sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for $n \gg 0$.

We begin with a special case: we will show that $\mathcal{L}^{\otimes n}$ is globally generated (i.e. base-point-free) for $n \gg 0$. To do this, it suffices to show that every closed point p has a neighborhood U so that there exists some N_p so that $n \geq N_p$, $\mathcal{L}^{\otimes n}$ is globally generated for all points of U_p. (Reason: by quasicompactness, every closed subset of X contains a closed point, by Exercise 6.1.E. So as p varies over the closed points of X, these U_p cover X. By quasicompactness again, we can cover X by a finite number of these U_p. Let N be the maximum of the corresponding N_p. Then for $n \geq N$, $\mathcal{L}^{\otimes n}$ is globally generated in each of these U_p, and hence on all of X.)

Let p be a closed point of X. For all n, $\mathfrak{m}_p \otimes \mathcal{L}^{\otimes n}$ is coherent (by our Noetherian hypotheses). By (e), there exists some n_0 so that for $n \geq n_0$, $H^1(X, \mathfrak{m}_p \otimes \mathcal{L}^{\otimes n}) = 0$. By the long exact sequence arising from the closed subscheme exact sequence

$$0 \to \mathfrak{m}_p \otimes \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}|_p \to 0,$$

we have that $\mathcal{L}^{\otimes n}$ is globally generated at p for $n \geq n_0$. By Exercise 16.3.C(b), there is an open neighborhood V_0 of p such that $\mathcal{L}^{\otimes n_0}$ is globally generated at all points of V_0. Thus $\mathcal{L}^{\otimes k n_0}$ is globally generated at all points of V_0 for all positive integers k (using Easy Exercise 16.3.B). For each $i \in \{1, \dots, n_0 - 1\}$, there is an open neighborhood V_i of p such that $\mathcal{L}^{\otimes (n_0 + i)}$ is globally generated at all points of V_i (again by Exercise 16.3.C(b)). We may take each V_i to be contained in V_0. By Easy Exercise 16.3.B, $\mathcal{L}^{\otimes (k n_0 + n_0 + i)}$ is globally generated at every point of V_i (as this is the case for $\mathcal{L}^{\otimes k n_0}$ and $\mathcal{L}^{\otimes (n_0 + i)}$). Thus in the open neighborhood $U_p := \cap_{i=0}^{n-1} V_i$, $\mathcal{L}^{\otimes n}$ is globally generated for $n \geq N_p := 2n_0$.

We have now shown that there exists some N such that for $n \geq N$, $\mathcal{L}^{\otimes n}$ is globally generated. Now suppose $\mathcal{F}$ is a coherent sheaf. To conclude the proof, we will show that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for $n \gg 0$. This argument has a similar flavor to what we have done so far, so we give it as an exercise.

20.6.G. EXERCISE. Suppose p is a closed point of X.

(a) Show that for $n \gg 0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated at p.

(b) Show that there exists an open neighborhood U_p of p such that for $n \gg 0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated at every point of U_p. Caution: while it is true that by Exercise 16.3.C(b), for each $n \gg 0$, there is some neighborhood V_n of p such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated there, it need not be true that

$$\cap_{n \gg 0} V_n \tag{20.6.3.1}$$

is an open set. You may need to use the fact that $\mathcal{L}^{\otimes n}$ is globally generated for $n \geq N$ to replace (20.6.3.1) by a finite intersection.

20.6.H. EXERCISE. Conclude the proof of Theorem 20.6.1 by showing that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for $n \gg 0$. □

20.6.4. *Aside: Serre's cohomological characterization of affineness.* Serre gave a characterization of affineness similar in flavor to Theorem 20.6.1. Because we won't use it, we omit the proof. (One is given in [**Ha**, Thm. III.3.7].)

20.6.5. Theorem (Serre's cohomological characterization of affineness). — *Suppose X is a Noetherian separated scheme. Then the following are equivalent.*

(a) *The scheme X is affine.*
(b) *For any quasicoherent sheaf $\mathcal{F}$ on X, $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*
(c) *For any coherent sheaf of ideals $\mathcal{I}$ on X, $H^1(X, \mathcal{I}) = 0$.*

Clearly (a) implies (b) implies (c) (the former from Property **(vi)** of §20.1) without any Noetherian assumptions, so the real substance is in the implication from (c) to (a).

Serre proved an analogous result in complex analytic geometry: Stein spaces are also characterized by the vanishing of cohomology of coherent sheaves.

20.7 Higher direct image sheaves

Cohomology groups were defined for $X \to \operatorname{Spec} A$ where the structure morphism is quasicompact and separated; for any quasicoherent $\mathcal{F}$ on X, we defined $H^i(X, \mathcal{F})$. We will now define a "relative" version of this notion, for quasicompact and separated morphisms $\pi : X \to Y$: for any quasicoherent $\mathcal{F}$ on X, we will define $R^i\pi_*\mathcal{F}$, a quasicoherent sheaf on Y. (Now would be a good time to do Exercise 2.6.H, the FHHF Theorem, if you haven't done it before.)

We have many motivations for doing this. In no particular order:

(1) It "globalizes" what we did before with cohomology.
(2) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of quasicoherent sheaves on X, then we know that $0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{G} \to \pi_*\mathcal{H}$ is exact, and higher pushforwards will extend this to a long exact sequence.
(3) We will later see that this will show how cohomology groups vary in families, especially in "nice" situations. Intuitively, if we have a nice family of varieties, and a family of sheaves on them, we could hope that the cohomology varies nicely in families, and in fact in "nice" situations, this is true. (As always, "nice" usually means "flat", whatever that means.

We will see that Euler characteristics are locally constant in proper flat families in §25.6, and the Cohomology and Base Change Theorem 25.7.5 will show that in particularly good situations, dimensions of cohomology groups are constant.)

All of the important properties of cohomology described in §20.1 will carry over to this more general situation. Best of all, there will be no extra work required.

In the notation $R^j f_* \mathcal{F}$ for higher pushforward sheaves, the "R" stands for "right derived functor", and corresponds to the fact that we get a long exact sequence in cohomology extending to the right (from the 0th terms). In Chapter 24, we will see that in good circumstances, if we have a left-exact functor, there is a long exact sequence going off to the right, in terms of right derived functors. Similarly, if we have a right-exact functor (e.g. if M is an A-module, then $\otimes_A M$ is a right-exact functor from the category of A-modules to itself), there may be a long exact sequence going off to the left, in terms of left derived functors.

Suppose $\pi : X \to Y$, and $\mathcal{F}$ is a quasicoherent sheaf on X. For each $\operatorname{Spec} A \subset Y$, we have A-modules $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})$. We will show that these patch together to form a quasicoherent sheaf. We need check only one fact: that this behaves well with respect to taking distinguished open sets. In other words, we must check that for each $f \in A$, the natural map $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F}) \to H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})_f$ (induced by the map of spaces in the opposite direction — H^i is contravariant in the space) is precisely the localization $\otimes_A A_f$. But this can be verified easily: let $\{U_i\}$ be an affine cover of $\pi^{-1}(\operatorname{Spec} A)$. We can compute $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})$ using the Čech complex (20.2.1.1). But this induces a cover $\operatorname{Spec} A_f$ in a natural way: If $U_i = \operatorname{Spec} A_i$ is an affine open for $\operatorname{Spec} A$, we define $U_i' = \operatorname{Spec}(A_i)_f$. The resulting Čech complex for $\operatorname{Spec} A_f$ is the localization of the Čech complex for $\operatorname{Spec} A$. As taking cohomology of a complex commutes with localization (as discussed in the FHHF Theorem, Exercise 2.6.H), we have defined a quasicoherent sheaf on Y by the characterization of quasicoherent sheaves in §14.3.3.

Define the **ith higher direct image sheaf** or the **ith (higher) pushforward sheaf** to be this quasicoherent sheaf.

20.7.1. Theorem. —

(a) $R^i \pi_*$ *is a covariant functor from the category of quasicoherent sheaves on* X *to the category of quasicoherent sheaves on* Y.

(b) *We can identify* $R^0 \pi_*$ *with* $\pi_* \mathcal{F}$.

(c) *(the* **long exact sequence of higher pushforward sheaves***) A short exact sequence* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *of sheaves on* X *induces a long exact sequence*

$$0 \longrightarrow R^0\pi_*\mathcal{F} \longrightarrow R^0\pi_*\mathcal{G} \longrightarrow R^0\pi_*\mathcal{H} \longrightarrow$$

$$R^1\pi_*\mathcal{F} \longrightarrow R^1\pi_*\mathcal{G} \longrightarrow R^1\pi_*\mathcal{H} \longrightarrow \cdots$$

of sheaves on Y.

(d) *(projective pushforwards of coherent are coherent: Grothendieck's coherence theorem for projective morphisms) If* π *is a projective morphism and* $\mathcal{O}_Y$ *is coherent on* Y *(this hypothesis is automatic for* Y *locally Noetherian), and* $\mathcal{F}$ *is a coherent sheaf on* X*, then for all* i, $R^i\pi_*\mathcal{F}$ *is a coherent sheaf on* Y.

Proof. Because it suffices to check each of these results on affine open sets, they all follow from the analogous statements in Čech cohomology (§20.1). $\square$

The following result is handy, and essentially immediate from our definition.

20.7.A. EASY EXERCISE. Show that if π is affine, then for $i > 0$, $R^i\pi_*\mathcal{F} = 0$.

This is in fact a characterization of affineness. Serre's criterion for affineness states that if f is quasicompact and separated, then f is affine if and only if f_* is an exact functor from the category of quasicoherent sheaves on X to the category of quasicoherent sheaves on Y. We won't use this fact.

20.7.2. *How higher pushforwards behave with respect to base change.*

20.7.B. EXERCISE (HIGHER PUSHFORWARDS AND BASE CHANGE). (a) Suppose $f : Z \to Y$ is any morphism, and $\pi : X \to Y$ as usual is quasicompact and separated. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let

(20.7.2.1)
$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Z \arrow[r, "f"] & Y \end{tikzcd}$$

be a fiber diagram. Describe a natural morphism $f^*(R^i\pi_*\mathcal{F}) \to R^i\pi'_*(f')^*\mathcal{F}$ of sheaves on Z. (Hint: the FHHF Theorem, Exercise 2.6.H.)
(b) (cohomology commutes with affine flat base change) If $f : Z \to Y$ is an affine morphism, and for a cover $\operatorname{Spec} A_i$ of Y, where $f^{-1}(\operatorname{Spec} A_i) = \operatorname{Spec} B_i$, B_i is a *flat* A-algebra (§2.6.10: $\otimes_A B_i$ is exact), and the diagram in (a) is a fiber diagram, show that the natural morphism of (a) is an isomorphism. (Exercise 20.2.G was a special case of this exercise. You can likely generalize this to non-affine morphisms — the Cohomology and Flat Base Change Theorem 25.2.8 — but we wait until Chapter 25 to discuss flatness at length.)

20.7.C. EXERCISE (CF. EXERCISE 17.3.G). Prove Exercise 20.7.B(a) *without* the hypothesis that (20.7.2.1) is a fiber diagram, but adding the requirement that π' is quasicompact and separated (just so our definition of $R^i\pi'_*$ applies). In the course of the proof, you will see a map arising in the Leray spectral sequence. (Hint: *use* Exercise 20.7.B(a).)

A useful special case of Exercise 20.7.B(a) is the following.

20.7.D. EXERCISE. If $y \in Y$, describe a natural morphism $R^i\pi_*i(Y, \pi_*\mathcal{F}) \otimes \kappa(y) \to H^i(\pi^{-1}(y), \mathcal{F}|_{\pi^{-1}(y)})$. (Hint: the FHHF Theorem, Exercise 2.6.H.)

Thus the fiber of the pushforward may not be the cohomology of the fiber, but at least it always maps to it. We will later see that in good situations this map is an isomorphism, and thus the higher direct image sheaf indeed "patches together" the cohomology on fibers (the Cohomology and Base Change Theorem 25.7.5).

20.7.E. EXERCISE (PROJECTION FORMULA, GENERALIZING EXERCISE 17.3.H). Suppose $\pi : X \to Y$ is quasicompact and separated, and $\mathcal{E}$, $\mathcal{F}$ are quasicoherent sheaves on X and Y respectively.

(a) Describe a natural morphism

$$(R^i\pi_*\mathcal{E}) \otimes \mathcal{F} \to R^i\pi_*(\mathcal{E} \otimes \pi^*\mathcal{F}).$$

(Hint: the FHHF Theorem, Exercise 2.6.H.)
(b) If $\mathcal{F}$ is locally free, show that this natural morphism is an isomorphism.

The following fact uses the same trick as Theorem 18.3.10 and Exercise 18.3.H.

20.7.3. Theorem (relative dimensional vanishing). — *If $f : X \to Y$ is a projective morphism and Y is Noetherian (or more generally $\mathcal{O}_Y$ is coherent over itself), then the higher pushforwards vanish in degree higher than the maximum dimension of the fibers.*

This is false without the projective hypothesis, as shown by the following exercise. In particular, you might hope that just as dimensional vanishing generalized from projective varieties to quasiprojective varieties (§20.2.7) that relative dimensional vanishing would generalize from projective morphisms to quasiprojective morphisms, but this is not the case.

20.7.F. EXERCISE. Consider the open immersion $\pi : \mathbb{A}^n - \{0\} \to \mathbb{A}^n$. By direct calculation, show that $R^{n-1}f_*\mathcal{O}_{\mathbb{A}^n-\{0\}} \neq 0$. (This calculation will remind you of the proof of the H^n part of Theorem 20.1.2, see also Remark 20.3.1.)

Proof of Theorem 20.7.3. Let m be the maximum dimension of all the fibers.

The question is local on Y, so we will show that the result holds near a point p of Y. We may assume that Y is affine, and hence that $X \hookrightarrow \mathbb{P}^n_Y$.

Let k be the residue field at p. Then $f^{-1}(p)$ is a projective k-scheme of dimension at most m. By Exercise 12.3.C we can find affine open sets $D(f_1)$, ..., $D(f_{m+1})$ that cover $f^{-1}(p)$. In other words, the intersection of $V(f_i)$ does not intersect $f^{-1}(p)$.

If $Y = \operatorname{Spec} A$ and $p = [\mathfrak{p}]$ (so $k = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$), then arbitrarily lift each f_i from an element of $k[x_0, \dots, x_n]$ to an element f_i' of $A_\mathfrak{p}[x_0, \dots, x_n]$. Let F be the product of the denominators of the f_i'; note that $F \notin \mathfrak{p}$, i.e. $p = [\mathfrak{p}] \in D(F)$. Then $f_i' \in A_F[x_0, \dots, x_n]$. The intersection of their zero loci $\cap V(f_i') \subset \mathbb{P}^n_{A_F}$ is a closed subscheme of $\mathbb{P}^n_{A_F}$. Intersect it with X to get another closed subscheme of $\mathbb{P}^n_{A_F}$. Take its image under f; as projective morphisms are closed, we get a closed subset of $D(F) = \operatorname{Spec} A_F$. But this closed subset does not include p; hence we can find an affine neighborhood $\operatorname{Spec} B$ of p in Y missing the image. But if f_i'' are the restrictions of f_i' to $B[x_0, \dots, x_n]$, then $D(f_i'')$ cover $f^{-1}(\operatorname{Spec} B)$; in other words, over $f^{-1}(\operatorname{Spec} B)$ is covered by $m+1$ affine open sets, so by the affine-cover vanishing theorem, its cohomology vanishes in degree at least $m+1$. But the higher-direct image sheaf is computed using these cohomology groups, hence the higher direct image sheaf $R^if_*\mathcal{F}$ vanishes on $\operatorname{Spec} B$ too. $\square$

20.7.G. EXERCISE (RELATIVE SERRE VANISHING, CF. THEOREM 20.1.3(II)). Suppose $\pi : X \to Y$ is a proper morphism of Noetherian schemes, and $\mathcal{L}$ is a π-ample invertible sheaf on X. Show that for any coherent sheaf $\mathcal{F}$ on X, for $m \gg 0$, $R^i\pi_*\mathcal{F} \otimes \mathcal{L}^{\otimes m} = 0$ for all $i > 0$.

20.8 ⋆ "Proper pushforwards of coherents are coherent", and Chow's lemma

The proofs in this section are starred because the results aren't absolutely necessary in the rest of our discussions, and may not be worth reading right now. But just knowing the statement Grothendieck's Coherence Theorem 20.8.1, (generalizing Theorem 20.7.1(d)) will allow you to immediately translate many of our arguments about projective schemes and morphisms to proper schemes and morphisms, and Chow's Lemma is a multi-purpose tool to extend results from the projective situation to the proper situation in general.

20.8.1. Grothendieck's Coherence Theorem. — *Suppose $\pi : X \to Y$ is a proper morphism of locally Noetherian schemes. Then for any coherent sheaf $\mathcal{F}$ on X, $R^i\pi_*\mathcal{F}$ is coherent on Y.*

The special case of $i = 0$ has already been mentioned a number of times.

20.8.A. EXERCISE. Recall that finite morphisms are affine (by definition) and proper. Use Theorem 20.8.1 to show that if $\pi : X \to Y$ is proper and affine and Y is Noetherian, then π is finite. (Hint: mimic the proof of the weaker result where proper is replaced by projective, Corollary 20.1.7.)

The proof of Theorem 20.8.1 requires two sophisticated facts. The first is the Leray Spectral Sequence. Suppose $f : X \to Y$ and $g : Y \to Z$ are quasicompact separated morphisms. Then for any quasicoherent sheaf $\mathcal{F}$ on X, there is a spectral sequence with E_2 term given by $R^p g_*(R^q f_*\mathcal{F})$ abutting to $R^{p+q}(g \circ f)_*\mathcal{F}$. Because this would be a reasonable (but hard) exercise in the case we need it (where Z is affine), we will feel comfortable using it. But because we will later prove it in Exercise 24.4.E (which applies in this situation because of Exercise 24.5.G), we won't prove it now.

We will also need Chow's Lemma.

20.8.2. Chow's Lemma. — *Suppose $\pi : X \to \operatorname{Spec} A$ is a proper morphism, and A is Noetherian. Then there exists $\rho : X' \to X$ which is surjective and projective, such that $\pi \circ \rho$ is also projective, and such that ρ is an isomorphism on a dense open subset of X.*

Many generalizations of results from projective to proper situations go through Chow's Lemma. We will prove this version, and state other versions of Chow's Lemma, in §20.8.3. Assuming these two facts, we now prove Theorem 20.8.1 in a series of exercises.

⋆ *Proof.* The question is local on Y, so we may assume Y is affine, say $Y = \operatorname{Spec} A$. We work by induction on $\dim \operatorname{Supp} \mathcal{F}$, with the base case when $\dim \operatorname{Supp} \mathcal{F} = -1$ (i.e. $\operatorname{Supp} \mathcal{F} = \varnothing$, i.e. $\mathcal{F} = 0$), which is obvious. So fix $\mathcal{F}$, and assume the result is known for all coherent sheaves with support of smaller dimension.

20.8.B. EXERCISE. Show that we may assume that $\operatorname{Supp} \mathcal{F} = X$. (Hint: the idea is to replace X by the **scheme-theoretic support** of $\mathcal{F}$, the smallest closed subscheme of X on which $\operatorname{Supp} \mathcal{F}$ "lives". More precisely, it is the smallest closed subscheme $i : W \hookrightarrow X$ such that there is a coherent sheaf $\mathcal{F}'$ on W, with $\mathcal{F} \cong i_*\mathcal{F}'$. Show that this notion makes sense, using the ideas of §9.3, by defining it on each affine open subset.)

We now invoke Chow's Lemma to construct a projective morphism $\rho : X' \to X$ that is an isomorphism on a dense open subset U of X (so $\dim X \setminus U < \dim X$), and such that $\pi \circ \rho : X' \to \operatorname{Spec} A$ is projective.

Then $\mathcal{G} = \rho^* \mathcal{F}$ is a coherent sheaf on X', $\rho_* \mathcal{F}$ is a coherent sheaf on X (by the projective case, Theorem 20.7.1(d)) and the adjunction map $\mathcal{F} \to \rho_* \mathcal{G} = \rho_* \rho^* \mathcal{F}$ is an isomorphism on U. The kernel $\mathcal{E}$ and cokernel $\mathcal{H}$ are coherent sheaves on X that are supported in smaller dimension:

$$0 \to \mathcal{E} \to \mathcal{F} \to \rho_* \mathcal{G} \to \mathcal{H} \to 0.$$

20.8.C. EXERCISE. By the inductive hypothesis, the higher pushforwards of $\mathcal{E}$ and $\mathcal{H}$ are coherent. Show that if all the higher pushforwards of $\rho_* \mathcal{G}$ are coherent, then the higher pushforwards of $\mathcal{F}$ are coherent.

So we are reduced to showing that the higher pushforwards of $\rho_* \mathcal{G}$ are coherent for any coherent $\mathcal{G}$ on X'.

The Leray spectral sequence for $X' \xrightarrow{\rho} X \xrightarrow{\pi} \operatorname{Spec} A$ has E_2 term given by $R^p \pi_* (R^q \rho_* \mathcal{G})$ abutting to $R^{p+q} (\pi \circ \rho)_* \mathcal{G}$. Now $R^q \rho_* \mathcal{G}$ is coherent by Theorem 20.7.1(d). Furthermore, as ρ is an isomorphism on a dense open subset U of X, $R^q \rho_* \mathcal{G}$ is zero on U, and is thus supported on the complement of U, whose dimension is *less than* that of X. Hence by our inductive hypothesis, $R^p f_* (R^q \phi_* \mathcal{G}')$ is coherent for all p, and all $q \geq 1$. The only possibly noncoherent sheaves on the E_2 page are in the row $q = 0$ — precisely the sheaves we are interested in. Also, by Theorem 20.7.1(d) applied to $\pi \circ \rho$, $R^{p+q} (\pi \circ \rho)_* \mathcal{F}$ is coherent.

20.8.D. EXERCISE. Show that $E_n^{p,q}$ is always coherent for any $n \geq 2$, $q > 0$. Show that $E_n^{p,0}$ is coherent for a given $n \geq 2$ if and only if $E_2^{p,0}$ is coherent. Show that $E_\infty^{p,q}$ is coherent, and hence that $E_2^{p,0}$ is coherent, thereby completing the proof of Theorem 20.8.1.

□

20.8.3. ⋆⋆ **Proof (and other statements) of Chow's Lemma.**

We use the properness hypothesis on $X \to S$ through each of its three constituent parts: finite type, separated, universally closed. The parts using separatedness are particularly tricky.

As X is Noetherian, it has finitely many irreducible components. Cover X with affine open sets $U_1, \dots, U_n$. We may assume that each U_i meets each irreducible component. (If some U_i does not meet an irreducible component Z, then take any affine open subset Z' of $Z - \overline{X - Z}$, and replace U_i by $U_i \cup Z'$.) Then $U := \cap_i U_i$ is a dense open subset of X. As each U_i is finite type over A, we can choose a closed immersion $U_i \subset \mathbb{A}_A^{n_i}$. Let $\overline{U}_i$ be the (scheme-theoretic) closure of U_i in $\mathbb{P}_A^{n-i}$.

Now we have the diagonal morphism $U \to X \times_A \prod \overline{U}_i$ (where the product is over $\operatorname{Spec} A$), which is a locally closed immersion (the composition of the closed immersion $U \hookrightarrow U^n$ with the open immersion $U^n \hookrightarrow X \times_A \prod \overline{U}_i$). Let X' be the scheme-theoretic closure of U in $X \times_A \prod \overline{U}_i$. Let ρ be the composed morphism

$X \to X \times_A \prod \overline{U}_i \to X$, so we have a diagram

$$\begin{tikzcd}
X' \arrow[d, hook, "\text{cl. imm.}"'] \arrow[dr, "\rho"] & \\
X \times_A \prod \overline{U}_i \arrow[r, "\text{proj.}"'] \arrow[d, "\text{proper}"'] & X \arrow[d, "\text{proper}"] \\
\prod \overline{U}_i \arrow[r, "\text{proj.}"'] \arrow[d, "\text{proj.}"'] & S \\
\operatorname{Spec} A &
\end{tikzcd}$$

(where the square is Cartesian). The morphism ρ is projective (as it is the composition of two projective morphisms and X is quasicompact, Exercise 18.3.E). We will conclude the argument by showing that $\rho^{-1}(U) = U$ (or more precisely, ρ is an isomorphism above U), and that $X' \to \prod \overline{U}_i$ is a closed immersion (from which the composition

$$X \to \prod \overline{U}_i \to \operatorname{Spec} A$$

is projective).

20.8.E. EXERCISE. Suppose $T_0,, \dots, T_n$ are *separated* schemes over A with isomorphic open sets, which we sloppily call V in each case. Then V is a locally closed subscheme of $T_0 \times \cdots \times T_n$. Let $\overline{V}$ be the closure of this locally closed subscheme. Show that

$$\begin{aligned}
V &\cong \overline{V} \cap (V \times_A T_1 \times_A \cdots \times_A T_n) \\
&= \overline{V} \cap (T_0 \times_A V \times_A T_2 \times_A \cdots \times_A T_n) \\
&= \cdots \\
&= \overline{V} \cap (T_0 \times_A \cdots \times_A T_{n-1} \times_A V).
\end{aligned}$$

(Hint for the first isomorphism: the graph of the morphism $V \to T_1 \times_A \cdots \times_A T_n$ is a closed immersion, as $T_1 \times_A \cdots \times_A T_n$ is separated over A, by Proposition 11.1.18. Thus the closure of V in $V \times_A T_1 \times_A \cdots \times_A T_n$ is V itself. Finally, the scheme-theoretic closure can be computed locally, essentially by Theorem 9.3.4.)

20.8.F. EXERCISE. Using (the idea behind) the previous exercise, show that $\rho^{-1}(U) = U$.

It remains to show that $X' \to \prod \overline{U}_i$ is a closed immersion. Now $X' \to \prod \overline{U}_i$ is closed (it is the composition of two closed maps), so it suffices to show that $X' \to \prod \overline{U}_i$ is a locally closed immersion.

20.8.G. EXERCISE. Let A_i be the closure of U in

$$B_i := X \times_A \overline{U}_1 \times_A \cdots \times_A U_i \times_A \cdots \overline{U}_n$$

(only the ith term is missing the bar), and let C_i be the closure of U in

$$D_i := \overline{U}_1 \times_A \cdots \times_A U_i \times_A \cdots \overline{U}_n.$$

Show that there is an isomorphism $A_i \to C_i$ induced by the projection $B_i \to D_i$. Hint: note that the section $D_i \to B_i$ of the projection $B_i \to D_i$, given informally by $(t_1, \dots, t_n) \mapsto (t_i, t_1, \dots, t_n)$, is a closed immersion, as it can be interpreted as the graph of a map to a separated scheme (over A). So U can be interpreted as a locally closed subscheme of D_i, which in turn can be interpreted as a closed subscheme of B_i. Thus the closure of U in D_i may be identified with its closure in B_i.

As the U_i cover X, the $\rho^{-1}(U_i)$ cover $\overline{X}$. But $\rho^{-1}(U_i) = A_i$ (closure can be be computed locally — the closure of U in B_i is the intersection of B_i with the closure $\overline{X}$ of U in $X \times_A \overline{U}_1 \times_A \cdots \overline{U}_n$).

Hence over each U_i, we get a closed immersion of $A_i \hookrightarrow D_i$, and thus $X' \to \prod \overline{U}_i$ is a locally closed immersion as desired. $\square$

20.8.4. *Other versions of Chow's Lemma.* We won't use these versions, but their proofs are similar to what we have already shown.

20.8.H. EXERCISE. By suitably crossing out lines in the proof above, weaken the hypothesis "$X \to \operatorname{Spec} A$ proper" to "$X \to \operatorname{Spec} A$ finite type and separated", at the expense of weakening the conclusion "$\pi \circ \rho$ is projective" to "$\pi \circ \rho$ is quasiprojective".

20.8.I. EXERCISE. Prove the generalization where $\operatorname{Spec} A$ is replaced by an arbitrary Noetherian scheme.

I intend to add other versions here later. If you have favorites (ideally ones you have used), please feel free to nominate them!

CHAPTER 21

Application: Curves

We now use what we have developed to study something explicit — curves. Throughout this chapter, we will assume that all curves are projective, geometrically integral, nonsingular curves over a field k. We will sometimes add the hypothesis that k is algebraically closed. Most people are happy with working over algebraically closed fields, and those people should ignore the adverb "geometrically".

We certainly don't need the massive machinery we have developed in order to understand curves, but with the perspective we have gained, the development is quite clean. The key ingredients we will need are as follows. We use a criterion for a morphism to be a closed immersion, that we prove in §21.1. We use the "black box" of Serre duality (to be proved in Chapter 27). In §21.2, we use this background to observe a very few useful facts, which we will use repeatedly. Finally, in the course of applying them to understand curves of various genera, we develop the theory of hyperelliptic curves in a hands-on way (§21.4), in particular proving a special case of the Riemann-Hurwitz formula.

If you are jumping into this chapter without reading much beforehand, you should skip §21.1 (taking Theorem 21.1.1 as a black box). Depending on your background, you may want to skip §21.2 as well (taking the crucial observations as a black box).

21.1 A criterion for a morphism to be a closed immersion

We will repeatedly use a criterion for when a morphism is a closed immersion, which is not special to curves. This is the hardest fact proved in this chapter. Before stating it, we recall some facts about closed immersions. Suppose $f : X \to Y$ is a closed immersion. Then f is projective, and it is injective on points. This is not enough to ensure that it is a closed immersion, as the example of the normalization of the cusp shows (Figure 10.3). Another example is the following.

21.1.A. EXERCISE (FROBENIUS). Suppose $\operatorname{char} k = p$, and π is the map $\pi : \mathbb{A}^1_k \to \mathbb{A}^1_k$ given by $x \mapsto x^p$. Show that π is a bijection on points, and even induces an isomorphism of residue fields on closed points, yet is not a closed immersion.

The additional information you need is that the tangent map is an isomorphism at all closed points.

21.1.B. EXERCISE. Show (directly, not invoking Theorem 21.1.1) that in the two examples described above (the normalization of a cusp and the Frobenius morphism), the tangent map is *not* an isomorphism at all closed points.

21.1.1. Theorem. — *Suppose* $k = \overline{k}$, *and* $f : X \to Y$ *is a projective morphism of finite-type* k*-schemes that is injective on closed points and injective on tangent vectors at closed points. Then* f *is a closed immersion.*

Remark: "injective on closed points and tangent vectors at closed points" means that f is unramified (under these hypotheses). (We will defined *unramified* in §23.4.5; in general unramified morphisms need not be injective.)

The example $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ shows that we need the hypothesis that k is algebraically closed in Theorem 21.1.1. Those allergic to algebraically closed fields should still pay attention, as we will use this to prove things about curves over k where k is *not* necessarily algebraically closed (see also Exercises 10.2.K and 21.1.E).

We need the hypothesis that the morphism be projective, as shown by the example of Figure 21.1. It is the normalization of the node, except we erase one of the preimages of the node. We map $\mathbb{A}^1$ to the plane, so that its image is a curve with one node. We then consider the morphism we get by discarding one of the preimages of the node. Then this morphism is an injection on points, and is also injective on tangent vectors, but it is not a closed immersion. (In the world of differential geometry, this fails to be an embedding because the map doesn't give a homeomorphism onto its image.)

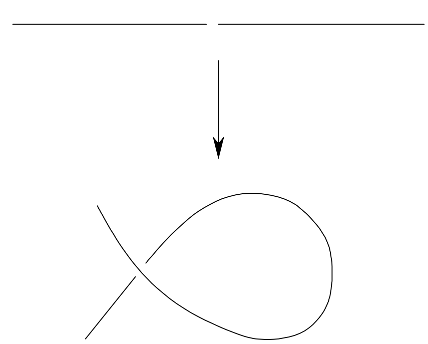


FIGURE 21.1. We need the projective hypothesis in Theorem 21.1.1

Theorem 21.1.1 appears to be fundamentally a statement about varieties, but it isn't. We will reduce it to the following result.

21.1.2. Theorem. — *Suppose* $f : X \to Y$ *is a finite morphism of Noetherian schemes whose degree at every point of* Y *(§14.7.5) is* 0 *or* 1. *Then* f *is a closed immersion.*

Once we know the meaning of "unramified", this will translate to: "unramified + finite = closed immersion for Noetherian schemes".

21.1.C. EXERCISE. Suppose $f : X \to Y$ is a finite morphism whose degree at every point of Y is 0 or 1. Show that f is injective on points (easy). If $x \in X$ is any point, show that f induces an isomorphism of residue fields $\kappa(f(x)) \to \kappa(x)$. Show that f induces an injection of tangent spaces. Thus key hypotheses of Theorem 21.1.1 are implicitly in the hypotheses of Theorem 21.1.2.

21.1.3. *Reduction of Theorem 21.1.1 to Theorem 21.1.2.* The property of being a closed immersion is local on the base, so we may assume that Y is affine, say $\operatorname{Spec} B$.

I next claim that f has finite fibers, not just finite fibers above closed points: the fiber dimension for projective morphisms is upper semicontinuous (Exercise 18.3.H), so the locus where the fiber dimension is at least 1 is a closed subset, so if it is non-empty, it must contain a closed point of Y. Thus the fiber over any point is a dimension 0 finite type scheme over that point, hence a finite set.

Hence f is a projective morphism with finite fibers, thus finite by Corollary 18.3.10.

But the degree of a finite morphism is upper semicontinuous, (§14.7.5), and is at most 1 at closed points of Y, hence is at most 1 at all points.

21.1.4. *Proof of Theorem 21.1.2. Reduction to Y affine.* The problem is local on Y, so we may assume Y is affine, say $Y = \operatorname{Spec} B$. Thus X is affine too, say $\operatorname{Spec} A$, and f corresponds to a ring morphism $B \to A$. We wish to show that this is a surjection of rings, or (equivalently) of B-modules.

Reduction to Y local. We will how that for any maximal ideal $\mathfrak{n}$ of B, $B_{\mathfrak{n}} \to A_{\mathfrak{n}}$ is a surjection of $B_{\mathfrak{n}}$-modules. (This implies that $B \to A$ is a surjection. Here is why: if K is the cokernel, so $B \to A \to K \to 0$, then we wish to show that $K = 0$. Now A is a finitely generated B-module, so K is as well, being the image of A. Thus $\operatorname{Supp} K$ is a closed set. If $K \neq 0$, then $\operatorname{Supp} K$ is non-empty, and hence contains a closed point $[\mathfrak{n}]$. Then $K_{\mathfrak{n}} \neq 0$, so from the exact sequence $B_{\mathfrak{n}} \to A_{\mathfrak{n}} \to K_{\mathfrak{n}} \to 0$, $B_{\mathfrak{n}} \to A_{\mathfrak{n}}$ is not a surjection.) Thus it remains to deal the case where Y is Spec of a local ring $(B, \mathfrak{n})$.

So far this argument is a straightforward sequence of reduction steps and facts we know well. But things now start to get subtle.

Then show that X is local, $X = \operatorname{Spec} A_{\mathfrak{m}}$. If $A_{\mathfrak{n}} = 0$, $B_{\mathfrak{n}}$ trivially surjects onto $A_{\mathfrak{n}}$, so assume $A_{\mathfrak{n}} \neq 0$. We next show that $A_{\mathfrak{n}} = A \otimes_B B_{\mathfrak{n}}$ is a local ring. Proof: $A_{\mathfrak{n}} \neq 0$, so $A_{\mathfrak{n}}$ *has* a prime ideal. Any point $\mathfrak{p}$ of $\operatorname{Spec} A_{\mathfrak{n}}$ maps to some point of $\operatorname{Spec} B_{\mathfrak{n}}$, which has $[\mathfrak{n}]$ in its closure. Thus by the Lying Over Theorem 8.2.5 ($\operatorname{Spec} A_{\mathfrak{n}} \to \operatorname{Spec} B_{\mathfrak{n}}$ is a finite morphism as it is obtained by base change from $\operatorname{Spec} A \to \operatorname{Spec} B$), there is a point $\mathfrak{q}$ in the closure of $\mathfrak{p}$ that maps to $[\mathfrak{n}]$. But by the "degree at most 1 at every point" hypothesis there is at most one point of $\operatorname{Spec} A_{\mathfrak{n}}$ mapping to $[\mathfrak{n}]$, which we denote $[\mathfrak{m}]$. Thus we have shown that $\mathfrak{m}$ contains all other prime ideals of $\operatorname{Spec} A_{\mathfrak{n}}$, so $A_{\mathfrak{n}}$ is a local ring.

Finally, we apply Nakayama twice. We complete the argument backwards, in order to motivate the clever double invocation of Nakayama. We wish to show that the sequence $B \to A \to 0$ of B-modules is exact. If the image of $1 \in B$ generates A as a B-module *modulo the maximal ideal $\mathfrak{n}$ of B*, we would be done, by Nakayama's lemma (using the local ring B). But we also know that $B/\mathfrak{n} \to A/\mathfrak{m}$ is an isomorphism, as f induces an isomorphism of residue fields (Exercise 21.1.C). So it suffices to show that $A/\mathfrak{m} = A/\mathfrak{n}$, i.e. that the injection $\mathfrak{n}A \to \mathfrak{m}A$ is also a surjection. By our Noetherian hypotheses, $\mathfrak{n}$ and $\mathfrak{m}$ are finitely generated A-modules. Now injectivity of tangent vectors (Exercise 21.1.C) means surjectivity of cotangent vectors, so $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is a surjection, hence $\mathfrak{n} \to \mathfrak{m}/\mathfrak{m}^2$ is a surjection, so $\mathfrak{n}A \to \mathfrak{m}A$ is a surjection modulo $\mathfrak{m}$. Hence by Nakayama's lemma using the local ring A, we indeed have that $\mathfrak{n}A_{\mathfrak{n}} = \mathfrak{m}A_{\mathfrak{n}}$. $\square$

21.1.D. EXERCISE. Use Theorem 21.1.1 to show that the dth Veronese morphism from $\mathbb{P}^n_k$, corresponding to the complete linear series $|\mathcal{O}_{\mathbb{P}^n_k}(d)|$, is a closed immersion. Do the same for the Segre morphism from $\mathbb{P}^m_k \times_{\operatorname{Spec} k} \mathbb{P}^n_k$. (This is just for practice for using this criterion. This is a weaker result than we had before; we have earlier checked both of these statements over an arbitrary base ring in Remark 9.2.8 and §10.5 respectively, and we are now checking it only over algebraically closed fields. However, see Exercise 21.1.E below.)

Exercise 10.2.K can be used to extend Theorem 21.1.1 to general fields k, not necessarily algebraically closed.

21.1.E. LESS IMPORTANT EXERCISE. Using the ideas from this section, prove that the dth Veronese morphism from $\mathbb{P}^n_{\mathbb{Z}}$ (over the integers!), is a closed immersion. (Again, we have done this before. This exercise is simply to show that these methods can easily extend to work more generally.)

21.2 A series of crucial observations

We are now ready to start understanding curves in a hands-on way. We will repeatedly make use of the following series of crucial remarks, and it will be important to have them at the tip of your tongue.

In what follows, C will be a projective, geometrically nonsingular, geometrically integral curve over a field k, and $\mathcal{L}$ is an invertible sheaf on C. (Often, what matters is integrality rather than geometric integrality, but most readers aren't worrying about this distinction, and those that are can weaken hypotheses as they see fit.)

21.2.1. Reminder: Serre duality. Serre duality (Theorem 20.4.5) on a geometrically irreducible nonsingular genus g curve C over k involves an invertible sheaf $\mathcal{K}$ (of degree $2g-2$, with g sections, Exercise 20.4.H), such that for any coherent sheaf $\mathcal{F}$ on C, $h^i(C, \mathcal{F}) = h^{1-i}(X, \mathcal{K} \otimes \mathcal{F}^\vee)$ for $i = 0, 1$. (Better: there is a duality between the two cohomology groups.)

21.2.2. Negative degree line bundles have no section. $h^0(C, \mathcal{L}) = 0$ if $\deg \mathcal{L} < 0$. Reason: $\deg \mathcal{L}$ is the number of zeros minus the number of poles (suitably counted) of any rational section (Important Exercise 20.4.C). If there is a regular section (i.e. with no poles), then this is necessarily non-negative. Refining this argument gives:

21.2.3. Degree 0 line bundles, and recognizing when they are trivial. $h^0(C, \mathcal{L}) = 0$ or 1 if $\deg \mathcal{L} = 0$, and if $h^0(C, \mathcal{L}) = 1$ then $\mathcal{L} \cong \mathcal{O}_C$. Reason: if there is a section s, it has no poles, and hence no zeros, because $\deg \mathcal{L} = 0$. Then $\operatorname{div} s = 0$, so $\mathcal{L} \cong \mathcal{O}_C(\operatorname{div} s) = \mathcal{O}_C$. (Recall how this works, cf. Important Exercise 15.2.E: s gives a trivialization for the invertible sheaf. We have a natural bijection for any open set $\Gamma(U, \mathcal{L}) \leftrightarrow \Gamma(U, \mathcal{O}_U)$, where the map from left to right is $s' \mapsto s'/s$, and the map from right to left is $f \mapsto sf$.) Conversely, for a geometrically integral projective variety, $h^0(\mathcal{O}) = 1$. (Exercise 20.1.C shows this for k algebraically closed, and Exercise 20.2.G shows that cohomology commutes with base field extension.)

Serre duality turns these statements about line bundles of degree at most 0 into statements about line bundles of degree at least $2g-2$.

21.2.4. We know $h^0(C, \mathcal{L})$ if the degree is sufficiently high. If $\deg \mathcal{L} > 2g-2$, then

$$h^0(C, \mathcal{L}) = \deg \mathcal{L} - g - 1. \tag{21.2.4.1}$$

So we know $h^0(C, \mathcal{L})$ if $\deg \mathcal{L} \gg 0$. *(This is important — remember this!)* Reason: $h^1(C, \mathcal{L}) = h^0(C, \mathcal{K} \otimes \mathcal{L}^\vee)$; but $\mathcal{K} \otimes \mathcal{L}^\vee$ has negative degree (as $\mathcal{K}$ has degree $2g-2$), and thus this invertible sheaf has no sections. The result then follows from the Riemann-Roch theorem 20.4.B.

21.2.A. USEFUL EXERCISE (RECOGNIZING $\mathcal{K}$ AMONG DEGREE $2g-2$ LINE BUNDLES). Suppose $\mathcal{L}$ is a degree $2g-2$ invertible sheaf. Show that it has $g-1$ or g sections, and it has g sections if and only if $\mathcal{L} \cong \mathcal{K}$.

21.2.5. Twisting $\mathcal{L}$ by a (degree 1) point changes h^0 by at most 1. Suppose p is any closed point of degree 1 (i.e. the residue field of p is k). Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 0$ or 1. (The twist of $\mathcal{L}$ by a divisor, such as $\mathcal{L}(-p)$, was defined in §15.2.8.) Reason: consider $0 \to \mathcal{O}_C(-p) \to \mathcal{O}_C \to \mathcal{O}|_p \to 0$, tensor with $\mathcal{L}$ (this is exact as $\mathcal{L}$ is locally free) to get

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{L}|_p \to 0.$$

Then $h^0(C, \mathcal{L}|_p) = 1$, so as the long exact sequence of cohomology starts off

$$0 \to H^0(C, \mathcal{L}(-p)) \to H^0(C, \mathcal{L}) \to H^0(C, \mathcal{L}|_p),$$

we are done.

21.2.6. A numerical criterion for $\mathcal{L}$ to be base-point-free. Suppose for this remark that k is algebraically closed, so *all* closed points have degree 1 over k. Then if $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 1$ for *all* closed points p, then $\mathcal{L}$ is base-point-free, and hence induces a morphism from C to projective space (Theorem 17.4.1). Reason: given any p, our equality shows that there exists a section of $\mathcal{L}$ that does not vanish at p.

21.2.7. Next, suppose p and q are distinct (closed) points of degree 1. Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 0, 1$, or 2 (by repeating the argument of Remark 21.2.5 twice). If $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then necessarily

$$h^0(C, \mathcal{L}) = h^0(C, \mathcal{L}(-p)) + 1 = h^0(C, \mathcal{L}(-q)) + 1 = h^0(C, \mathcal{L}(-p-q)) + 2. \tag{21.2.7.1}$$

Then the linear series $\mathcal{L}$ separates points p and q, i.e. the corresponding map f to projective space satisfies $f(p) \neq f(q)$. Reason: there is a hyperplane of projective space passing through p but not passing through q, or equivalently, there is a section of $\mathcal{L}$ vanishing at p but not vanishing at q. This is because of the last equality in (21.2.7.1).

21.2.8. By the same argument as above, if p is a (closed) point of degree 1, then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2p)) = 0, 1$, or 2. I claim that if this is 2, then map corresponds to $\mathcal{L}$ (which is already seen to be base-point-free from the above) separates the tangent vectors at p. To show this, we need to show that the cotangent map is *surjective*. To show surjectivity onto a one-dimensional vector space, I just need to show that the map is non-zero. So I need to give a function on the target vanishing at the image of p that pulls back to a function that vanishes at p to order 1 but not 2. In other words, we want a section of $\mathcal{L}$ vanishing at p to order 1 but not 2. But that is the content of the statement $h^0(C, \mathcal{L}(-p)) - h^0(C, \mathcal{L}(-2p)) = 1$.

21.2.9. Criterion for $\mathcal{L}$ to be very ample. Combining some of our previous comments: suppose C is a curve over an *algebraically closed* field k, and $\mathcal{L}$ is an invertible sheaf such that for *all* closed points p and q, *not necessarily distinct*, $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then $\mathcal{L}$ gives a *closed immersion into projective space*, as it separates points and tangent vectors, by Theorem 21.1.1.

21.2.B. EXERCISE. Suppose that k is algebraically closed, so the previous remark applies. Show that $C \setminus \{p\}$ is affine. (Hint: Show that if $k \gg 0$, then $\mathcal{O}(kp)$ is base-point-free and has at least two linearly independent sections, one of which has divisor kp. Use these two sections to map to $\mathbb{P}^1$ so that the set-theoretic preimage of ∞ is p. Argue that the map is finite, and that $C \setminus \{p\}$ is the preimage of $\mathbb{A}^1$.)

21.2.10. Conclusion. We can combine much of the above discussion to give the following useful fact. If k is algebraically closed, then $\deg \mathcal{L} \geq 2g$ implies that $\mathcal{L}$ is base-point-free (and hence determines a morphism to projective space). Also, $\deg \mathcal{L} \geq 2g + 1$ implies that this is in fact a closed immersion (so $\mathcal{L}$ is very ample). Remember this!

21.2.C. EXERCISE. Show that an invertible sheaf $\mathcal{L}$ on projective, nonsingular integral curve over $\overline{k}$ is ample if and only if $\deg \mathcal{L} > 0$.

(This can be extended to curves over general fields using Exercise 21.2.D below.) Thus there is a blunt purely numerical criterion for ampleness of line bundles on curves. This generalizes to projective varieties of higher dimension; this is called *Nakai's criterion for ampleness*, Theorem 22.3.1.

21.2.D. EXERCISE. Show that the statements in §21.2.10 even without the hypothesis that k is algebraically closed. (Hint: to show one of the facts about some curve C and line bundle $\mathcal{L}$, consider instead $C \otimes_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. Then show that if the pullback of $\mathcal{L}$ here has sections giving you one of the two desired properties, then there are sections downstairs with the same properties. You may want to use facts that we have used, such as the fact that base-point-freeness is independent of extension of base field, Exercise 20.2.H, or that the property of an affine morphism over k being a closed immersion holds if and only if it does after an extension of k, Exercise 10.2.K.)

21.2.E. EXERCISE (ON A PROJECTIVE NONSINGULAR INTEGRAL CURVE, AMPLE = POSITIVE DEGREE). Suppose $\mathcal{L}$ is an invertible sheaf on a projective, geometrically nonsingular, geometrically integral curve C (over k). Show that $\mathcal{L}$ is ample if and only if it has positive degree. (This was promised in Exercise 20.6.F.)

We are now ready to take these facts and go to the races.

21.3 Curves of genus 0

We are now ready to (in some form) answer the question: what are the curves of genus 0?

In §7.5.8, we saw a genus 0 curve (over a field k) that was *not* isomorphic to $\mathbb{P}^1$: $x^2+y^2+z^2=0$ in $\mathbb{P}^2_{\mathbb{R}}$. (It has genus 0 by (20.5.3.1).) We have already observed that this curve is *not* isomorphic to $\mathbb{P}^1_{\mathbb{R}}$, because it doesn't have an $\mathbb{R}$-valued point. On the other hand, we haven't seen a genus 0 curve over an algebraically closed field with this property. This is no coincidence: the lack of an existence of a k-valued point is the only obstruction to a genus 0 curve being $\mathbb{P}^1$.

21.3.1. Proposition. — *Suppose C is genus 0, and C has a k-valued (degree 1) point. Then $C \cong \mathbb{P}^1_k$.*

Thus we see that all genus 0 (integral, nonsingular) curves over an algebraically closed field are isomorphic to $\mathbb{P}^1$.

Proof. Let p be the point, and consider $\mathcal{L} = \mathcal{O}(p)$. Then $\deg \mathcal{L} = 1$, so we can apply what we know above: first, $h^0(C, \mathcal{L}) = 2$ (Remark 21.2.4), and second, these two sections give a closed immersion into $\mathbb{P}^1_k$ (Remark 21.2.10). But the only closed immersion of a curve into the integral curve $\mathbb{P}^1_k$ is an isomorphism! $\square$

As a bonus, Proposition 21.3.1 implies that $x^2+y^2+z^2=0$ in $\mathbb{P}^2_{\mathbb{R}}$ has no *line bundles* of degree 1 over $\mathbb{R}$; otherwise, we could just apply the above argument to the corresponding line bundle. This example shows us that over a non-algebraically closed field, there can be genus 0 curves that are not isomorphic to $\mathbb{P}^1_k$. The next result lets us get our hands on them as well.

21.3.2. Claim. — *All genus 0 curves can be described as conics in $\mathbb{P}^2_k$.*

Proof. Any genus 0 curve has a degree -2 line bundle — the canonical bundle $\mathcal{K}$. Thus any genus 0 curve has a degree 2 line bundle: $\mathcal{L} = \mathcal{K}^\vee$. We apply Remark 21.2.10: $\deg \mathcal{L} = 2 \geq 2g+1$, so this line bundle gives a closed immersion into $\mathbb{P}^2$. $\square$

21.3.A. EXERCISE. Suppose C is a genus 0 curve (projective, geometrically integral and nonsingular). Show that C has a point of degree at most 2. (The degree of a point was defined in §6.3.8.)

The geometric means of finding Pythagorean triples presented in §7.5.7 looked quite different, but was really the same. There was a genus 0 curve C (a plane conic) with a k-valued point p, and we proved that it was isomorphic to $\mathbb{P}^1_k$. The line bundle used to show the isomorphism wasn't the degree 1 line bundle $\mathcal{O}_C(p)$; it was the degree 1 line bundle $\mathcal{O}_{\mathbb{P}^2}(1)|_C \otimes \mathcal{O}_C(-p)$.

We will use the following result later.

21.3.3. Proposition. — *Suppose C is not isomorphic to $\mathbb{P}^1_k$ (with no restrictions on the genus of C), and $\mathcal{L}$ is an invertible sheaf of degree 1. Then $h^0(C, \mathcal{L}) < 2$.*

Proof. Otherwise, let s_1 and s_2 be two (independent) sections. As the divisor of zeros of s_i is the degree of $\mathcal{L}$, each vanishes at a single point p_i (to order 1). But $p_1 \neq p_2$ (or else s_1/s_2 has no poles or zeros, i.e. is a constant function, i.e. s_1 and

s_2 are dependent). Thus we get a map $C \to \mathbb{P}^1$ which is base-point-free. This is a finite degree 1 map of nonsingular curves, which induces a degree 1 extension of function fields, i.e. an isomorphism of function fields, which means that the curves are isomorphic. But we assumed that C is not isomorphic to $\mathbb{P}^1_k$. $\square$

21.3.4. Corollary. — *If* C *is a projective nonsingular geometrically integral curve over* k*, and* p *and* q *are degree* 1 *points, then* $\mathcal{O}_C(p) \cong \mathcal{O}_C(q)$ *if and only if* $p = q$.

21.3.B. EXERCISE. Show that if k is algebraically closed, then C has genus 0 if and only if all degree 0 line bundles are trivial.

21.4 Hyperelliptic curves

We next discuss an important class of curves, the hyperelliptic curves. In this section, we assume k is algebraically closed of characteristic not 2. (These hypotheses can be relaxed, at some cost.)

A (projective nonsingular irreducible) genus g curve C is **hyperelliptic** if it admits a double cover of (i.e. degree 2, necessarily finite, morphism to) $\mathbb{P}^1_k$. For convenience, when we say C is hyperelliptic, we will implicitly have in mind a *choice* of double cover $\pi : C \to \mathbb{P}^1$. (We will later see that if $g \geq 2$, then there is at most one such double cover, Proposition 21.4.7, so this is not a huge assumption.) The map π is called the **hyperelliptic map**.

By Exercise 18.4.D, the preimage of any closed point p of $\mathbb{P}^1$ consists of either one or two points. If $\sharp(\pi^{-1}p) = 1$, we say p is a **branch point**, and $\pi^{-1}p$ is a **ramification point** of π. (The notion of ramification will be defined more generally in §23.4.5.)

21.4.1. Theorem (hyperelliptic Riemann-Hurwitz formula). — *Suppose* $k = \overline{k}$ *and* $\operatorname{char} k \neq 2$, $\pi : C \to \mathbb{P}^1_k$ *is a double cover by a projective nonsingular irreducible genus* g *curve over* k. *Then* π *has* $2g + 2$ *branch points.*

This is a special case of the Riemann-Hurwitz formula, which we will state and prove in §23.5. You may have already heard about genus 1 complex curves double covering $\mathbb{P}^1$, branched over 4 points.

To prove Theorem 21.4.1, we prove the following.

21.4.2. Proposition. — *Assume* $\operatorname{char} k \neq 2$ *and* $k = \overline{k}$. *Given* n *distinct points* $p_1, \dots, p_r \in \mathbb{P}^1$*, there is precisely one double cover branched at precisely these points if* r *is even, and none if* r *is odd.*

Proof. Pick points 0 and ∞ of $\mathbb{P}^1$ distinct from the r branch points. All r branch points are in $\mathbb{P}^1 - \infty = \mathbb{A}^1 = \operatorname{Spec} k[x]$. Suppose we have a double cover of $\mathbb{A}^1$, $C' \to \mathbb{A}^1$, where x is the coordinate on $\mathbb{A}^1$. This induces a quadratic field extension K over $k(x)$. As $\operatorname{char} k \neq 2$, this extension is Galois. Let $\sigma : K \to K$ be the Galois involution. Let y be an element of K such that $\sigma(y) = -y$, so 1 and y form a basis for K over the field $k(x)$, and are eigenvectors of σ. Now $\sigma(y^2) = y^2$, so $y^2 \in k(x)$. We can replace y by an appropriate $k(x)$-multiple so that y^2 is a polynomial, with no repeated factors, and monic. (This is where we use the hypothesis that k is algebraically closed, to get leading coefficient 1.)

Thus $y^2 = x^N + a_{N-1}x^{N-1} + \cdots + a_0$, where the polynomial on the right (call it $f(x)$) has no repeated roots. The Jacobian criterion (in the guise of Exercise 13.2.D) implies that this curve C_0' in $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$ is nonsingular. Then C_0' is normal and has the same function field as C. Thus C_0' and C' are both normalizations of $\mathbb{A}^1$ in the finite field extension generated by y, and hence are isomorphic. Thus we have identified C' in terms of an explicit equation.

The branch points correspond to those values of x for which there is exactly one value of y, i.e. the roots of $f(x)$. In particular, $N = n$, and $f(x) = (x-p_1)\cdots(x-p_r)$, where the p_i are interpreted as elements of $\overline{k}$.

Having mastered the situation over $\mathbb{A}^1$, we return to the situation over $\mathbb{P}^1$. We will examine the branched cover over the affine open set $\mathbb{P}^1 \setminus \{0\} = \operatorname{Spec} k[u]$, where $u = 1/x$. The previous argument applied to $\operatorname{Spec} k[u]$ rather than $\operatorname{Spec} k[x]$ shows that any such double cover must be of the form

$$\begin{aligned} C'' &= \operatorname{Spec} k[z,u]/(z^2 - (u - 1/p_1)\cdots(u-1/p_r)) = \operatorname{Spec} k[z,u]/(z^2 - u^r f(1/u)) \\ &\to \operatorname{Spec} k[u] = \mathbb{A}^1. \end{aligned}$$

So if there is a double cover over all of $\mathbb{P}^1$, it must be obtained by gluing C'' to C' over the gluing of $\operatorname{Spec} k[x]$ to $\operatorname{Spec} k[u]$ to obtain $\mathbb{P}^1$.

Thus in $K(C)$, we must have

$$z^2 = u^r f(1/u) = f(x)/x^r = y^2/x^r$$

from which $z^2 = y^2/x^r$.

If r is even, considering $K(C)$ as generated by y and x, there are two possible values of z: $z = \pm y^2/x^{r/2}$. After renaming z by $-z$ if necessary, there is a single way of gluing these two patches together (we choose the positive square root).

If r is odd, the result follows from Exercise 21.4.A below. $\square$

21.4.A. Exercise. Show that x does not have a square root in the field $k(x)[y]/(y^2 - f(x))$, where f is a polynomial with non-zero roots $p_1, \dots, p_r$. (Possible hint: why is $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$?)

For future reference, we collect here our explicit (two-affine) description of the hyperelliptic cover $C \to \mathbb{P}^1$.

(21.4.2.1)
$$\begin{tikzcd} \operatorname{Spec} k[x,y]/(y^2 - f(x)) \arrow[r, dash, "z=y/x^{r/2}", "y=z/u^{r/2}"'] \arrow[d] & \operatorname{Spec} k[u,z]/(z^2 - u^r f(1/u)) \arrow[d] \\ \operatorname{Spec} k[x] \arrow[r, dash, "u=1/x", "x=1/u"'] & \operatorname{Spec} k[u] \end{tikzcd}$$

21.4.3. *If k is not algebraically closed.* If k is not algebraically closed (but of characteristic not 2), the above argument shows that if we have a double cover of $\mathbb{A}^1$, then it is of the form $y^2 = af(x)$, where f is monic, and $a \in k^\times/(k^\times)^2$. You may be able to use this to show that (assuming the $k^\times \neq (k^\times)^2$) a double cover is *not* determined by its branch points. Moreover, this failure is classified by $k^\times/(k^\times)^2$. Thus we have lots of curves that are not isomorphic over k, but become isomorphic over $\overline{k}$. These are often called *twists* of each other.

(In particular, once we define elliptic curves, you will be able to show that there exist two elliptic curves over $\mathbb{Q}$ with the same j-invariant, that are not isomorphic, see Exercise 21.8.D.)

21.4.4. *Back to proving the hyperelliptic Riemann-Hurwitz formula, Theorem 21.4.1.* Our explicit description of the unique double cover of $\mathbb{P}^1$ branched over r different points will allow us to compute the genus, thereby completing the proof of Theorem 21.4.1.

We continue the notation (21.4.2.1) of the proof of Proposition 21.4.2. Suppose $\mathbb{P}^1$ has affine cover by $\operatorname{Spec} k[x]$ and $\operatorname{Spec} k[u]$, with $u = 1/x$, as usual. Suppose $C \to \mathbb{P}^1$ is a double cover, given by $y^2 = f(x)$ over $\operatorname{Spec} k[x]$, where f has degree r, and $z^2 = u^r f(1/u)$. Then C has an affine open cover by $\operatorname{Spec} k[x,y]/(y^2 - f(x))$ and $\operatorname{Spec} k[u,z]/(z^2 - u^r f(1/u))$. The corresponding Čech complex for $\mathcal{O}_C$ is

$$0 \longrightarrow k[x,y]/(y^2 - f(x)) \times k[u,z]/(z^2 - u^r f(u)) \xrightarrow{d}$$

$$\left(k[x,y]/(y^2 - f(x))\right)_x \longrightarrow 0.$$

The degree 1 part of the complex has basis consisting of monomials $x^n y^\epsilon$, where $n \in \mathbb{Z}$ and $\epsilon = 0$ or 1. To compute the genus $g = h^1(C, \mathcal{O}_C)$, we must compute $\operatorname{coker} d$. We can use the first factor $k[x,y]/(y^2 - f(x))$ to hit the monomials $x^n y^\epsilon$ where $n \in \mathbb{Z}^{\geq 0}$, and $\epsilon = 0$ or 1. The image of the second factor is generated by elements of the form $u^m z^\epsilon$, where $m \geq 0$ and $\epsilon = 0$ or 1. But $u^m z^\epsilon = x^{-m}(y/x^{r/2})^\epsilon$. By inspection, the cokernel has basis generated by monomials $x^{-1}y$, $x^{-2}y$, $\dots$, $x^{-r/2+1}y$, and thus has dimension $r/2 - 1$. Hence $g = r/2 - 1$, from which Theorem 21.4.1 follows. $\square$

21.4.5. *Curves of every genus.* As a consequence of the hyperelliptic Riemann-Hurwitz formula (Theorem 21.4.1), we see that there are curves of every genus $g \geq 0$ over an algebraically closed field of characteristic 0: to get a curve of genus g, consider the branched cover branched over $2g+2$ distinct points. The unique genus 0 curve is of this form, and we saw above that every genus 2 curve is of this form. We will soon see that every genus 1 curve (reminder: over an algebraically closed field!) is too (§21.8.5). But it is too much to hope that all curves are of this form, and we will soon see (§21.6.2) that there are genus 3 curves that are *not* hyperelliptic, and we will get heuristic evidence that "most" genus 3 curves are not hyperelliptic. We will later give vague evidence (that can be made precise) that "most" genus g curves are not hyperelliptic if $g > 2$ (§21.7.1).

We can also classify hyperelliptic curves. Hyperelliptic curves of genus g correspond to precisely $2g+2$ points on $\mathbb{P}^1$ modulo S_{2g+2}, and modulo automorphisms of $\mathbb{P}^1$. Thus "the space of hyperelliptic curves" has dimension

$$2g + 2 - \dim \operatorname{Aut} \mathbb{P}^1 = 2g - 1.$$

This is not a well-defined statement, because we haven't rigorously defined "the space of hyperelliptic curves" — an example of a *moduli space*. For now, take it as a plausibility statement. It is also plausible that this space is irreducible and reduced — it is the image of something irreducible and reduced.

21.4.B. EXERCISE. Verify that a curve C of genus at least 1 admits a degree 2 cover of $\mathbb{P}^1$ if and only if it admits a degree 2 invertible sheaf $\mathcal{L}$ with $h^0(C, \mathcal{L}) = 2$. Possibly in the course of doing this, verify that if C is a curve, and $\mathcal{L}$ has a degree 2 invertible sheaf with at least 2 (linearly independent) sections, then $\mathcal{L}$ has precisely two sections, and that this $\mathcal{L}$ is base-point-free and gives a hyperelliptic map.

21.4.6. Proposition. — *If $\mathcal{L}$ corresponds to a hyperelliptic cover $C \to \mathbb{P}^1$, then $\mathcal{L}^{\otimes(g-1)} \cong \mathcal{K}_C$.*

Proof. Compose the hyperelliptic map with the $(g-1)$th Veronese map:

$$C \xrightarrow{\mathcal{L}} \mathbb{P}^1 \xrightarrow{\mathcal{O}_{\mathbb{P}^1}(g-1)} \mathbb{P}^{g-1}.$$

The composition corresponds to $\mathcal{L}^{\otimes(g-1)}$. This invertible sheaf has degree $2g-2$. The pullback $H^0(\mathbb{P}^{g-1}, \mathcal{O}(1)) \to H^0(C, \mathcal{L}^{\otimes(g-1)})$ is injective because the image of C in $\mathbb{P}^{g-1}$ (a rational normal curve) is nondegenerate: if there were a hyperplane $s \in H^0(\mathbb{P}^{g-1}, \mathcal{O}(1))$ that pulled back to 0 on C, then the image of C would lie in that hyperplane, yet a rational normal curve cannot. Thus $\mathcal{L}^{\otimes(g-1)}$ has at least g sections. But by Exercise 21.2.A, the only invertible sheaf of degree $2g-2$ with (at least) g sections is the canonical sheaf. $\square$

21.4.7. Proposition (a genus ≥ 2 curve can be hyperelliptic in "only one way"). — *Any curve C of genus at least 2 admits at most one double cover of $\mathbb{P}^1$. More precisely, if $\mathcal{L}$ and $\mathcal{M}$ are two degree two line bundles yielding maps $C \to \mathbb{P}^1$, then $\mathcal{L} \cong \mathcal{M}$.*

Proof. If C is hyperelliptic, then we can recover the hyperelliptic map by considering the canonical linear series given by $\mathcal{K}$ (the *canonical map*, which we will use again repeatedly in the next few sections): it is a double cover of a degree $g-1$ rational normal curve (by the previous proposition), which is isomorphic to $\mathbb{P}^1$. This double cover is the hyperelliptic cover (also by the proof of the previous proposition). Thus we have uniquely recovered the map $C \to \mathbb{P}^1$, and this map must be induced by $\mathcal{L}$ and $\mathcal{M}$, from which $\mathcal{L} \cong \mathcal{M}$ (recall Theorem 17.4.1, relating maps to projective space and line bundles). $\square$

21.5 Curves of genus 2

21.5.1. *The reason for leaving genus 1 for later.* It might make most sense to jump to genus 1 at this point, but the theory of elliptic curves is especially rich and subtle, so we will leave it for §21.8.

In general, curves have quite different behaviors (topologically, arithmetically, geometrically) depending on whether $g = 0$, $g = 1$, or $g \geq 2$. This trichotomy extends to varieties of higher dimension. We already have some inkling of it in the case of curves. Arithmetically, genus 0 curves can have lots and lots of points, genus 1 curves can have lots of points, and by Faltings' Theorem (Mordell's Conjecture) any curve of genus at least 2 has at most finitely many points. (Thus even before Wiles' proof of the Taniyama-Shimura conjecture, we knew that $x^n + y^n = z^n$ in $\mathbb{P}^2$ has at most finitely many rational solutions for $n \geq 4$, as such curves have genus $\binom{n-1}{2} > 1$, see (20.5.3.1).) In the language of differential geometry, Riemann surfaces of genus 0 are positively curved, Riemann surfaces of genus 1 are flat, and Riemann surfaces of genus 1 are negatively curved. It is a fact that curves of genus at least 2 have finite automorphism groups (see for example [**ACGH**]), while curves of genus 1 have some automorphisms (a one-dimensional family, see Question 21.8.15), and the unique curve of genus 0 over an algebraically closed field has a three-dimensional automorphism group (see Exercises 17.4.B and 17.4.C).

21.5.2. *Back to curves of genus 2.*

Over an algebraically closed field, we saw in §21.3 that there is only one genus 0 curve. In §21.4 that there are hyperelliptic curves of genus 2. How can we get a hold of curves of genus 2? For example, are they all hyperelliptic? "How many" are there? We now tackle these questions.

Fix a curve C of genus $g = 2$. Then $\mathcal{K}$ is degree $2g - 2 = 2$, and has 2 sections (Exercise 21.2.A). I claim that $\mathcal{K}$ is base-point-free. We may assume k is algebraically closed, as base-point-freeness is independent of field extension of k (Exercise 20.2.H). If $\mathcal{K}$ is not base-point-free, then if p is a base point, then $\mathcal{K}(-p)$ is a degree 1 invertible sheaf with 2 sections, which Proposition 21.3.3 shows is impossible. Thus we canonically constructed a double cover $C \to \mathbb{P}^1$ (unique up to automorphisms of $\mathbb{P}^1$, which we studied in Exercises 17.4.B and 17.4.C). Conversely, any double cover $C \to \mathbb{P}^1$ arises from a degree 2 invertible sheaf with at least 2 sections, so if $g(C) = 2$, this invertible sheaf must be the canonical bundle (by the easiest case of Proposition 21.4.6).

Hence we have a natural bijection between genus 2 curves and genus 2 double covers of $\mathbb{P}^1$ (up to automorphisms of $\mathbb{P}^1$). If the characteristic is not 2, the hyperelliptic Riemann-Hurwitz formula (Theorem 21.4.1) shows that the double cover is branched over $2g + 2 = 6$ geometric points. In particular, we have a "three-dimensional space of genus 2 curves". This isn't rigorous, but we can certainly show that there are an infinite number of non-isomorphic genus 2 curves.

21.5.A. EXERCISE. Fix an algebraically closed field k of characteristic 0. Show that there are an infinite number of (pairwise) non-isomorphic genus 2 curves k.

21.5.B. EXERCISE. Show that every genus 2 curve (over any field) has finite automorphism group.

21.6 Curves of genus 3

Suppose C is a curve of genus 3. Then $\mathcal{K}$ has degree $2g - 2 = 4$, and has $g = 3$ sections.

21.6.1. Claim. — *$\mathcal{K}$ is base-point-free, and hence gives a map to $\mathbb{P}^2$.*

Proof. We check base-point-freeness by working over the algebraic closure $\overline{k}$. For any point p, by Riemann-Roch,

$$h^0(C, \mathcal{K}(-p)) - h^0(C, \mathcal{O}(p)) = \deg(\mathcal{K}(-p)) - g + 1 = 3 - 3 + 1 = 1.$$

But $h^0(C, \mathcal{O}(p)) = 1$ by Proposition 21.3.3, so

$$h^0(C, \mathcal{K}(-p)) = 2 = h^0(C, \mathcal{K}) - 1.$$

Thus p is not a base-point of $\mathcal{K}$ for any p, so by Criterion 21.2.6 $\mathcal{K}$ is base-point-free. □

The next natural question is: Is this a closed immersion? Again, we can check over algebraic closure. We use our "closed immersion test" (again, see our useful facts). If it *isn't* a closed immersion, then we can find two points p and q (possibly identical) such that

$$h^0(C, \mathcal{K}) - h^0(C, \mathcal{K}(-p - q)) = 1 \text{ or } 0,$$

i.e. $h^0(C, \mathcal{K}(-p-q)) = 2$. But by Serre duality, this means that $h^0(C, \mathcal{O}(p+q)) = 2$. We have found a degree 2 divisor with 2 sections, so C is hyperelliptic. (Indeed, I could have skipped that sentence, and made this observation about $\mathcal{K}(-p - q)$, but I've done it this way in order to generalize to higher genus.) Conversely, if C is hyperelliptic, then we already know that $\mathcal{K}$ gives a double cover of a nonsingular conic in $\mathbb{P}^2$, and hence $\mathcal{K}$ does not give a closed immersion.

Thus we conclude that if (and only if) C is not hyperelliptic, then the canonical map describes C as a degree 4 curve in $\mathbb{P}^2$.

Conversely, any quartic plane curve is canonically embedded. Reason: the curve has genus 3 (see (20.5.3.1)), and is mapped by an invertible sheaf of degree 4 with 3 sections. But by Exercise 21.2.A, the only invertible sheaf of degree $2g - 2$ with g sections is $\mathcal{K}$.

In particular, each non-hyperelliptic genus 3 curve can be described as a quartic plane curve in only one way (up to automorphisms of $\mathbb{P}^2$).

In conclusion, there is a bijection between non-hyperelliptic genus 3 curves, and plane quartics up to projective linear transformations.

21.6.2. *Remark.* In particular, as there exist nonsingular plane quartics (Exercise 13.2.J), there exist non-hyperelliptic genus 3 curves.

21.6.A. EXERCISE. Give a heuristic (non-rigorous) argument that the nonhyperelliptic curves of genus 3 form a family of dimension 6. (Hint: Count the dimension of the family of nonsingular quartics, and quotient by $\operatorname{Aut} \mathbb{P}^2 = PGL(3)$.)

The genus 3 curves thus seem to come in two families: the hyperelliptic curves (a family of dimension 5), and the nonhyperelliptic curves (a family of dimension 6). This is misleading — they actually come in a single family of dimension 6.

In fact, hyperelliptic curves are naturally limits of nonhyperelliptic curves. We can write down an explicit family. (This explanation necessarily requires some hand-waving, as it involves topics we haven't seen yet.) Suppose we have a hyperelliptic curve branched over $2g + 2 = 8$ points of $\mathbb{P}^1$. Choose an isomorphism of $\mathbb{P}^1$ with a conic in $\mathbb{P}^2$. There is a nonsingular quartic meeting the conic at precisely those 8 points. (This requires Bertini's theorem 26.5.2, which we haven't yet discussed, so we omit the argument.) Then if f is the equation of the conic, and g

is the equation of the quartic, then $f^2 + t^2 g$ is a family of quartics that are nonsingular for most t (nonsingularity is an open condition, as we will see). The $t = 0$ case is a double conic. Then it is a fact that if you normalize the family, the central fiber (above $t = 0$) turns into our hyperelliptic curve. Thus we have expressed our hyperelliptic curve as a limit of nonhyperelliptic curves.

21.6.B. UNIMPORTANT EXERCISE. A (projective) curve (over a field k) admitting a degree 3 cover of $\mathbb{P}^1$ is called **trigonal**. Show that every non-hyperelliptic genus 3 complex curve is trigonal, by taking the quartic model in $\mathbb{P}^2$, and projecting to $\mathbb{P}^1$ from any point on the curve. Do this by choosing coordinates on $\mathbb{P}^2$ so that p is at $[0; 0; 1]$. (After doing this, you may find Remark 19.4.8 more enlightening. But you certainly don't need the machinery of blowing up to solve the problem.)

21.7 Curves of genus 4 and 5

We begin with two exercises in general genus, then specialize to genus 4.

21.7.A. EXERCISE. Assume $k = \overline{k}$ (purely to avoid distraction — feel free to remove this hypothesis). Suppose C is a genus g curve. Show that if C is not hyperelliptic, then the canonical bundle gives a closed immersion $C \hookrightarrow \mathbb{P}^{g-1}$. (In the hyperelliptic case, we have already seen that the canonical bundle gives us a double cover of a rational normal curve.) Hint: follow the genus 3 case. Such a curve is called a **canonical curve**, and this closed immersion is called the **canonical embedding** of C.

21.7.B. EXERCISE. Suppose C is a curve of genus $g > 1$, over a field k that is not algebraically closed. Show that C has a closed point of degree at most $2g - 2$ over the base field. (For comparison: if $g = 1$, for any n, there is a genus 1 curve over $\mathbb{Q}$ with no point of degree less than n!)

We next consider nonhyperelliptic curves C of genus 4. Note that $\deg \mathcal{K} = 6$ and $h^0(C, \mathcal{K}) = 4$, so the canonical map expresses C as a sextic curve in $\mathbb{P}^3$. We shall see that all such C are complete intersections of quadric surfaces and cubic surfaces, and conversely all nonsingular complete intersections of quadrics and cubics are genus 4 non-hyperelliptic curves, canonically embedded.

By (21.2.4.1) (Riemann-Roch and Serre duality),

$$h^0(C, \mathcal{K}^{\otimes 2}) = \deg \mathcal{K}^{\otimes 2} - g + 1 = 12 - 4 + 1 = 9.$$

We have the restriction map $H^0(\mathbb{P}^3, \mathcal{O}(2)) \to H^0(C, \mathcal{K}^{\otimes 2})$, and $\dim \operatorname{Sym}^2 \Gamma(C, \mathcal{K}) = \binom{4+1}{2} = 10$. Thus there is at least one quadric in $\mathbb{P}^3$ that vanishes on our curve C. Translation: C lies on at least on quadric Q. Now quadrics are either double planes, or the union of two planes, or cones, or nonsingular quadrics. (They corresponds to quadric forms of rank 1, 2, 3, and 4 respectively.) But C can't lie in a plane, so Q must be a cone or nonsingular. In particular, Q is irreducible.

Now C can't lie on *two* (distinct) such quadrics, say Q and Q'. Otherwise, as Q and Q' have no common components (they are irreducible and not the same!), $Q \cap Q'$ is a curve (not necessarily reduced or irreducible). By Bézout's theorem (Exercise 20.5.M), $Q \cap Q'$ is a curve of degree 4. Thus our curve C, being of degree

6, cannot be contained in $Q \cap Q'$. (If you don't see why directly, Exercise 20.5.H might help.)

We next consider cubic surfaces. By (21.2.4.1) again, $h^0(C, \mathcal{K}^{\otimes 3}) = \deg \mathcal{K}^{\otimes 3} - g + 1 = 18 - 4 + 1 = 15$. Now $\dim \operatorname{Sym}^3 \Gamma(C, \mathcal{K})$ has dimension $\binom{4+2}{3} = 20$. Thus C lies on at least a 5-dimensional vector space of cubics. Now a 4-dimensional subspace come from multiplying the quadric Q by a linear form $(?w + ?x + ?y + ?z)$. But hence there is still one cubic K whose underlying form is not divisible by the quadric form Q (i.e. K doesn't contain Q.) Then K and Q share no component, so $K \cap Q$ is a complete intersection containing C as a closed subscheme. Now $K \cap Q$ and C are both degree 6 (the former by Bézout's theorem, Exercise 20.5.M, and the latter because C is embedded by a degree 6 line bundle, Exercise 20.5.K). Also, $K \cap Q$ and C both have arithmetic genus 4 (the former by Exercise 20.5.S). These two invariants determine the (linear) Hilbert polynomial, so $K \cap Q$ and C have the same Hilbert polynomial. Hence $C = K \cap Q$ by Exercise 20.5.H.

We now show the converse, and that any nonsingular complete intersection C of a quadric surface with a cubic surface is a canonically embedded genus 4 curve. By Exercise 20.5.S, such a complete intersection has genus 4.

21.7.C. EXERCISE. Show that $\mathcal{O}_C(1)$ has at least 4 sections. (Translation: C doesn't lie in a hyperplane.)

The only degree $2g-2$ invertible sheaf with (at least) g sections is the canonical sheaf (Exercise 21.2.A), so $\mathcal{O}_C(1) \cong \mathcal{K}_C$, and C is indeed canonically embedded.

21.7.D. EXERCISE. Give a heuristic argument suggesting that the nonhyperelliptic curves of genus 4 "form a family of dimension 9".

On to genus 5!

21.7.E. EXERCISE. Suppose C is a nonhyperelliptic genus 5 curve. Show that the canonical curve is degree 8 in $\mathbb{P}^4$. Show that it lies on a three-dimensional vector space of quadrics (i.e. it lies on 3 linearly independent independent quadrics). Show that a nonsingular complete intersection of 3 quadrics is a canonical(ly embedded) genus 5 curve.

Unfortunately, not all canonical genus 5 curves are the complete intersection of 3 quadrics in $\mathbb{P}^4$. But in the same sense that most genus 3 curves can be described as plane quartics, most canonical genus 5 curves are complete intersections of 3 quadrics, and most genus 5 curves are non-hyperelliptic. The correct way to say this is that there is a dense Zariski-open locus in the moduli space of genus 5 curves consisting of nonhyperelliptic curves whose canonical embedding is cut out by 3 quadrics.

(Those nonhyperelliptic genus 5 canonical curves not cut out by a three-dimensional vector space of quadrics are precisely the trigonal curves, see Exercise 21.6.B. The triplets of points mapping to the same point of $\mathbb{P}^1$ under the trigonal map turn out to lie on a line in the canonical map. Any quadric vanishing along those 3 points must vanish along the line — basically, any quadratic polynomial with three zeros must be the zero polynomial.)

21.7.F. EXERCISE. Assuming the discussion above, count complete intersections of three quadrics to give a heuristic argument suggesting that the curves of genus 5 "form a family of dimension 12".

We have now understood curves of genus 3 through 5 by thinking of canonical curves as complete intersections. Sadly our luck has run out.

21.7.G. EXERCISE. Show that if $C \subset \mathbb{P}^{g-1}$ is a canonical curve of genus $g \geq 6$, then C is *not* a complete intersection. (Hint: Bézout's theorem, Exercise 20.5.M.)

21.7.1. *Some discussion on curves of general genus.* However, we still have some data. If $\mathcal{M}_g$ is this ill-defined "moduli space of genus g curves", we have heuristics to find its dimension for low g. In genus 0, over an algebraically closed field, there is only genus 0 curve (Proposition 21.3.1), so it appears that $\dim \mathcal{M}_0 = 0$. In genus 1, over an algebraically closed field, we will soon see that the elliptic curves are classified by the j-invariant (Exercise 21.8.C), so it appears that $\dim \mathcal{M}_1 = 1$. We have also informally computed $\dim \mathcal{M}_2 = 3$, $\dim \mathcal{M}_3 = 6$, $\dim \mathcal{M}_4 = 9$, $\dim \mathcal{M}_5 = 12$. What is the pattern? In fact in some strong sense it was known by Riemann that $\dim \mathcal{M}_g = 3g - 3$ for $g > 1$. What goes wrong in genus 0 and genus 1? As a clue, recall our insight when discussing Hilbert functions (§20.5) that whenever some function is "eventually polynomial", we should assume that it "wants to be polynomial", and there is some better function (usually an Euler characteristic) that *is* polynomial, and that cohomology-vanishing ensures that the original function and the better function "eventually agree". Making sense of this in the case of $\mathcal{M}_g$ is far beyond the scope of our current discussion, so we will content ourselves by observing the following facts. *Every* nonsingular curve of genus greater than 1 has a finite number of automorphisms — a zero-dimensional automorphism group. *Every* nonsingular curve of genus 1 has a one-dimensional automorphism group (see Question 21.8.15). And the only nonsingular curve of genus 0 has a three-dimensional automorphism group (Exercise 17.4.C). (See Aside 23.4.9 for more discussion.) So notice that for all $g \geq 0$,

$$\dim \mathcal{M}_g - \dim \operatorname{Aut} C_g = 3g - 3$$

where $\operatorname{Aut} C_g$ means the automorphism group of any curve of genus g.

In fact, in the language of stacks (or orbifolds), it makes sense to say that the dimension of the moduli space of (projective smooth geometrically irreducible) genus 0 curves is -3, and the dimension of the moduli space of genus 1 curves is 0.

21.8 Curves of genus 1

Finally, we come to the very rich case of curves of genus 1. We will present the theory by thinking about line bundles of steadily increasing degree.

21.8.1. Line bundles of degree 0.

Suppose C is a genus 1 curve. Then $\deg \mathcal{K}_C = 2g-2 = 0$ and $h^0(C, \mathcal{K}_C) = g = 1$ (by Exercise 21.2.A). But the only degree 0 invertible sheaf with a section is the structure sheaf (§21.2.3), so we conclude that $\mathcal{K}_C \cong \mathcal{O}_C$.

We move on to line bundles of higher degree. Next, note that if $\deg \mathcal{L} > 0$, then Riemann-Roch and Serre duality (21.2.4.1) give

$$h^0(C, \mathcal{L}) = \deg \mathcal{L} - g + 1 = \deg \mathcal{L}.$$

21.8.2. Line bundles of degree 1.

Each degree 1 (k-valued) point q determines a line bundle $\mathcal{O}(q)$, and two distinct points determine two distinct line bundles (as a degree 1 line bundle has only one section, up to scalar multiples). Conversely, any degree 1 line bundle $\mathcal{L}$ is of the form $\mathcal{O}(q)$ (as $\mathcal{L}$ has a section — then just take its divisor of zeros), and it is of this form in one and only one way.

Thus we have a canonical bijection between degree 1 line bundles and degree 1 (closed) points. (If k is algebraically closed, as all closed points have residue field k, this means that we have a canonical bijection between degree 1 line bundles and closed points.)

Define an **elliptic curve** to be a genus 1 curve E with a choice of k-valued point p. The choice of this point should always be considered part of the definition of an elliptic curve — "elliptic curve" is not a synonym for "genus 1 curve". (Note: a genus 1 curve need not have any k-valued points at all! For example, you can show that $x^3+2y^3+4z^3 = 0$ in $\mathbb{P}^2_{\mathbb{Q}}$ has no $\mathbb{Q}$-points. Even faster once you are comfortable with double covers of $\mathbb{P}^1$, the genus 1 curve compactifying $y^2 = x^4 + 1$ in $\mathbb{A}^2_{\mathbb{Q}}$ has no $\mathbb{R}$-points, and hence no $\mathbb{Q}$-points. Of course, if $k = \overline{k}$, then any closed point is k-valued, by the Nullstellensatz 4.2.2.) We will often denote elliptic curves by E rather than C.

If (E, p) is an elliptic curve, then there is a canonical bijection between the set of degree 0 invertible sheaves (up to ismorphism) and the set of degree 1 points of E: simply the twist the degree 1 line bundles by $\mathcal{O}(-p)$. Explicitly, the bijection is given by

$$\mathcal{L} \longmapsto \operatorname{div}(\mathcal{L}(p))$$

$$\mathcal{O}(q-p) \longleftarrow\!\shortmid\; q$$

But the degree 0 invertible sheaves form a group (under tensor product), so have proved:

21.8.3. Proposition (the group law on the degree 1 points of an elliptic curve). — *The above bijection defines an abelian group structure on the degree 1 points of an elliptic curve, where p is the identity.*

From now on, we will identify closed points of E with degree 0 invertible sheaves on E without comment.

For those familiar with the complex analytic picture, this isn't surprising: E is isomorphic to the complex numbers modulo a lattice: $E \cong \mathbb{C}/\Lambda$.

This is currently just a bijection of sets. Given that E has a much richer structure (it has a generic point, and the structure of a variety), this is a sign that there should be a way of defining some *scheme* $\operatorname{Pic}^0(E)$, and that this should be an isomorphism of schemes. We will soon show (Theorem 21.8.13) that this group structure on the degree 1 points of E comes from a group variety structure on E.

21.8.4. *Aside: The Mordell-Weil Theorem, group, and rank.* This is a good excuse to mention the *Mordell-Weil Theorem*: for any elliptic curve E over $\mathbb{Q}$, the $\mathbb{Q}$-points of E form a *finitely generated* abelian group, often called the *Mordell-Weil group*. By the classification of finitely generated abelian groups, the $\mathbb{Q}$-points are a direct sum of a torsion part, and of a free $\mathbb{Z}$-module. The rank of the $\mathbb{Z}$-module is called the *Mordell-Weil rank*.

21.8.5. Line bundles of degree 2.

Note that $\mathcal{O}_E(2p)$ has 2 sections, so E admits a double cover of $\mathbb{P}^1$ (Exercise 21.4.B). One of the branch points is $2p$: one of the sections of $\mathcal{O}_E(2p)$ vanishes to p of order 2, so there is a point of $\mathbb{P}^1$ consists of p (with multiplicity 2). Assume now that $k = \overline{k}$ and $\operatorname{char} k \neq 2$, so we can use the hyperelliptic Riemann-Hurwitz formula (Theorem 21.4.1), which implies that E has 4 branch points (p and three others). Conversely, given 4 points in $\mathbb{P}^1$, there exists a unique double cover branched at those 4 points (Proposition 21.4.2). Thus elliptic curves correspond to 4 distinct points in $\mathbb{P}^1$, where one is marked p, up to automorphisms of $\mathbb{P}^1$. Equivalently, by placing p at ∞, elliptic curves correspond to 3 points in $\mathbb{A}^1$, up to affine maps $x \mapsto ax + b$.

21.8.A. EXERCISE. Show that the other three branch points are precisely the (non-identity) 2-torsion points in the group law. (Hint: if one of the points is q, show that $\mathcal{O}(2q) \cong \mathcal{O}(2p)$, but $\mathcal{O}(q)$ is not congruent to $\mathcal{O}(p)$.)

Thus (if the $\operatorname{char} k \neq 2$ and $k = \overline{k}$) every elliptic curve has precisely four 2-torsion points. If you are familiar with the complex picture $E \cong \mathbb{C}/\Lambda$, this isn't surprising.

21.8.6. *Follow-up remark.* An elliptic curve with *full level n-structure* is an elliptic curve with an isomorphism of its n-torsion points with $(\mathbb{Z}/n)^2$. (This notion has problems if n is divisible by $\operatorname{char} k$.) Thus an elliptic curve with *full level 2 structure* is the same thing as an elliptic curve with an ordering of the three other branch points in its degree 2 cover description. Thus (if $k = \overline{k}$) these objects are parametrized by the λ-line, which we discuss below.

Follow-up to the follow-up. There is a notion of moduli spaces of elliptic curves with full level n structure. Such moduli spaces are smooth curves (where this is interpreted appropriately — they are stacks), and have smooth compactifications. A *weight k level n modular form* is a section of $\mathcal{K}^{\otimes k}$ where $\mathcal{K}$ is the canonical sheaf of this moduli space ("modular curve").

21.8.7. *The cross-ratio and the j-invariant.* If the three other points are temporarily labeled q_1, q_2, q_3, there is a unique automorphism of $\mathbb{P}^1$ taking p, q_1, q_2 to $(\infty, 0, 1)$ respectively (as $\operatorname{Aut} \mathbb{P}^1$ is three-transitive, Exercise 17.4.C). Suppose that q_3 is taken to some number λ under this map, where necessarily $\lambda \neq 0, 1, \infty$.

The value λ is called the **cross-ratio** of the four-points (p, q_1, q_2, q_3) of $\mathbb{P}^1$ (first defined by Clifford, but implicitly known since the time of classical Greece).

21.8.B. EXERCISE. Show that isomorphism class of four ordered distinct points on $\mathbb{P}^1$, up to projective equivalence (automorphisms of $\mathbb{P}^1$), are classified by the cross-ratio.

We have not defined the notion of *moduli space*, but the previous exercise illustrates the fact that $\mathbb{P}^1 - \{0, 1, \infty\}$ (the image of the cross-ratio map) is the moduli space for four ordered distinct points of $\mathbb{P}^1$ up to projective equivalence.

Notice:

- If we had instead sent p, q_2, q_1 to $(\infty, 0, 1)$, then q_3 would have been sent to $1 - \lambda$.
- If we had instead sent p, q_1, q_3 to $(\infty, 0, 1)$, then q_2 would have been sent to $1/\lambda$.
- If we had instead sent p, q_3, q_1 to $(\infty, 0, 1)$, then q_2 would have been sent to $1 - 1/\lambda = (\lambda - 1)/\lambda$.
- If we had instead sent p, q_2, q_3 to $(\infty, 0, 1)$, then q_2 would have been sent to $1/(1 - \lambda)$.
- If we had instead sent p, q_3, q_2 to $(\infty, 0, 1)$, then q_2 would have been sent to $1 - 1/(1 - \lambda) = \lambda/(\lambda - 1)$.

Thus these six values (which correspond to S_3) yield the same elliptic curve, and this elliptic curve will (upon choosing an ordering of the other 3 branch points) yield one of these six values.

This is fairly satisfactory already. To check if two elliptic curves (E, p), (E', p') over $k = \overline{k}$ are isomorphic, we write both as double covers of $\mathbb{P}^1$ ramified at p and p' respectively, then order the remaining branch points, then compute their respective λ's (say λ and λ' respectively), and see if they are related by one of the six numbers above:

$$\lambda' = \lambda,\ 1 - \lambda,\ (\lambda - 1)/\lambda,\ 1/(1 - \lambda),\ \text{or}\ \lambda/(\lambda - 1). \tag{21.8.7.1}$$

It would be far more convenient if, instead of a "six-valued invariant" λ, there were a single invariant (let's call it j), such that $j(\lambda) = j(\lambda')$ if and only if one of the equalities of (21.8.7.1) holds. This j-function should presumably be algebraic, so it would give a map j from the λ-line $\mathbb{A}^1 - \{0, 1\}$ to the $\mathbb{A}^1$. By the Curve-to-projective Extension Theorem 17.5.1, this would extend to a morphism $j : \mathbb{P}^1 \to \mathbb{P}^1$. By Exercise 18.4.D, because this is (for most λ) a 6-to-1 map, the degree of this cover is 6 (or more correctly, at least 6).

We can make this dream more precise as follows. The elliptic curves over k corresponds to k-valued points of $\mathbb{P}^1 - \{0, 1, \lambda\}$, modulo the action of S_3 on λ given above. Consider the subfield K of $k(\lambda)$ fixed by S_3. Then $k(\lambda)/K$ is necessarily Galois, and a degree 6 extension. We are hoping that this subfield is of the form $k(j)$, and if so, we would obtain the j-map $\mathbb{P}^1 \to \mathbb{P}^1$ as described above. One could show that K is finitely generated over k, and then invoke Lüroth's theorem, which we will soon prove in Example 23.5.6; but we won't need this.

Instead, we will just hunt for such a j. Note that λ should satisfy a sextic polynomial over $k(\lambda)$ (or more precisely given what we know right now, a polynomial of degree at least six), as for each j-invariant, there are six values of λ in general.

As you are undoubtedly aware, there is such a j-invariant. Here is the formula for the j-invariant that everyone uses:

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}. \tag{21.8.7.2}$$

You can readily check that $j(\lambda) = j(1/\lambda) = j(1 - \lambda) = \cdots$, and that as j has a degree 6 numerator and degree < 6 denominator, j indeeds determines a degree

6 map from $\mathbb{P}^1$ (with coordinate λ) to $\mathbb{P}^1$ (with coordinate j). But this complicated-looking formula begs the question: where did this formula come from? How did someone think of it? We will largely answer this, but we will ignore the 2^8 (which, as you might imagine, arises from characteristic 2 issues, and in order to invoke the results of §21.4 we have been assuming char $k \neq 2$).

Rather than using the formula handed to us, let's try to guess what j is. We won't expect to get the same formula as (21.8.7.2), but our answer should differ by an automorphism of the j-line ($\mathbb{P}^1$) — we will get $j' = (aj+b)/(cj+d)$ for some a, b, c, d.

We are looking for some $j'(\lambda)$ such that $j'(\lambda) = j'(1/\lambda) = \cdots$. Hence we want some expression in λ that is invariant under this S_3-action. A first possibility would be to take the product of the six numbers

$$\lambda \cdot (1-\lambda) \cdot \frac{1}{\lambda} \cdot \frac{\lambda-1}{\lambda} \cdot \frac{1}{1-\lambda} \cdot \frac{\lambda}{\lambda-1}$$

This is silly, as the product is obviously 1.

A better idea is to add them all together:

$$\lambda + (1-\lambda) + \frac{1}{\lambda} + \frac{\lambda-1}{\lambda} + \frac{1}{1-\lambda} + \frac{\lambda}{\lambda-1}$$

This also doesn't work, as they add to 3 — the six terms come in pairs adding to 1.

(Another reason you might realize this can't work: if you look at the sum, you will realize that you will get something of the form "degree at most 3" divided by "degree at most 2" (before cancellation). Then if $j' = p(\lambda)/q(\lambda)$, then λ is a root of a cubic over j. But we said that λ should satisfy a sextic over j'. The only way we avoid a contradiction is if $j' \in k$.)

But you will undoubtedly have another idea immediately. One good idea is to take the second symmetric function in the six roots. An equivalent one that is easier to do by hand is to add up the squares of the six terms. Even before doing the calculation, we can see that this will work: it will clearly produce a fraction whose numerator and denominator have degree at most 6, and it is not constant, as when λ is some fixed small number (say 1/2), the sum of squares is some small real number, while when λ is a large real number, the sum of squares will have to be some large real number (different from the value when $\lambda = 1/2$).

When you add up the squares by hand (which is not hard), you will get

$$j' = \frac{2\lambda^6 - 6\lambda^5 + 9\lambda^4 - 8\lambda^3 + 9\lambda^2 - 6\lambda + 2}{\lambda^2(\lambda-1)^2}.$$

Indeed $k(j) \cong k(j')$: you can check (again by hand) that

$$2j/2^8 = \frac{2\lambda^6 - 6\lambda^5 + 12\lambda^4 - 14\lambda^3 + 12\lambda^2 - 6\lambda + 2}{\lambda^2(\lambda-1)^2}.$$

Thus $2j/2^8 - j' = 3$.

21.8.C. EXERCISE. Explain why genus 1 curves over an algebraically closed field are classified by j-invariant.

21.8.D. EXERCISE. Give (with proof) two genus 1 curves over $\mathbb{Q}$ with the same j-invariant that are not isomorphic. (Hint: §21.4.3.)

21.8.8. Line bundles of degree 3.

In the discussion of degree 2 line bundles 21.8.5, we assumed $\operatorname{char} k \neq 2$ and $k = \overline{k}$, in order to invoke the Riemann-Hurwitz formula. In this section, we will start with no assumptions, and add them as we need them. In this way, you will see what partial results hold with weaker assumptions.

Consider the degree 3 invertible sheaf $\mathcal{O}_E(3p)$. By Riemann-Roch (21.2.4.1), $h^0(E, \mathcal{O}_E(3p)) = \deg(3p) - g + 1 = 3$. As $\deg E > 2g$, this gives a closed immersion (Remark 21.2.10 and Exercise 21.2.D). Thus we have a closed immersion $E \hookrightarrow \mathbb{P}^2_k$ as a cubic curve. Moreover, there is a line in $\mathbb{P}^2_k$ meeting E at point p with multiplicity 3, corresponding to the section of $\mathcal{O}(3p)$ vanishing precisely at p with multiplicity 3. (A line in the plane meeting a smooth curve with multiplicity at least 2 is a *tangent line*, see Definition 13.2.7. A line in the plane meeting a smooth curve with multiplicity at least 3 is said to be a **flex line**, and that point is a **flex point** of the curve.)

Choose projective coordinates on $\mathbb{P}^2_k$ so that p maps to $[0;1;0]$, and the flex line is the line at infinity $z = 0$. Then the cubic is of the following form:

$$\begin{array}{ccccccccccc}
?\,x^3 & + & 0\,x^2y & + & 0\,xy^2 & + & 0\,y^3 & & & & \\
 + & ?\,x^2z & + & ?\,xyz & + & ?\,y^2z & & & & & =0 \\
 & + & ?\,xz^2 & + & ?\,yz^2 & & & & & & \\
 & & + & ?\,z^3 & & & & & & &
\end{array}$$

The co-efficient of x is not 0 (or else this cubic is divisible by z). Dividing the entire equation by this co-efficient, we can assume that the coefficient of x^3 is 1. The coefficient of y^2z is not 0 either (or else this cubic is singular at $x = z = 0$). We can scale z (i.e. replace z by a suitable multiple) so that the coefficient of y^2z is 1. If the characteristic of k is not 2, then we can then replace y by $y + ?x + ?z$ so that the coefficients of xyz and yz^2 are 0, and if the characteristic of k is not 3, we can replace x by $x + ?z$ so that the coefficient of x^2z is also 0. In conclusion, if $\operatorname{char} k \neq 2, 3$, the elliptic curve may be written

$$y^2z = x^3 + ax^2z + bz^3. \tag{21.8.8.1}$$

This is called the **Weierstrass normal form** of the curve.

We see the hyperelliptic description of the curve (by setting $z = 1$, or more precisely, by working in the distinguished open set $z \neq 0$ and using inhomogeneous coordinates). In particular, we can compute the j-invariant should we want to.

21.8.E. EXERCISE. Show that the flexes of the cubic are the 3-torsion points in the group E. ("Flex" was defined in §21.8.8: it is a point where the tangent line meets the curve with multiplicity at least 3 at that point. In fact, if k is algebraically closed and $\operatorname{char} k \neq 3$, there are nine of them. This won't be surprising if you are familiar with the complex story, $E = \mathbb{C}/\Lambda$.)

21.8.9. The group law, geometrically.

The group law has a beautiful classical description in terms of the Weierstrass form. Consider Figure 21.2. In the Weierstrass coordinates, the origin p is the only point of E meeting the line at infinity ($z = 0$); in fact the line at infinity corresponds to the tautological section of $\mathcal{O}(3p)$. If a line meets E at three points p_1, p_2, p_3, then

$$\mathcal{O}(p_1 + p_2 + p_3) \cong \mathcal{O}(3p)$$

from which (in the group law) $p_1 + p_2 + p_3 = 0$.

Hence to find the inverse of a point s, we consider the intersection of E with the line sp; $-s$ is the third point of intersection. To find the sum of two points q and r, we consider the intersection of E with the line qr, and call the third points s. We then compute $-s$ by connecting s to p, obtaining $q + r$.

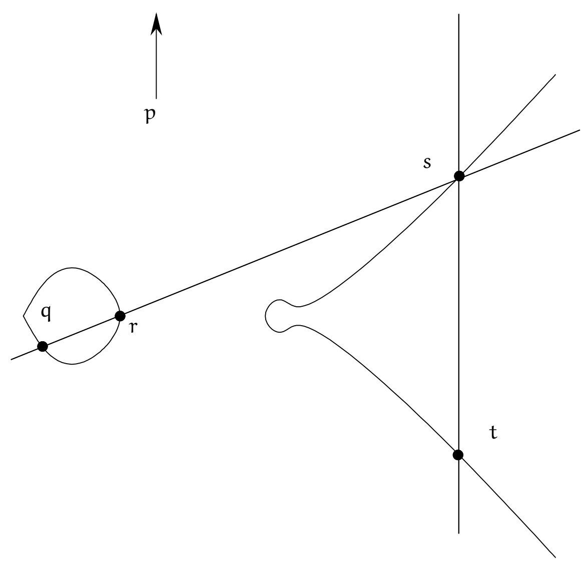


FIGURE 21.2. The group law on the elliptic curve, geometrically

We could give this description of a group law on a cubic curve in Weierstrass normal form to anyone familiar with the notion of projective space, and the notion of a group, but we would then have to prove that the construction we are giving indeed defines a group. In particular, we would have to prove associativity, which is not a priori clear. But in this case, we have already established that the degree 1 points form a group, by giving a bijection to $\operatorname{Pic}^0 E$, and we are merely interpreting the group law on $\operatorname{Pic}^0 E$.

Note that this description works even in characteristic 2 and 3; we don't need the cubic to be in Weierstrass normal form, and we need only that $\mathcal{O}(3p)$ gives a closed immersion into $\mathbb{P}^2$.

21.8.10. Elliptic curves are group varieties.

We initially described the group law on the degree 1 points of an algebraic curve in a rather abstract way. From that definition, it was not clear that over $\mathbb{C}$ the group operations (addition, inverse) are continuous. But the explicit description in terms of the Weierstrass cubic makes this clear. In fact we can observe even more: addition and inverse are algebraic in general. Better yet, elliptic curves are group varieties.

(This is a clue that $\mathrm{Pic}^0(E)$ really wants to be a scheme, and not just a group. Once the notion of "moduli space of line bundles on a variety" is made precise, this can be shown.)

We begin with the inverse case, as a warm-up.

21.8.11. Proposition. — *If* $\mathrm{char}\, k \neq 2, 3$*, there is a morphism of* k*-varieties* $E \to E$ *sending a (degree* 1*) point to its inverse, and this construction behaves well under field extension of* k*.*

In other words, the "inverse map" in the group law actually arises from a morphism of schemes — it isn't just a set map. (You are welcome to think through the two remaining characteristics, and to see that essentially the same proof applies. But the proof of Theorem 21.8.13 will give you a better sense of how to proceed.)

Proof. In characteristic not 2 or 3, it is the map (the hyperelliptic involution) $y \mapsto -y$ of the Weierstrass normal form. $\square$

The algebraic description of addition would be a big mess if we were to write it down. We will be able to show algebraicity by a trick — not by writing it down explicitly, but by thinking through how we *could* write it down explicitly. The main part of the trick is the following proposition. We give it in some generality just because it can be useful, but you may prefer to assume that $k = \overline{k}$ and C is a nonsingular cubic.

21.8.12. Proposition. — *Suppose* $C \subset \mathbb{P}^2_k$ *is a geometrically integral cubic curve (so in particular* C *contains no lines). Let* C^{ns} *be the nonsingular points of* C*. There is a unique morphism* $t : C^{ns} \times C^{ns} \to C^{ns}$ *such that*

(a) *if* p *and* q *are distinct nonsingular* k*-valued points of* C*, then* $t(p, q)$ *is obtained by intersecting the line* $\overline{pq}$ *with* C*, and taking the third "residual" point of intersection with* C*. More precisely,* $\overline{pq}$ *will meet* C *at three points with multiplicity (Exercise 9.2.E), including* p *and* q*;* $t(p, q)$ *is the third point.*
(b) *this property remains true after extension to* $\overline{k}$*.*

Furthermore, if p *is a* k*-valued point of* C^{ns}*, then* $t(2p)$ *is where the tangent line* ℓ *to* C *at* p *meets* C *again. More precisely,* ℓ *will meet* C *at three points with multiplicity, which includes* p *with multiplicity* 2*;* $t(p, p)$ *is the third point.*

We will need property (b) because C may have few enough k-valued points (perhaps none!) that the morphism t can not be determined by its behavior on them. In the course of the proof, we will see that (b) can be extended to "this property remains true after any field extension of k".

Proof. We first show (in this paragraph) that if p and q are distinct nonsingular points, then the third point r of intersection of $\overline{pq}$ with C is also nonsingular. If $r = p$ or $r = q$, we are done. Otherwise, the cubic obtained by restricting C to $\overline{pq}$ has three distinct (hence reduced, i.e. multiplicity 1) roots, p, q, and r. Thus $C \cap \overline{pq}$ is nonsingular at r, so r is a nonsingular point of C by the slicing criterion for nonsingularity, Exercise 13.2.B.

We now assume that $k = \overline{k}$, and leave the general case to the end. Fix p, q, and r, where $p \neq q$, and r is the "third" point of intersection of $\overline{pq}$ with C. We will describe a morphism $t_{p,q}$ in a neighborhood of $(p,q) \in C^{ns} \times C^{ns}$. By Exercise 11.2.B, showing that morphisms of varieties over $\overline{k}$ are determined by their behavior on closed ($\overline{k}$-valued) points, that these morphisms glue together (uniquely) to give a morphism t, completing the proof in the case $k = \overline{k}$.

Choose projective coordinates on $\mathbb{P}^2$ in such a way that $U_0 \cong \operatorname{Spec} k[x_1, x_2]$ contains p, q, and r, and the line $\overline{pq}$ is not "vertical". More precisely, in $\operatorname{Spec} k[x_1, x_2]$, say $p = (p_1, p_2)$ (in terms of "classical coordinates" — more pedantically, $p = [(x_1 - p_1, x_2 - p_2)]$), $q = (q_1, q_2)$, $r = (r_1, r_2)$, and $p_1 \neq q_1$. In these coordinates, the curve C is cut out by some cubic, which we also sloppily denote C: $C(x_1, x_2) = 0$.

Now if $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$ are in $C \cap U_0$, we attempt to compute the third point of intersection of $\overline{PQ}$ with C, in a way that works on an open subset of $C \times C$ that includes (p, q). To do this explicitly requires ugly high school algebra, but because we know how it looks, we will be able to avoid dealing with any details!

The line $\overline{PQ}$ is given by $x_2 = mx_1 + b$, where $m = \frac{P_2 - Q_2}{P_1 - Q_1}$ and $b = P_2 - mP_1$ are both rational functions of P and Q. Then m and b are defined for all P and Q such that $P_1 \neq Q_1$ (and hence for a neighborhood of (p, q), as $p_1 \neq q_1$, and as $P_1 \neq Q_1$ is an open condition).

Now we solve for $C \cap \overline{PQ}$, by substituting $x_2 = mx_1 + b$ into C, to get $C(x_1, mx_1 + b)$. This is a cubic in x_1, say

$$\gamma(x_1) = Ax_1^3 + BX_1^2 + Cx_1 + D = 0.$$

The coefficients of γ are rational functions of P_1, P_2, Q_1, and Q_2. The cubic γ has 3 roots (with multiplicity) so long as $A \neq 0$, which is an open algebraic condition on m and b, and hence on P_1, P_2, Q_1, Q_2. As $P, Q \in C \cap \overline{PQ} \cap U_0$, P_1 and Q_1 are two of the roots of $\gamma(x_1) = 0$. The sum of the roots of $\gamma(x_1) = 0$ is $-B/A$ (by Viète's formula), so the third root of γ is $R_1 := -B/A - P_1 - Q_1$. Thus if we take $R_2 = mR_1 + b$, we have found the third points of intersection of $\overline{PQ}$ with C (which happily lies in U_0) We have thus described a morphism from the open subset of $(C^{ns} \cap U_0) \times (C^{ns} \cap U_0)$, containing (p, q), that does what we want. (Precisely, the open subset is defined by $A \neq 0$, which can be explicitly unwound.) We have thus completed the proof of Proposition 21.8.12 (except for the last paragraph) for $k = \overline{k}$. (Those who believe they are interested only in algebraically closed fields can skip ahead.)

We extend this to Proposition 21.8.12 for every field k except $\mathbb{F}_2$. Suppose $U_0[x_1, x_2] = \operatorname{Spec} k[x_1, x_2]$ is any affine open subset of $\mathbb{P}^2_k$, along with choice of coordinates. (The awkward notation "$[x_1, x_2]$" is there to emphasize that the particular coordinates are used in the construction.) Then the construction above gives a morphism *defined over* k from an open subset of $(C^{ns} \cap U_0[x_1, x_2]) \times (C^{ns} \cap U_0[x_1, x_2])$ (note that all of the hypothetical algebra was done over k), that sends P and Q to the third points of intersection of $\overline{PQ}$ with C. Note that this construction commutes with any field extension, as the construction is insensitive to the field we are working over. Thus after base change to the algebraic closure, the map also has the property that it takes as input two points, and spits out the third point of intersection of the line with the cubic. Furthermore, all of these maps (as $U_0[x_1, x_2]$

varies over all complements U_0 of lines "with k-coefficients", and choices of coordinates on U_0) can be glued together: they agree on their pairwise overlaps (as after base change to $\overline{k}$ they are the same, by our previous discussion, and two maps that are the same after base change to $\overline{k}$ were the same to begin with by Exercise 10.2.J), and this is what is required to glue them together (Exercise 7.2.A).

We can geometrically interpret the open subset $(C^{ns} \cap U_0[x_1, x_2]) \times (C^{ns} \cap U_0[x_1, x_2])$ by examining the construction: it is defined in the locus $\{P = (P_1, P_2), Q = (Q_1, Q_2)\}$ where (i) $P_1 \neq Q_1$, and (ii) the third point of intersection R of $\overline{PQ}$ with C also lies in U_0.

So which points (P, Q) of $C^{ns} \times C^{ns}$ are missed? Condition (i) isn't important; if (P, Q) satisfies (ii) but not (i), we can swap the roles of x_1 and x_2, and (P, Q) will then satisfy (i). The only way (P, Q) can not be covered by one of these open sets is if there is *no* U_0 (a complement of a line defined over k) that includes P, Q, and R.

21.8.F. EXERCISE. Use $|k| > 2$ to show that there is a linear form on $\mathbb{P}^2$ with coefficients in k that misses P, Q, and R. (This is sadly *not* true if $k = \mathbb{F}_2$ — do you see why?)

21.8.G. EXERCISE. Prove the last statement of Proposition 21.8.12.

21.8.H. $\star\star$ UNIMPORTANT EXERCISE. Complete the proof by dealing with the case $k = \mathbb{F}_2$. Hint: first produce the morphism t over $\mathbb{F}_4$. The goal is then to show that this t is really "defined over" $\mathbb{F}_2$ ("descends to" $\mathbb{F}_2$). The morphism t is initially described locally by considering the complement of a line defined over $\mathbb{F}_4$ (and then letting the line vary). Instead, look at the map by looking at the complement of a line and its "conjugate". The complement of the line and its conjugate is an affine $\mathbb{F}_2$-variety. The partially-defined map t on this affine variety is a priori defined over $\mathbb{F}_4$, and is preserved by conjugation. Show that this partially defined map is "really" defined over $\mathbb{F}_2$. (If you figure out what all of this means, you will have an important initial insight into the theory of "descent".)

$\square$

We can now use this to define the group variety structure on E.

21.8.13. Theorem. — *Suppose (E, p) is an elliptic curve (a nonsingular genus 1 curve over k, with a k-valued point p). Take the Weierstrass embedding of E in $\mathbb{P}^2_k$, via the complete linear series $|\mathcal{O}_E(3p)|$. Define the k-morphism $e : \operatorname{Spec} k \to E$ by sending $\operatorname{Spec} k$ to p. Define the k-morphism $i : E \to E$ via $q \mapsto t(p, q)$, or more precisely, as the composition*

$$E \xrightarrow{(id, e)} E \times E \xrightarrow{t} E.$$

Define the k-morphism $m : E \times E \to$ via $(q, r) \mapsto t(p, t(q, r))$. Then (E, e, i, m) is a group variety over k.

By the construction of t, all of these morphisms "commute with arbitrary base extension".

Proof. We need to check that various pairs of morphisms described in §7.6.3 axioms (i)–(iii) are equal. For example, in axiom (iii), we need to show that $m \circ (i, id) = m \circ (id, i)$; all of the axioms are clearly of this sort.

Assume first that $k = \overline{k}$. Then each of these pairs of morphisms agree as maps of $\overline{k}$-points: Pic E is a group, and under the bijection between Pic E and E of Proposition 21.8.3, the group operations translate into the maps described in the statement of Theorem 21.8.13 by the discussion of §21.8.9.

But morphisms of $\overline{k}$-varieties are determined by their maps on the level of $\overline{k}$-points (Exercise 11.2.B), so each of these pairs of morphisms are the same.

For general k, we note that from the $\overline{k}$ case, these morphisms agree after base change to the algebraic closure. Then Exercise 10.2.J, they must agree to begin with.

21.8.14. *Features of this construction.* The most common derivation of the properties of an elliptic curve are to describe it as a cubic, and describe addition using the explicit construction with lines. Then one has to work hard to prove that the multiplication described is associative.

Instead, we started with something that was patently a group (the degree 0 line bundles). We interpreted the maps used in the definition of the group (addition and inverse) geometrically using our cubic interpretation of elliptic curves. This allowed us to see that these maps were algebraic.

As a bonus, we see that in some (as yet unprecise) sense, the Picard group of an elliptic curve wants to be an algebraic variety.

21.8.I. EXERCISE. Suppose p and q are k-points of a genus 1 curve E. Show that there is an automorphism of E sending p to q.

21.8.J. EXERCISE. Suppose (E, p) is an elliptic curve over an algebraically closed field k of characteristic not 2. Show that the automorphism group of (E, p) is isomorphic to $\mathbb{Z}/2$, $\mathbb{Z}/4$, or $\mathbb{Z}/6$. (An automorphism of an elliptic curve (E, p) over $k = \overline{k}$ is an automorphism of E fixing p scheme-theoretically, or equivalently, fixing the k-valued points by Exercise 11.2.B.) Hint: reduce to the question of automorphisms of $\mathbb{P}^1$ fixing a point ∞ and a set of distinct three points $\{p_1, p_2, p_3\} \in \mathbb{P}^1 \setminus \{\infty\}$. (The algebraic closure of k is not essential, so feel free to remove this hypothesis, using Exercise 10.2.J.)

21.8.15. *Vague question.* What are the possible automorphism groups of a genus 1 curve over an algebraically closed k of characteristic not 2? You should be able to convince yourself that the group has "dimension 1".

21.8.K. IMPORTANT EXERCISE: A DEGENERATE ELLIPTIC CURVE. Consider the genus 1 curve $C \subset \mathbb{P}^2_k$ given by $y^2 z = x^3 + x^2 z$, with the point $p = [0; 1; 0]$. Emulate the above argument to show that $C \setminus \{[0; 0; 1]\}$ is a group variety. Show that it is isomorphic to $\mathbb{G}_m$ (the multiplicative group scheme Spec $k[t, t^{-1}]$, see Exercise 7.6.C) with coordinate $t = y/x$, by showing an isomorphism of schemes, and showing that multiplication and inverse in both group varieties agree under this isomorphism.

21.8.L. EXERCISE: AN EVEN MORE DEGENERATE ELLIPTIC CURVE. Consider the genus 1 curve $C \subset \mathbb{P}^2_k$ given by $y^2 z = x^3$, with the point $p = [0; 1; 0]$. Emulate the above argument to show that $C \setminus \{[0;0;1]\}$ is a group variety. Show that it is isomorphic to $\mathbb{A}^1$ (with additive group structure) with coordinate $t = y/x$, by showing an isomorphism of schemes, and showing that multiplication/addition and inverse in both group varieties agree under this isomorphism.

21.8.16. Degree 4 line bundles. You have probably forgotten that we began by studying line bundles degree by degree. The story doesn't stop in degree 3. In the same way that we showed that a canonically embedded nonhyperelliptic curve of genus 4 is the complete intersection in $\mathbb{P}^3_k$ of a quadric and a cubic (§21.7), we can show the following.

21.8.M. EXERCISE. Show that the complete linear series for $\mathcal{O}(4p)$ embeds E in $\mathbb{P}^3$ as the complete intersection of two quadrics. (Hint: Show the image of E is contained in at least 2 linearly independent quadrics. Show that neither can be reducible, so they share no components. Use Bézout's theorem, Exercise 20.5.M.)

The beautiful structure doesn't stop with degree 4, but it gets more complicated. For example, the degree 5 embedding is not a complete intersection (of hypersurfaces), but is the complete intersection of $G(2,5)$ under its Plücker embedding with a five hyperplanes (or perhaps better, a codimension 5 linear space). In seemingly different terminology, its equations are 4×4 Pfaffians of a general 5×5 skew-symmetric matrix of linear forms, although I won't say what this means.

21.9 Counterexamples and pathologies from elliptic curves

We now give some fun counterexamples using our understanding of elliptic curves. The main extra juice elliptic curves give us comes from the fact that elliptic curves are the simplest varieties with "continuous Picard groups".

21.9.1. An example of a scheme that is factorial, but such that no affine open neighborhood of any point has ring that is a unique factorization domain.

Suppose E is an elliptic curve over $\mathbb{C}$ (or some other uncountable field). Consider $p \in E$. The local ring $\mathcal{O}_{E,p}$ is a discrete valuation ring and hence a unique factorization domain. Then an open neighborhood of E is of the form $E - q_1 - \cdots - q_n$. I claim that its Picard group is nontrivial. Recall the exact sequence:

$$\mathbb{Z}^{\oplus n} \xrightarrow{(a_1,\dots,a_n)\mapsto a_1q_1+\cdots+a_nq_n} \operatorname{Pic} E \longrightarrow \operatorname{Pic}(E - q_1 - \cdots - q_n) \longrightarrow 0 .$$

But the group on the left is countable, and the group in the middle is uncountable, so the group on the right is non-zero.

21.9.2. Counterexamples using the existence of a non-torsion point.

We next give a number of counterexamples using the existence of a non-torsion point of a complex elliptic curve. We show the existence of such a point.

We have a "multiplication by n" map $[n] : E \to E$, which sends p to np. If $n = 0$, this has degree 0. If $n = 1$, it has degree 1. Given the complex picture of a torus, you might not be surprised that the degree of $\times n$ is n^2. If $n = 2$, we have almost shown that it has degree 4, as we have checked that there are precisely 4 points q such that $2p = 2q$. All that really shows is that the degree is at least 4. (We could check by hand that the degree is 4 is we really wanted to.)

21.9.3. Proposition. — *Suppose E is an elliptic curve over a field k of characteristic not 2. For each $n > 0$, the "multiplication by n" map has positive degree. In other words, there are only a finite number of n torsion points, and the $[n] \neq [0]$.*

Proof. We may assume $k = \overline{k}$, as the degree of a map of curves is independent of field extension.

We prove the result by induction; it is true for $n = 1$ and $n = 2$.

If n is odd, then assume otherwise that $nq = 0$ for all closed points q. Let r be a non-trivial 2-torsion point, so $2r = 0$. But $nr = 0$ as well, so $r = (n - 2[n/2])r = 0$, contradicting $r \neq 0$.

If n is even, then $[\times n] = [\times 2] \circ [\times(n/2)]$, and by our inductive hypothesis both $[\times 2]$ and $[\times(n/2)]$ have positive degree. □

In particular, the total number of torsion points on E is countable, so if k is an uncountable field, then E has an uncountable number of closed points (consider an open subset of the curve as $y^2 = x^3 + ax + b$; there are uncountably many choices for x, and each of them has 1 or 2 choices for y).

21.9.4. Corollary. — *If E is a curve over an uncountable algebraically closed field of characteristic not 2 (e.g. $\mathbb{C}$), then E has a non-torsion point.*

Proof. For each n, there are only finitely many n-torsion points. Thus there are (at most) countably many torsion points. The curve E has uncountably many closed points. (One argument for this: take a double cover $\pi : E \to \mathbb{P}^1$. Then $\mathbb{P}^1$ has uncountably many closed points, and π is surjective on closed points. □

In a sense we can make precise using cardinalities, almost all points on E are non-torsion. We use this to create some interesting pathologies.

21.9.5. An example of an affine open subset of an affine scheme that is not a distinguished open set.

We can use this to construct an example of an affine scheme X and an affine open subset Y that is not distinguished in X. Let $X = E - p$, which is affine (see Exercise 21.2.B, or better, note that the linear series $\mathcal{O}(3p)$ sends E to $\mathbb{P}^2$ in such a way that the "line at infinity" meets E only at p; then $E - p$ has a closed immersion into the affine scheme $\mathbb{A}^2$).

Let q be another point on E so that $q - p$ is non-torsion. Then $E - p - q$ is affine (Exercise 21.2.B). Assume that it is distinguished. Then there is a function f on $E - p$ that vanishes on q (to some positive order d). Thus f is a rational function on E that vanishes at q to order d, and (as the total number of zeros minus poles of f is 0) has a pole at p of order d. But then $d(p - q) = 0$ in $\operatorname{Pic}^0 E$, contradicting our assumption that $p - q$ is non-torsion.

21.9.6. A Picard group that has no chance of being a scheme.

We informally observed that the Picard group of an elliptic curve "wants to be" a scheme (see §21.8.14). This is true of projective (and even proper) varieties in general. On the other hand, if we work over $\mathbb{C}$, the affine scheme $E - p -$

q (in the language of §21.9.5 above) has a Picard group that can be interpreted as $\mathbb{C}$ modulo a lattice modulo a non-torsion point (e.g. $\mathbb{C}/\langle 1, i, \pi\rangle$). This has no reasonable interpretation as a manifold, let alone a variety. So the fact that the Picard group of proper varieties turns out to be a scheme should be seen as quite remarkable.

21.9.7. Example of a variety with non-finitely-generated ring of global sections. We next show an example of a complex variety whose ring of global sections is not finitely generated. This is related to Hilbert's fourteenth problem, although I won't say how.

We begin with a preliminary exercise.

21.9.A. EXERCISE. Suppose X is a scheme, and L is the total space of a line bundle corresponding to invertible sheaf $\mathcal{L}$, so $L = \operatorname{Spec} \oplus_{n \geq 0} (\mathcal{L}^\vee)^{\otimes n}$. (This construction first appeared in Definition 18.1.4.) Show that $H^0(L, \mathcal{O}_L) = \oplus H^0(X, (\mathcal{L}^\vee)^{\otimes n})$. (Possible hint: choose a trivializing cover for $\mathcal{L}$. Rhetorical question: can you figure out the more general statement if $\mathcal{L}$ is a rank r locally free sheaf?)

Let E be an elliptic curve over some ground field k, $\mathcal{N}$ a degree 0 non-torsion invertible sheaf on E, and $\mathcal{P}$ a positive-degree invertible sheaf on E. Then $H^0(E, \mathcal{N}^m \otimes \mathcal{P}^n)$ is nonzero if and only if either (i) $n > 0$, or (ii) $m = n = 0$ (in which case the sections are elements of k).

21.9.B. EASY EXERCISE. Show that the ring $R = \oplus_{m,n \geq 0} H^0(E, \mathcal{N}^m \otimes \mathcal{P}^n)$ is not finitely generated.

21.9.C. EXERCISE. Let X be the total space of the vector bundle associated to $(\mathcal{N} \oplus \mathcal{P})^\vee$ over E. Show that the ring of global sections of X is R, and hence is not finitely generated. (Hint: interpret X as a line bundle over a line bundle over E.)

CHAPTER 22

$\star$ Application: A glimpse of intersection theory

The only reason this Chapter appears after Chapter 21 is because we will use Exercise 21.2.E.

22.1 Intersecting n line bundles with an n-dimensional variety

Throughout this chapter, X will be a k-variety; in most applications, X will be projective. The central tool in this chapter is the following.

22.1.1. *Definition: intersection product, or intersection number.* Suppose $\mathcal{F}$ is a coherent sheaf on X with proper support (automatic if X is proper) of dimension at most n, and $\mathcal{L}_1, \dots, \mathcal{L}_n$ are invertible sheaves on X. Let $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F})$ be the signed sum over the 2^n subsets of $\{1, \dots, n\}$

$$\sum_{\{i_1,\dots,i_m\}\subset\{1,\dots,n\}} (-1)^m \chi(\mathcal{L}_{i_1}^\vee \otimes \cdots \otimes \mathcal{L}_{i_m}^\vee \otimes \mathcal{F}). \tag{22.1.1.1}$$

We call this the *intersection of* $\mathcal{L}_1, \dots, \mathcal{L}_n$ *with* $\mathcal{F}$. (Never forget that whenever we write $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F})$, we are implicitly assuming that $\dim \operatorname{Supp} \mathcal{F} \leq n$.) The case we will find most useful is if $\mathcal{F}$ is the structure sheaf of a subscheme Y (of dimension at most n). In this case, we may write it $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot Y)$. If the $\mathcal{L}_i$ are all the same, say $\mathcal{L}$, one often writes $(\mathcal{L}^n \cdot \mathcal{F})$ or $(\mathcal{L}^n \cdot Y)$. (Be very careful with this confusing notation: $\mathcal{L}^n$ does not mean $\mathcal{L}^{\otimes n}$.) In some circumstances the convention is to omit the parentheses.

We will prove many things about the intersection product in this chapter. One fact is left until we study flatness (Exercise 25.6.5): that it is "deformation-invariant" — that it is constant in "nice" families.

22.1.A. EXERCISE (REALITY CHECK). Show that if $\mathcal{L}_1 \cong \mathcal{O}_X$ then $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) = 0$.

The following exercise suggests that the intersection product might be interesting, as it "interpolates" between two useful notions: the degree of a line bundle on a curve, and Bezout's theorem.

22.1.B. EXERCISE.
(a) If X is a curve, and $\mathcal{L}$ is an invertible sheaf on X, show that $(\mathcal{L} \cdot X) = \deg_X \mathcal{L}$.
(b) Suppose k is an infinite field, $X = \mathbb{P}^N$, and Y is a dimension n subvariety of X. If $H_1, \dots, H_n$ are generally chosen hypersurfaces of degrees $d_1, \dots, d_n$ respectively (so $\dim(H_1 \cap \cdots \cap H_n \cap Y) = 0$ by Exercise 12.3.C(d)), then by Bezout's theorem

(Exercise 20.5.M),

$$\deg(H_1 \cap \cdots \cap H_n \cap Y) = d_1 \cdots d_n \deg(Y).$$

Show that

$$(\mathcal{O}_X(H_1) \cdots \mathcal{O}_X(H_n) \cdot Y) = d_1 \cdots d_n \deg(Y).$$

We now describe some of the properties of the intersection product. In the course of proving Exercise 22.1.B(b) you will in effect solve the following exercise.

22.1.C. EXERCISE. Suppose D is an effective Cartier divisor on X that restricts to an effective Cartier divisor on Y (i.e. remains not locally a zerodivisor on Y). Show that

$$(\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot \mathcal{O}(D) \cdot Y) = (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot D).$$

More generally, if D is an effective Cartier divisor on X that does not meet any associated points of $\mathcal{F}$, show that

$$(\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot \mathcal{O}(D) \cdot \mathcal{F}) = (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot \mathcal{F}|_D).$$

22.1.2. *Definition.* For this reason, if D is an effective Cartier divisor, in the symbol for the intersection product, we often writes D instead of $\mathcal{O}(D)$. We interchangeably think of intersecting divisors rather than line bundles. For example, we will discuss the special case of intersection theory on a surface in §22.2, and when we intersect two curves C and D, we will write the intersection as $(C \cdot D)$ or even $C \cdot D$.

22.1.D. EXERCISE. Show that the intersection product (22.1.1.1) is preserved by field extension of k.

22.1.3. Proposition. — *Assume X is projective. For fixed $\mathcal{F}$, the intersection product $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F})$ is a symmetric multilinear function of the $\mathcal{L}_1, \dots, \mathcal{L}_n$.*

We remark that Proposition 22.1.3 is true without projective hypotheses. For an argument in the proper case, see [**Kl**, Prop. 2]. Unlike most extensions to the proper case, this is not just an application of Chow's lemma; it involves a different approach, involving a beautiful trick called *dévissage*.

Proof. Symmetry is clear. By Exercise 22.1.D, we may assume that k is infinite (e.g. algebraically closed). We now prove the result by induction on n.

22.1.E. EXERCISE (BASE CASE). Prove the result when $n = 1$. (Hint: Exercise 20.4.M.)

We now assume the result for when the support of the coherent sheaf has dimension less than n.

We now use a trick. We wish to show that (for arbitrary $\mathcal{L}_1, \mathcal{L}_1', \mathcal{L}_2, \dots, \mathcal{L}_n$,

$$(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) + (\mathcal{L}_1' \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) - ((\mathcal{L}_1 \otimes \mathcal{L}_1') \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) \tag{22.1.3.1}$$

is 0.

22.1.F. EXERCISE. Rewrite (22.1.3.1) as

$$(\mathcal{L}_1 \cdot \mathcal{L}_1' \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}). \tag{22.1.3.2}$$

(There are now $n+1$ line bundles appearing in the product, but this does not contradict the definition of the intersection product, as $\dim \operatorname{Supp} \mathcal{F} \leq n < n+1$.)

22.1.G. EXERCISE. Use the inductive hypothesis to show that (22.1.3.1) is 0 if $\mathcal{L}_n \cong \mathcal{O}(D)$ for D an effective Cartier divisor missing the associated points of $\mathcal{F}$.

In particular, if $\mathcal{L}_n$ is very ample, then (22.1.3.1) is 0, as Exercise 20.5.C shows that there exists a section of $\mathcal{L}_n$ missing the associated points of $\mathcal{F}$.

By the symmetry of its incarnation as (22.1.3.2), expression (22.1.3.1) vanishes if $\mathcal{L}_1$ is very ample. Let $\mathcal{A}$ and $\mathcal{B}$ be any two very ample line bundles on X. Then by substituting $\mathcal{L}_1 = \mathcal{B}$ and $\mathcal{L}_1' = \mathcal{A} \otimes \mathcal{B}^\vee$, using the vanishing of (22.1.3.1), we have

$$(22.1.3.3) \qquad (\mathcal{A} \otimes \mathcal{B}^\vee \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) = (\mathcal{A} \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) - (\mathcal{B} \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F})$$

Both summands on the right side of (22.1.3.3) are linear in $\mathcal{L}_n$, so the same is true of the left side. But by Exercise 16.3.H, *any* invertible sheaf on X may be written in the form $\mathcal{A} \otimes \mathcal{B}^\vee$ ("as the difference of two very amples"), so $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F})$ is linear in $\mathcal{L}_n$, and thus (by symmetry) in each of the $\mathcal{L}_i$. (An interesting feature of this argument is that we intended to show linearity in $\mathcal{L}_1$, and ended up showing linearity in $\mathcal{L}_n$.) $\square$

We have an added bonus arising from the proof.

22.1.H. EXERCISE. Show that if $\dim \operatorname{Supp} \mathcal{F} < n+1$, and $\mathcal{L}_1$, $\mathcal{L}_1'$, $\mathcal{L}_2$, …, $\mathcal{L}_n$ are invertible sheaves on X, then (22.1.3.2) vanishes. In other words, the intersection product of $n+1$ invertible sheaves with a coherent sheaf $\mathcal{F}$ vanishes if the $\dim \operatorname{Supp} \mathcal{F} < n+1$.

22.1.4. Proposition. — *The intersection product depends only on the numerical equivalence classes of the $\mathcal{L}_i$.*

We prove Proposition 22.1.4 when X is projective, as we use the fact that every line bundles is the difference two very ample line bundles in both the proof of Proposition 22.1.3 and in the proof of Proposition 22.1.4 itself.

Proof if X *is projective.* Suppose $\mathcal{L}_1$ is numerically equivalent to $\mathcal{L}_1'$, and $\mathcal{L}_2, \dots, \mathcal{L}_n$, and $\mathcal{F}$ are arbitrary. We wish to show that $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) = (\mathcal{L}_1' \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F})$. By Exercise 22.1.D, we may assume that k is infinite (e.g. algebraically closed). We proceed by induction on n. The case $n = 1$ follows from Exercise 20.4.M (as all proper curves are projective, Exercise 20.6.E). We assume that $n > 1$, and assume the result for "smaller n". By multilinearity of the intersection product, and the fact that each $\mathcal{L}_n$ maybe written as the "difference" of two very ample invertible sheaves (Exercise 16.3.H), it suffices to prove the result in the case when $\mathcal{L}_n$ is very ample. We may write $\mathcal{L}_n = \mathcal{O}(D)$, where D is an effective Cartier divisor missing the associated points of $\mathcal{F}$ (Exercise 20.5.C). Then and the inductive hypothesis,

$$\begin{aligned}
(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) &= (\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_{n-1} \cdot \mathcal{F}|_D) \quad \text{(Ex. 22.1.C)} \\
&= (\mathcal{L}_1' \cdot \mathcal{L}_2 \cdots \mathcal{L}_{n-1} \cdot \mathcal{F}|_D) \quad \text{(inductive hyp.)} \\
&= (\mathcal{L}_1' \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) \quad \text{(Ex. 22.1.C).}
\end{aligned}$$

$\square$

22.1.5. *Asymptotic Riemann-Roch.*

Recall that if Y is a proper curve, $\chi(Y, \mathcal{L}^{\otimes m}) = m \deg_Y \mathcal{L} + \chi(Y, \mathcal{O}_Y)$ (see (20.4.8.1)) is a linear polynomial in m, whose leading term is an intersection product. This generalizes.

22.1.I. EXERCISE (ASYMPTOTIC RIEMANN-ROCH). Suppose $\mathcal{F}$ is a coherent sheaf with $\dim \operatorname{Supp} \mathcal{F} \leq n$. Show that $\chi(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F})$ is a polynomial in $\mathcal{L}$ of degree m of degree at most n. Show that the coefficient of m^n in this polynomial (the "leading term") is $(\mathcal{L}^n \cdot \mathcal{F})/n!$. Hint: Exercise 22.1.H implies that $(\mathcal{L}^{n+1} \cdot (\mathcal{L}^{\otimes i} \otimes \mathcal{F})) = 0$. (Careful with this notation: $\mathcal{L}^{n+1}$ doesn't mean $\mathcal{L}^{\otimes(n+1)}$, it means $\mathcal{L} \cdot \mathcal{L} \cdots \mathcal{L}$ with $n+1$ factors.) Expand this out using (22.1.1.1) to get a recursion for $\chi(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F})$. Your argument may resemble the proof of polynomiality of the Hilbert polynomial, Theorem 20.5.1, so you may find further hints there. Exercise 20.5.E in particular might help.

Thus if because of a "vanishing theorem" (such as Serre vanishing, Theorem 20.1.3(ii)), we know that $h^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}) = 0$ for $m \gg 0$ and $i > 0$, then we know $h^0(X, \mathcal{L}^{\otimes m})$. In the proof of Nakai's criterion (Theorem 22.3.1), we will do something along these lines, but a little weaker and a little cleverer.

We know all the coefficients of this polynomial if X is a curve, by Riemann-Roch (see (20.4.8.1)), or basically by definition. We will know/interpret all the coefficients if X is a nonsingular projective surface and $\mathcal{F}$ is an invertible sheaf when we prove Riemann-Roch for surfaces (Exercise 22.2.B(b)). To understand the general case, we need the theory of Chern classes. The result is the Hirzebruch-Riemann-Roch Theorem, which can be further generalized to the celebrated Grothendieck-Riemann-Roch Theorem.

22.1.J. EXERCISE (THE PROJECTION FORMULA). Suppose $\pi : X_1 \to X_2$ is a projective morphism of projective schemes (over a field k) of the same dimension n, and $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are invertible sheaves on X_2. Show that $(\pi^*\mathcal{L}_1 \cdots \pi^*\mathcal{L}_n) = \deg(X_1/X_2)(\mathcal{L}_1 \cdots \mathcal{L}_n)$. (The first intersection is on X_1, and the second is on X_2.) Hint: argue that by the multilinearity of the intersection product, it suffices to deal with the case where the $\mathcal{L}_i$ are very ample. Then choose sections of each $\mathcal{L}_i$, all of whose intersection lies in the locus where π has "genuine degree $\deg d$". (In fact, the result holds with projective replaced with proper.) A better hint will be added later.

22.1.6. *Remark: A more general projection formula.* Suppose $\pi : X_1 \to X_2$ is a proper morphism of proper varieties, and $\mathcal{F}$ is a coherent sheaf on X_1 with $\dim \operatorname{Supp} \mathcal{F} \leq n$ (so $\dim \operatorname{Supp} \pi_*\mathcal{F} \leq n$). Suppose also that $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are invertible sheaves on X_2. Then

$$(\pi^*\mathcal{L}_1 \cdots \pi^*\mathcal{L}_n \cdot \mathcal{F}) = (\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \pi_*\mathcal{F}).$$

This is called the **projection formula** (and generalizes, in a nonobvious way, Exercise 22.1.J). Because we won't use this version of the projection formula, we omit the proof. One is given in [**Kl2**, B.15].

22.1.K. EXERCISE (INTERSECTING WITH AMPLE LINE BUNDLES). Suppose X is a projective k-variety, and $\mathcal{L}$ is an ample line bundle on X. Show that for any subvariety Y of X of dimension n, $(\mathcal{L}^n \cdot Y) > 0$. (Hint: use Proposition 22.1.3 and Theorem 16.3.11 to reduce to the case where $\mathcal{L}$ is very ample. Then show that

$(\mathcal{L}^n \cdot Y) = \deg Y$ in the embedding into projective space induced by the linear system $|\mathcal{L}|$.)

Nakai's criterion (Theorem 22.3.1) states that this characterizes ampleness.

22.1.7. ⋆⋆ *Cohomological interpretation in the complex projective case, generalizing Exercise 20.4.G.* If $k = \mathbb{C}$, we can interpret $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot Y)$ as the degree of

$$c_1((\mathcal{L}_1)_{an}) \cup \cdots \cup c_1((\mathcal{L}_n)_{an}) \cap [Y_{an}] \tag{22.1.7.1}$$

in $H_0(Y_{an}, \mathbb{Z})$. (Recall $c_1((\mathcal{L}_i)_{an}) \in H^2(X_{an}, \mathbb{Z})$, as discussed in Exercise 20.4.G.) One way of proving this is to use multilinearity of both the intersection product and (22.1.7.1) to reduce to the case where the $\mathcal{L}_n$ is very ample, so $\mathcal{L}_n \cong \mathcal{O}(D)$, where D restricts to an effective Cartier divisor E on Y. Then show that if $\mathcal{L}$ is an analytic line bundle on Y_{an} with non-zero section E_{an}, then $c_1(\mathcal{L}) \cap [Y_{an}] = [E_{an}]$. Finally, use induction on n and Exercise 22.1.C.

22.2 Intersection theory on a surface

We now apply the general machinery of §22.1 to the case of a nonsingular projective surface X. (What matters is that is X is Noetherian and factorial, so $\operatorname{Pic} X \to \operatorname{Cl} X$ is an isomorphism, Proposition 15.2.7. Recall that nonsingular schemes are factorial by the Auslander-Buchsbaum Theorem 13.3.1.)

22.2.A. EXERCISE/DEFINITION. Suppose C and D are effective divisors on X (curves).
(a) Show that

$$\deg_C \mathcal{O}_X(D)|_C \tag{22.2.0.2}$$
$$= (\mathcal{O}(C) \cdot \mathcal{O}(D) \cdot X) \tag{22.2.0.3}$$
$$= \deg_D \mathcal{O}_X(C)|_D. \tag{22.2.0.4}$$

We call this the **intersection number** of C and D, and denote it $C \cdot D$.
(b) If C and D have no components in common, show that

$$C \cdot D = h^0(C \cap D, \mathcal{O}_{C \cap D})$$

where $C \cap D$ is the scheme-theoretic intersection of C and D on X.

We thus have three descriptions of the intersection number (22.2.0.2)–(22.2.0.4), each with advantages and disadvantages. The Euler characteristic description (22.2.0.3) is remarkably useful (for example, in the exercises below), but the geometry is obscured. The definition $\deg_C \mathcal{O}_X(D)|_C$, (22.2.0.2) is not obviously symmetric in C and D. The definition $h^0(C \cap D, \mathcal{O}_{C \cap D})$ is clearly local — to each point of $C \cap D$, we have a vector space. For example, we know that in $\mathbb{A}^2_k$, $y - x^2 = 0$ meets the x-axis with multiplicity 2, because h^0 of the scheme-theoretic intersection $(k[x, y]/(y - x^2, y))$ has dimension 2. (This h^0 is also the *length* of the dimension 0 scheme, but we won't use this terminology.)

By Proposition 22.1.3, the intersection number induces a bilinear "intersection form"

$$\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}. \tag{22.2.0.5}$$

By Asymptotic Riemann-Roch (Exercise 22.1.I), $\chi(X, \mathcal{O}(nD))$ is a quadratic polynomial in n.

You can verify that Exercise 22.2.A recovers Bézout's theorem for plane curves (see Exercise 20.5.M), using $\chi(\mathbb{P}^2, \mathcal{O}(n)) = (n+2)(n+1)-2$ (from Theorem 20.1.2).

Before getting to a number of interesting explicit examples, we derive a couple of fundamental theoretical facts.

22.2.B. EXERCISE. Assuming Serre duality for X (Theorem 20.4.5), prove the following for a smooth projective surface X. (We are mixing divisor and invertible sheaf notation, so be careful. Here K_X is a divisor corresponding to $\mathcal{K}_X$.)
(a) *(sometimes called the adjunction formula)* $C \cdot (K_X + C) = 2p_a(C) - 2$.
(b) *(Riemann-Roch for surfaces)* $\chi(\mathcal{O}_X(D)) = D \cdot (D - K_X)/2 + \chi(\mathcal{O}_X)$ (cf. Riemann-Roch for curves, Exercise 20.4.B).

22.2.1. Two explicit examples: $\mathbb{P}^1 \times \mathbb{P}^1$ and $\operatorname{Bl}_p \mathbb{P}^2$.

22.2.C. EXERCISE: $X = \mathbb{P}^1 \times \mathbb{P}^1$. Recall from Exercise 15.2.N that $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}\ell \times \mathbb{Z}m$, where ℓ is the curve $\mathbb{P}^1 \times \{0\}$ and m is the curve $\{0\} \times \mathbb{P}^1$. Show that the intersection form (22.2.0.5) is given by $\ell \cdot \ell = m \cdot m = 0$, $\ell \cdot m = 1$. (Hint: You can compute the cohomology groups of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ using Exercise 20.3.D, but it is much faster to use Exercise 22.2.A(b).) What is the class of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ in terms of these generators?

22.2.D. EXERCISE: THE BLOWN UP PROJECTIVE PLANE. (You absolutely needn't have read Chapter 19 to do this exercise!) Let $X = \operatorname{Bl}_p \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2_k$ at a k-valued point (the origin, say) p — see Exercise 10.2.M, which describes the blow-up of $\mathbb{A}^2_k$, and "compactify". Interpret $\operatorname{Pic} X$ is generated (as an abelian group) by ℓ and e, where ℓ is a line not passing through the origin, and e is the exceptional divisor. Show that the intersection form (22.2.0.5) is given by $\ell \cdot \ell = 1$, $e \cdot e = -1$, and $\ell \cdot e = 0$. Hence show that $\operatorname{Pic} X \cong \mathbb{Z}\ell \times \mathbb{Z}e$ (as promised in the aside in Exercise 15.2.O). In particular, the exceptional divisor has negative self-intersection.

22.2.2. *Hint.* Here is a possible hint to get the intersection form in Exercise 22.2.D. The scheme-theoretic preimage in $\operatorname{Bl}_p \mathbb{P}^2$ of a line through the origin is the scheme-theoretic union of the exceptional divisor e and the "proper transform" m of the line through the origin. Show that $\ell = e+m$ in $\operatorname{Pic} \operatorname{Bl}_p \mathbb{P}^2$ (writing the Picard group law additively). Show that $\ell \cdot m = e \cdot m = 1$ and $m \cdot m = 0$.

22.2.E. EXERCISE. Show that the blown up projective plane $\operatorname{Bl}_p \mathbb{P}^2$ in Exercise 22.2.D is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, perhaps considering their (isomorphic) Picard groups, and identifying which classes are effective (represented by effective divisors). (This is an example of a pair of smooth projective birational surfaces that have isomorphic Picard groups, but which are not isomorphic. This exercise shows that $\mathbb{F}_0$ is not isomorphic to $\mathbb{F}_1$, as promised in Definition 18.2.2)

22.2.F. EXERCISE (CF. EXERCISE 20.4.R). Show that the nef cone (Exercise 20.4.Q) of $\operatorname{Bl}_p \mathbb{P}^2$ is generated by ℓ and m. Hint: show that ℓ and m are nef. By intersecting line bundles with the *curves* e and ℓ, show that nothing outside the cone spanned by ℓ and m are nef. (Side remark: note that as in Exercise 20.4.R, linear series corresponding to the boundaries of the cone give "interesting contractions".)

22.2.G. EXERCISE: A NONPROJECTIVE SURFACE. Show the existence of a proper nonprojective surface over a field as follows, parallelling the construction of a proper nonprojective threefold in §17.4.8. Take two copies of the blown up projective plane $\mathrm{Bl}_p \mathbb{P}^2$, gluing ℓ on the first to e on the second, and e on the second to ℓ on the first. Hint: show that if $\mathcal{L}$ is a line bundle having positive degree on each effective curve, then $\mathcal{L} \cdot \ell > \mathcal{L} \cdot e$, using $\ell = e + m$ from Hint 22.2.2.

22.2.3. Fibrations.

Suppose $\pi : X \to B$ is a morphism from a projective surface to a nonsingular curve and $b \in B$ is a closed point. Let $F = \pi^* b$. Then $\mathcal{O}_X(F) = \pi^* \mathcal{O}_B(b)$, which is isomorphic to $\mathcal{O}$ on F. Thus $F \cdot F = \deg_F \mathcal{O}_X(F) = 0$: "the self-intersection of a fiber is 0". The same argument works without X being nonsingular, as long as you phrase it properly: $(\pi^* \mathcal{O}_X(b))^2 = 0$.

22.2.H. EXERCISE. Suppose E is an elliptic curve, with origin p. On $E \times E$, let Δ be the diagonal. By considering the "difference" map $E \times E \to E$, for which $\pi^* p = \Delta$, show that $\Delta^2 = 0$. Show that $N^1_{\mathbb{Q}}(X)$ has rank at least 3. Show that in general for schemes X and Y, $\operatorname{Pic} X \times \operatorname{Pic} Y \to \operatorname{Pic}(X \times Y)$ (defined by pulling back and tensoring) need not be isomorphism; the case of $X = Y = \mathbb{P}^1$ is misleading.

Remark: $\dim_{\mathbb{Q}} N^1_{\mathbb{Q}}(E \times E)$ is always 3 or 4. It is 4 if there is a nontrivial endomorphism from E to itself (i.e. not just multiplication by some n); the additional class comes from the graph of this endomorphism.

Our next goal is to describe the self-intersection of a curve on a ruled surface (Exercise 22.2.J). To set this up, we have a useful preliminary result.

22.2.I. EXERCISE (THE NORMAL BUNDLE TO A SECTION OF $\mathcal{P}$roj OF A RANK 2 VECTOR BUNDLE. Suppose X is a scheme, and and $\mathcal{V}$ is a rank 2 locally free sheaf on C. Explain how the short exact sequences

$$0 \to \mathcal{S} \to \mathcal{V} \to \mathcal{Q} \to 0 \qquad (22.2.3.1)$$

on X, where $\mathcal{S}$ and $\mathcal{Q}$ have rank 1, correspond to the sections $\sigma : X \to \mathbb{P}\mathcal{V}$ to the projection $\mathbb{P}\mathcal{V} \to X$. Show that the normal bundle to $\sigma(X)$ in $\mathbb{P}\mathcal{V}$ is $\mathcal{Q} \otimes \mathcal{S}^\vee$. (A generalization is stated in §23.3.7.) Hint: (i) For simplicity, it is convenient to assume $\mathcal{S} = \mathcal{O}_X$, by replacing $\mathcal{V}$ by $\mathcal{V} \otimes \mathcal{S}^\vee$, as the statement of the problem respects tensoring by an invertible sheaf (see Exercise 18.2.G). (ii) Assume now (*with* loss of generality) that $\mathcal{Q} \cong \mathcal{O}_X$. Then describe the section as $\sigma : X \to \mathbb{P}^1 \times X$, with X mapping to the 0 section. Describe an isomorphism of $\mathcal{O}_X$ with the normal bundle to $\sigma(X) \to \mathbb{P}^1 \times X$. (Do *not* just say that the normal bundle "is trivial".) (iii) Now consider the case where $\mathcal{Q}$ is general. Choose trivializing neighborhoods U_i of $\mathcal{Q}$, and let g_{ij} be the the transition function for $\mathcal{Q}$. On the overlap between two trivializing neighborhoods $U_i \cap U_j$, determine how your two isomorphisms of $\mathcal{O}_X$ with $N_{\sigma(X)/\mathbb{P}^1_X}$ with $\mathcal{O}_X$ from (ii) (one for U_i, one for U_j) are related. In particular, show that they differ by g_{ij}.

22.2.J. EXERCISE (SELF-INTERSECTIONS OF SECTIONS OF RULED SURFACES). Suppose C is a nonsingular curve, and $\mathcal{V}$ is a rank 2 locally free sheaf on C. Then $\mathbb{P}\mathcal{V}$ is a ruled surface (Definition 18.2.2). Fix a section σ of $\mathbb{P}\mathcal{V}$ corresponding to a filtration (22.2.3.1). Show that $\sigma(C) \cdot \sigma(C) = \deg_C \mathcal{Q} \otimes \mathcal{S}^\vee$.

22.2.4. The Hirzebruch surfaces $\mathbb{F}_n = \mathcal{P}\mathrm{roj}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$.

Recall the definition of the Hirzebruch surface $\mathbb{F}_n = \mathcal{P}\mathrm{roj}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ in Definition 18.2.2. It is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$; let $\pi : \mathbb{F}_n \to \mathbb{P}^1$ be the structure morphism. Using Exercise 22.2.J, corresponding to

$$0 \to \mathcal{O}(n) \to \mathcal{O} \oplus \mathcal{O}(n) \to \mathcal{O} \to 0,$$

we have a section of π of self-intersection $-n$; call it $E \subset \mathbb{F}_n$. Similarly, corresponding to

$$0 \to \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(n) \to \mathcal{O}(n) \to 0,$$

we have a section $C \subset \mathbb{F}_n$ of self-intersection n. Let p be any k-valued point of $\mathbb{P}^1$, and let $F = \pi^* p$.

22.2.K. EXERCISE. Show that $\mathcal{O}(F)$ is independent of the choice of p.

22.2.L. EXERCISE. Show that $\operatorname{Pic} \mathbb{F}_n$ is generated by E and F. In the course of doing this, you will develop "local charts" for $\mathbb{F}_n$, which will help you solve later exercises.

22.2.M. EXERCISE. Compute the intersection matrix on $\operatorname{Pic} \mathbb{F}_n$. Show that E and F are independent, and thus $\operatorname{Pic} \mathbb{F}_n \cong \mathbb{Z}E \oplus \mathbb{Z}F$. Calculate C in terms of E and F.

22.2.N. EXERCISE. Show how to identify $\mathbb{F}_n \setminus E$, along with the structure map π, with the total space of the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$, with C as the 0-section. Similarly show how to identify $\mathbb{F}_n \setminus C$ with the total space of the line bundle $\mathcal{O}(-n)$ on $\mathbb{P}^1$; with E as the 0-section.

22.2.O. EXERCISE. Show that $h^0(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(C)) > 1$. (As $\mathcal{O}_{\mathbb{F}_n}(C)$ has a section — namely C — we have that $h^0(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(C)) \geq 1$.) One way to proceed is to write down another section using local charts for $\mathbb{F}_n$.

22.2.P. EXERCISE. Show that every effective curve on $\mathbb{F}_n$ is a non-negative linear combination of E and F. (Conversely, it is clear that for every nonnegative a and b, $\mathcal{O}(aE + bF)$ has a section, corresponding to the effective curve "$aE + bF$". The extension of this to $N^1_{\mathbb{Q}}$ is called the *effective cone*, and this notion, extended to proper varieties more general, can be very useful. This exercise shows that E and F generate the effective cone of $\mathbb{F}_n$.) Hint: show that because "F moves", any effective curve must intersect F nonnegatively, and similarly because "C moves" (Exercise 22.2.O), any effective curve must intersect C nonnegatively. If $\mathcal{O}(aE+bF)$ has a section corresponding to an effective curve D, what does this say about a and b?

22.2.Q. EXERCISE. By comparing effective cones, and the intersection pairing, show that the $\mathbb{F}_n$ are pairwise nonisomorphic.

This is difficult to do otherwise, and foreshadows the fact that nef and effective cones are useful tools in classifying and understanding varieties general. In particular, they are central to the minimal model program.

22.2.R. EXERCISE. Show that the nef cone of $\mathbb{F}_n$ is generated by C and F. (We will soon see that by Kleiman's criterion for ampleness, Theorem 22.3.7, that the ample cone is the interior of this cone, so we have now identified the ample line bundles on $\mathbb{F}_n$.)

22.2.S. EXERCISE. We have seen earlier (Exercises 22.2.F and 20.4.R) that the boundary of the nef cone give "interesting contractions". What are the maps given by the two linear series corresponding to $\mathcal{O}(F)$ and $\mathcal{O}(C)$? After this series of exercises, you may wish to revisit Exercises 22.2.C-22.2.F, and interpret them as special cases: $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1 \cong \mathrm{Bl}_p \mathbb{P}^2$.

22.2.5. Blow-ups in general.
Exercise 22.2.D is a special case of the following.

22.2.T. EXERCISE. Suppose X is a nonsingular projective surface over k, and p is a k-valued point. Let $\beta : \mathrm{Bl}_p X \to X$ be the blow-up morphism, and let $E = E_p X$ be the exceptional divisor. Consider the exact sequence

$$\mathbb{Z} \xrightarrow{\gamma : 1 \mapsto [E]} \operatorname{Pic} \mathrm{Bl}_p X \xrightarrow{\alpha} \operatorname{Pic}(\mathrm{Bl}_p X \setminus E) \longrightarrow 0$$

(from (15.2.6.2)). Note that $\mathrm{Bl}_p X \setminus E = X \setminus p$. Show that $\operatorname{Pic}(X \setminus p) = \operatorname{Pic} X$. Show that $\beta^* : \operatorname{Pic} X \to \operatorname{Pic} \mathrm{Bl}_p X$ gives a section to α. Use §19.3.5 to show that $E^2 = -1$, and from that show that γ is an injection. Conclude that $\operatorname{Pic} \mathrm{Bl}_p X \cong \operatorname{Pic} X \oplus \mathbb{Z}$. Describe how to find the intersection matrix on $N^1_{\mathbb{Q}}(\mathrm{Bl}_p X)$ from that of $N^1_{\mathbb{Q}}(X)$.

22.2.U. EXERCISE. Suppose D is an effective Cartier divisor (a curve) on X. Let $\mathrm{mult}_p D$ be the multiplicity of D at p (Exercise 19.4.J), and let D^{pr} be the proper transform of D. Show that $\pi^* D = D^{pr} + (\mathrm{mult}_p D) E$ as effective Cartier divisors. More precisely, show that the product of the local equation for D^{pr} and the $(\mathrm{mult}_p D)$th power of the local equation for E is the local equation for $\pi^* D$, and hence that (i) $\pi^* D$ is an effective Cartier divisor, and (ii) $\pi^* \mathcal{O}_X(D) \cong \mathcal{O}_{\mathrm{Bl}_p X}(D^{pr}) \otimes \mathcal{O}_{\mathrm{Bl}_p X}(E)^{\otimes(\mathrm{mult}_p D)}$. (A special case is the equation $\ell = e + m$ in Hint 22.2.2.)

22.3 ⋆⋆ Nakai and Kleiman's criteria for ampleness

Exercise 22.1.K stated that if X is projective k-variety, and $\mathcal{L}$ is an ample line bundle on X, then for any subvariety Y of X of dimension n, $(\mathcal{L}^n \cdot Y) > 0$. Nakai's criterion states that this is a characterization:

22.3.1. Theorem (Nakai's criterion for ampleness). — *If $\mathcal{L}$ is an invertible sheaf on a projective k-scheme X, and for every subvariety Y of X of dimension n, $(\mathcal{L}^n \cdot Y) > 0$, then $\mathcal{L}$ is ample.*

22.3.2. *Remarks.* We note that X need only be proper for this result to hold ([**Kl**, Thm. III.1.1]).

Before proving Nakai's theorem, we point out some consequences related to §20.4.10. By Proposition 22.1.4, $(\mathcal{L}^n \cdot Y)$ depends only on the numerical equivalence class of $\mathcal{L}$, so ampleness is a numerical property. As a result, the notion of ampleness makes sense on $N^1_{\mathbb{Q}}(X)$. As the tensor product of two ample invertible sheaves is ample (Exercise 16.3.L), the ample $\mathbb{Q}$-line bundles in $N^1_{\mathbb{Q}}(X)$ form a cone, called the **ample cone** of X.

22.3.3. Proposition. — *If X is a projective k-scheme, the ample cone is open.*

22.3.4. *Warning.* In the course of this proof, we introduce a standard, useful, but confusing convention suggested by the multilinearity of the intersection product: we write tensor product of invertible sheaves *additively*. This is because we want to deal with intersections on the $\mathbb{Q}$-vector space $N^1_{\mathbb{Q}}(X)$. So for example by $((a\mathcal{L}_1 + b\mathcal{L}_1') \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F})$ $(a, b \in \mathbb{Q})$, we mean $a(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) + b(\mathcal{L}_1' \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F})$. (Some people try to avoid confusion by using divisors rather than line bundles, as we add divisors when we "multiply" the corresponding line bundles. This is psychologically helpful, but may add more confusion, as one then has to worry about the whether and why and how and when line bundles correspond to divisors.)

Proof. Suppose $\mathcal{A}$ is an ample invertible sheaf on X. We will describe a small open neighborhood of $[\mathcal{A}]$ in $N^1_{\mathbb{Q}}(X)$ consisting of ample $\mathbb{Q}$-line bundles. Choose invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_n$ on X whose classes form a basis of $N^1_{\mathbb{Q}}(X)$. By Exercise 16.3.J, there is some m such that $\mathcal{A}^{\otimes m} \otimes \mathcal{L}_i$ and $\mathcal{A}^{\otimes m} \otimes \mathcal{L}_i^{\vee}$ are both very ample for all n. Thus (in the additive notation of Warning 22.3.4), $\mathcal{A} + \frac{1}{m}\mathcal{L}_i$ and $\mathcal{A} - \frac{1}{m}\mathcal{L}_i$ are both ample. As the ample $\mathbb{Q}$-line bundles form a cone, it follows that $\mathcal{A} + \epsilon_1\mathcal{L}_1 + \cdots + \epsilon_n\mathcal{L}_n$ is ample for $|\epsilon_i| \leq 1/m$. $\square$

22.3.5. *Proof of Nakai's criterion, Theorem 22.3.1.* We prove Nakai's criterion in several steps.

22.3.A. UNIMPORTANT EXERCISE. Prove the case where $\dim X = 0$.

Step 1: initial reductions. Suppose $\mathcal{L}$ satisfies the hypotheses of the Theorem; we wish to show that $\mathcal{L}$ is ample. By Exercises 20.6.C and 20.6.D, we may assume that X is integral. Moreover, we can work by induction on dimension, so we can assume that $\mathcal{L}$ is ample on any closed subvariety. The base case is dimension 1, which was done in Exercise 21.2.E.

Step 2: sufficiently high powers of $\mathcal{L}$ have sections. We show that $H^0(X, \mathcal{L}^{\otimes m}) \neq 0$ for $m \gg 0$.

Our plan is as follows. By Asymptotic Riemann-Roch (Exercise 22.1.I), $\chi(X, \mathcal{L}^{\otimes m}) = m^n(\mathcal{L}^n)/n! + \cdots$ grows (as a function of m) without bound. A plausible means of attack is to show that $h^i(X, \mathcal{L}^{\otimes m}) = 0$ for $i > 0$ and $m \gg 0$. We won't do that, but will do something similar.

By Exercise 16.3.H, $\mathcal{L}$ is the difference of two very ample line bundles, say $\mathcal{L} \cong \mathcal{A} \otimes \mathcal{B}^{-1}$ with $\mathcal{A} = \mathcal{O}(A)$ and $\mathcal{B} = \mathcal{O}(B)$. From $0 \to \mathcal{O}(-A) \to \mathcal{O} \to \mathcal{O}_A \to 0$ we have

$$0 \to \mathcal{L}^{\otimes m}(-B) \to \mathcal{L}^{\otimes(m+1)} \to \mathcal{L}^{\otimes(m+1)}|_A \to 0. \tag{22.3.5.1}$$

From $0 \to \mathcal{O}(-B) \to \mathcal{O} \to \mathcal{O}_B \to 0$, we have

$$0 \to \mathcal{L}^{\otimes m}(-B) \to \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m}|_B \to 0. \tag{22.3.5.2}$$

Choose m large enough so that both $\mathcal{L}^{\otimes(m+1)}|_A$ and $\mathcal{L}^{\otimes m}|_B$ have vanishing higher cohomology (i.e. $h^{>0} = 0$ for both; use the inductive hypothesis, and Serre vanishing, Theorem 20.1.3(ii)). This implies that for $i \geq 2$,

$$\begin{aligned} H^i(X, \mathcal{L}^{\otimes m}) &\cong H^i(X, \mathcal{L}^{\otimes m}(-B)) \quad \text{(long exact sequence for (22.3.5.2))} \\ &\cong H^i(X, \mathcal{L}^{\otimes m+1}) \quad \text{(long exact sequence for (22.3.5.1))} \end{aligned}$$

so the higher cohomology stabilizes (is constant) for large m. From

$$\chi(X, \mathcal{L}^{\otimes m}) = h^0(X, \mathcal{L}^{\otimes m}) - h^1(X, \mathcal{L}^{\otimes m}) + \text{constant},$$

$H^0(\mathcal{L}^{\otimes m}) \neq 0$ for $m \gg 0$, completing Step 2.

So by replacing $\mathcal{L}$ by a suitably large multiple (ampleness is independent of taking tensor powers, Theorem 16.3.11), we may assume $\mathcal{L}$ has a section D. We now use D as a crutch.

Step 3: $\mathcal{L}^{\otimes m}$ is globally generated for $m \gg 0$.

As D is effective, $\mathcal{L}^{\otimes m}$ is globally generated on the complement of D: we have a section vanishing on that big open set. Thus any base locus must be contained in D. Consider the short exact sequence

$$0 \to \mathcal{L}^{\otimes(m-1)} \to \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m}|_D \to 0 \tag{22.3.5.3}$$

Now $\mathcal{L}|_D$ is ample by our inductive hypothesis. Choose m so large that $H^1(X, \mathcal{L}^{\otimes m}|_D) = 0$ (Serre vanishing, Theorem 20.1.3(b)). From the exact sequence associated to (22.3.5.3),

$$\phi_m : H^1(X, \mathcal{L}^{\otimes(m-1)}) \to H^1(X, \mathcal{L}^{\otimes m})$$

is surjective for $m \gg 0$. Using the fact that the $H^1(X, \mathcal{L}^{\otimes m})$ are finite-dimensional vector spaces, as m grows, $H^1(X, \mathcal{L}^{\otimes m})$ must eventually stabilize, so the ϕ_m are isomorphisms for $m \gg 0$.

Thus for large m, from the long exact sequence in cohomology for (22.3.5.3), $H^0(X, \mathcal{L}^{\otimes m}) \to H^0(X, \mathcal{L}^{\otimes m}|_D)$ is surjective for $m \gg 0$. But $H^0(X, \mathcal{L}^{\otimes m}|_D)$ has no base points by our inductive hypothesis (applied to D), i.e. for any point p of D there is a section of $\mathcal{L}^{\otimes m}|_D$ not vanishing at p, so $H^0(X, \mathcal{L}^{\otimes m})$ has no base points on D either, completing Step 3.

Step 4. Thus $\mathcal{L}$ is a base-point-free line bundle with positive degree on each curve (by hypothesis of Theorem 22.3.1), so by Exercise 20.6.B we are done. □

The following result is the key to proving Kleiman's numerical criterion of ampleness, Theorem 22.3.7.

22.3.6. Kleiman's Theorem. — *Suppose X is a projective k-scheme. If $\mathcal{L}$ is a nef invertible sheaf on X, then $(\mathcal{L}^k \cdot V) \geq 0$ for every irreducible subvariety $V \subset X$ of dimension k.*

As usual, this extends to the proper case ([**Kl**, Thm. IV.2.1]). And as usual, we postpone the proof until after we appreciate the consequences.

22.3.B. EXERCISE.
(a) Suppose X is a projective k-scheme, $\mathcal{H}$ is ample, and $\mathcal{L}$ is nef. Show that $\mathcal{L} + \epsilon\mathcal{H}$ is ample for all $\epsilon \in \mathbb{Q}^+$. (Hint: use Nakai: $((\mathcal{L} + \epsilon\mathcal{H})^k \cdot V) > 0$. This may help you appreciate the additive notation.)
(b) Conversely, if $\mathcal{L}$ and $\mathcal{H}$ are any two invertible sheaves such that $\mathcal{L} + \epsilon\mathcal{H}$ is ample for all sufficiently small $\epsilon > 0$, show that $\mathcal{L}$ is nef. (Hint: $\lim_{\epsilon \to 0}$.)

22.3.7. Theorem (Kleiman's numerical criterion for ampleness). — *Suppose X is a projective k-scheme.*

(a) *The nef cone is the closure of the ample cone.*
(b) *The ample cone is the interior of the nef cone.*

Proof. (a) Ample invertible sheaves are nef (Exercise 20.4.P(e)), and the nef cone is closed (Exercise 20.4.Q), so the closure of the ample cone is contained in the cone. Conversely, each nef element of $N^1_{\mathbb{Q}}(X)$ is the limit of ample classes by Exercise 22.3.B, so the nef cone is contained in the closure of the ample cone.

(b) As the ample cone is open (Proposition 22.3.3), the ample cone is contained in the interior of the nef cone. Conversely, suppose $\mathcal{L}$ is in the interior of the nef cone, and $\mathcal{H}$ is any ample class. Then $\mathcal{L} - \epsilon\mathcal{H}$ is nef for all small enough positive ϵ. Then by Exercise 22.3.B, $\mathcal{L} = (\mathcal{L} - \epsilon\mathcal{H}) + \epsilon\mathcal{H}$ is ample. $\square$

Suitably motivated, we prove Kleiman's Theorem 22.3.6.

Proof. We may immediately reduce to the case where X is irreducible and reduced. We work by induction on $n := \dim X$. The base case $n = 1$ is obvious. So we assume that $(\mathcal{L}^{\dim V} \cdot V) \geq 0$ for all irreducible V not equal to X. We need only show that $(\mathcal{L}^n \cdot X) \geq 0$.

Fix some very ample $\mathcal{H}$ on X. Consider $P(t) := ((\mathcal{L} + t\mathcal{H})^n \cdot X) \in N^1_{\mathbb{Q}}(X)$, a polynomial in t. We wish to show that $P(0) \geq 0$. Assume otherwise that $P(0) < 0$.

22.3.C. Exercise. Show that $(\mathcal{L}^k \cdot \mathcal{H}^{n-k} \cdot X) \geq 0$ for all $k < n$. (Hint: use the inductive hypothesis).

Thus $P(t)$ has a negative constant term, and the remaining terms are positive, so $P(t)$ has precisely one positive real root t_0.

22.3.D. Exercise. Show that for (rational) $t > t_0$, $\mathcal{L} + t\mathcal{H}$ is ample. (Hint: use Nakai's criterion; and use the inductive hypothesis for all but the "leading term".)

Now let $Q(t) := (\mathcal{L} \cdot (\mathcal{L} + t\mathcal{H})^{n-1} \cdot X)$ and $R(t) := (t\mathcal{H} \cdot (\mathcal{L} + t\mathcal{H})^{n-1} \cdot X)$, so $P(t) = Q(t) + R(t)$.

22.3.E. Exercise. Show that $Q(t) \geq 0$ for all rational $t \geq t_0$. Hint (which you will have to make sense of): It suffices to show this for $t > t_0$. Then $(\mathcal{L} + t\mathcal{H})$ is ample, so for N sufficiently large, $N(\mathcal{L} + t\mathcal{H})$ is very ample. Use the idea of the proof of Proposition 22.1.4 to intersect X with $n - 1$ divisors in the class of $N(\mathcal{L} + t\mathcal{H})$ so that "$((N(\mathcal{L} + t\mathcal{H}))^{n-1} \cdot X)$ is an effective curve C". Then $(\mathcal{L} \cdot C) \geq 0$ as $\mathcal{L}$ is nef.

22.3.F. Exercise. Show that $R(t_0) > 0$. (Hint: expand out the polynomial, and show that all the terms are positive.)

Thus $P(t_0) > 0$ as desired. $\square$

CHAPTER 23

Differentials

23.1 Motivation and game plan

Differentials are an intuitive geometric notion, and we are going to figure out the right description of them algebraically. The algebraic manifestation is somewhat non-intuitive, so it is helpful to understand differentials first in terms of geometry. Also, although the algebraic statements are odd, none of the proofs are hard or long. You will notice that this topic could have been done as soon as we knew about morphisms and quasicoherent sheaves. We have usually introduced new ideas through a number of examples, but in this case we will spend a fair amount of time discussing theory, and only then get to a number of examples.

Suppose X is a "smooth" k-variety. We would like to define a tangent bundle. We will see that the right way to do this will easily apply in much more general circumstances.

• We will see that cotangent is more "natural" for schemes than tangent bundle. This is similar to the fact that the Zariski *cotangent* space is more natural than the *tangent space* (i.e. if A is a ring and $\mathfrak{m}$ is a maximal ideal, then $\mathfrak{m}/\mathfrak{m}^2$ is "more natural" than $(\mathfrak{m}/\mathfrak{m}^2)^\vee$), as we have repeatedly discussed since §13.1. In both cases this is because we are understanding "spaces" via their (sheaf of) functions on them, which is somehow dual to the geometric pictures you have of spaces in your mind.

So we will define the cotangent sheaf first. An element of the (co)tangent space will be called a **(co)tangent vector**.

• Our construction will automatically apply for general X, even if X is not "smooth" (or even at all nice, e.g. finite type). The cotangent sheaf won't be locally free, but it will still be a quasicoherent sheaf.

• Better yet, this construction will naturally work "relatively". For *any* $\pi : X \to Y$, we will define $\Omega_\pi = \Omega_{X/Y}$, a quasicoherent sheaf on X, the sheaf of *relative differentials*. The fiber of this sheaf at a point will be the cotangent vectors of the fiber of the map. This will specialize to the earlier case by taking $Y = \operatorname{Spec} k$. The idea is that this glues together the cotangent sheaves of the fibers of the family. Figure 23.1 is a sketch of the relative tangent space of a map $X \to Y$ at a point $p \in X$ — it is the tangent to the fiber. (The tangent space is easier to draw than the cotangent space!) An element of the relative (co)tangent space is called a **vertical** or **relative (co)tangent vector**.

Thus the central concept of this chapter is the cotangent sheaf $\Omega_\pi = \Omega_{X/Y}$ for a morphism $\pi : X \to Y$ of schemes. A good picture to have in your mind is the following. If $f : X \to Y$ is a map of smooth manifolds whose fibers are

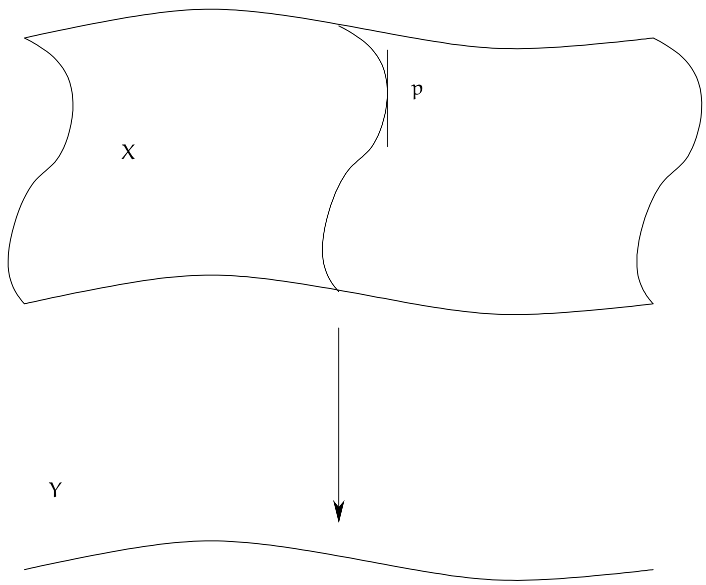


FIGURE 23.1. The relative tangent space of a morphism $X \to Y$ at a point p

manifolds (a submersion), you might hope that the tangent spaces to the fibers at each point $p \in X$ might fit together to form a vector bundle. This is the relative tangent bundle (of π), and its dual is $\Omega_{X/Y}$ (see Figure 23.1). Even if you are not geometrically minded, you will find this useful. (For an arithmetic example, see Exercise 23.2.F.)

23.2 Definitions and first properties

23.2.1. The affine case: three definitions.

We first study the affine case. Suppose A is a B-algebra, so we have a morphism of rings $\phi : B \to A$ and a morphism of schemes $\operatorname{Spec} A \to \operatorname{Spec} B$. I will define an A-module $\Omega_{A/B}$ in three ways. This is called the **module of relative differentials** or the **module of Kähler differentials**. The module of differentials will be defined to be this module, as well as a map $d : A \to \Omega_{A/B}$ satisfying three properties.

(i) **additivity.** $da + da' = d(a + a')$
(ii) **Leibniz.** $d(aa') = a\, da' + a' da$
(iii) **triviality on pullbacks.** $db = 0$ for $b \in \phi(B)$.

These properties will not be surprising if you have seen differentials in any other context.

23.2.A. TRIVIAL EXERCISE. Show that d is B-linear. (In general it will not be A-linear.)

23.2.B. EXERCISE. Prove the quotient rule: if $b = as$, then $da = (s\ db - b\ ds)/s^2$.

23.2.C. EXERCISE. State and prove the chain rule for $d(f(g))$ where f is a polynomial with B-coefficients, and $g \in A$. (As motivation, think of the case $B = k$. So for example, $da^n = na^{n-1}da$, and more generally, if f is a polynomial in one variable, $df(a) = f'(a)\ da$, where f' is defined formally: if $f = \sum c_i x^i$ then $f' = \sum c_i i x^{i-1}$.)

I will give you three definitions of the module of Kähler differentials, which will soon "sheafifiy" to the sheaf of relative differentials. The first definition is a concrete hands-on definition. The second is by universal property. And the third will globalize well, and will allow us to define $\Omega_{X/Y}$ conveniently in general.

23.2.2. First definition of differentials: explicit description. We define $\Omega_{A/B}$ to be finite A-linear combinations of symbols "da" for $a \in A$, subject to the three rules (i)–(iii) above. For example, take $A = k[x, y]$, $B = k$. Then a sample differential is $3x^2\ dy + 4\ dx \in \Omega_{A/B}$. We have identities such as $d(3xy^2) = 3y^2\ dx + 6xy\ dy$.

23.2.3. Key fact. Note that if A is generated over B (as an algebra) by $x_i \in A$ (where i lies in some index set, possibly infinite), subject to some relations r_j (where j lies in some index set, and each is a polynomial in the x_i), then the A-module $\Omega_{A/B}$ is generated by the dx_i, subject to the relations (i)—(iii) and $dr_j = 0$. In short, we needn't take every single element of A; we can take a generating set. And we needn't take every single relation among these generating elements; we can take generators of the relations.

23.2.D. EXERCISE. Verify Key fact 23.2.3. (If you wish, use the affine conormal exact sequence, Theorem 23.2.11, to verify it; different people prefer to work through the theory in different orders. Just take care not to make any circular arguments.)

In particular:

23.2.4. Proposition. — *If A is a finitely generated B-algebra, then $\Omega_{A/B}$ is a finite type (i.e. finitely generated) A-module. If A is a finitely presented B-algebra, then $\Omega_{A/B}$ is a finitely presented A-module.*

Recall (§8.3.13) that an algebra A is *finitely presented* over another algebra B if it can be expressed with finite number of generators and finite number of relations:

$$A = B[x_1, \dots, x_n]/(r_1(x_1, \dots, x_n), \dots, r_j(x_1, \dots, x_n)).$$

If A is Noetherian, then finitely presented is the same as finite type, as the "finite number of relations" comes for free, so most of you will not care.

Let's now see some examples. Among these examples are three particularly important building blocks for ring maps: adding free variables; localizing; and taking quotients. If we know how to deal with these, we know (at least in theory) how to deal with any ring map. (They were similarly useful in understanding the fibered product in practice, in §10.2.)

23.2.5. Example: taking a quotient. If $A = B/I$, then $\Omega_{A/B} = 0$: $da = 0$ for all $a \in A$, as each such a is the image of an element of B. This should be believable; in this case, there are no "vertical tangent vectors".

23.2.6. Example: adding variables. If $A = B[x_1, \dots, x_n]$, then $\Omega_{A/B} = A dx_1 \oplus \cdots \oplus A dx_n$. (Note that this argument applies even if we add an arbitrarily infinite number of indeterminates.) The intuitive geometry behind this makes the answer very reasonable. The cotangent bundle of affine n-space should indeed be free of rank n.

23.2.7. Explicit example: an affine plane curve. Consider the plane curve $y^2 = x^3 - x$ in $\mathbb{A}^2_k$, where the characteristic of k is not 2. Let $A = k[x,y]/(y^2 - x^3 + x)$ and $B = k$. By Key fact 23.2.3, the module of differentials $\Omega_{A/B}$ is generated by dx and dy, subject to the relation

$$2y\ dy = (3x^2 - 1)\ dx.$$

Thus in the locus where $y \neq 0$, dx is a generator (as dy can be expressed in terms of dx). We conclude that where $y \neq 0$, $\widetilde{\Omega_{A/B}}$ is isomorphic to the trivial line bundle (invertible sheaf). Similarly, in the locus where $3x^2 - 1 \neq 0$, dy is a generator. These two loci cover the entire curve, as solving $y = 0$ gives $x^3 - x = 0$, i.e. $x = 0$ or ± 1, and in each of these cases $3x^2 - 1 \neq 0$. We have shown that $\widetilde{\Omega_{A/B}}$ is an invertible sheaf.

We can interpret dx and dy geometrically. Where does the differential dx vanish? The previous paragraph shows that it doesn't vanish on the patch where $2y \neq 0$. On the patch where $3x^2 - 1 \neq 0$, where dy is a generator, $dx = (2y/(3x^2 - 1))dx$ from which we see that dx vanishes precisely where $y = 0$. You should find this believable from the picture. We have shown that $dx = 0$ precisely where the curve has a vertical tangent vector (see Figure 21.2 for a picture). Once we can pull back differentials (Exercise 23.2.I(a) or Theorem 23.2.25, we can interpret dx as the pullback of a differential on the x-axis to $\operatorname{Spec} A$ (pulling back along the projection to the x-axis). When we do that, using the fact that dx doesn't vanish on the x-axis, we can interpret the locus where $dx = 0$ as the locus where the projection map branches. (Can you compute where $dy = 0$, and interpret it geometrically?)

This discussion applies to plane curves more generally. Suppose $A = k[x,y]/f(x,y)$, where for convenience $k = \overline{k}$. Then the same argument as the one given above shows that $\widetilde{\Omega_{A/k}}$ is free of rank 1 on the open set $D(\partial f/\partial x)$, and also on $D(\partial f/\partial y)$. If $\operatorname{Spec} A$ is a nonsingular curve, then these two sets cover all of $\operatorname{Spec} A$. (Exercise 13.2.D — basically the Jacobian criterion — gives nonsingularity at the closed point. Furthermore, the curve must be reduced, or else as the nonreduced locus is closed, it would be nonreduced at a closed point, contradicting nonsingularity. Finally, reducedness at a generic point is equivalent to nonsingularity (basically, a scheme whose underlying set is a point is reduced if and only if it is nonsingular — do you see why?). Alternatively, we could invoke a big result, Fact 13.3.8, to get nonsingularity at the generic point from nonsingularity at the closed points.)

Conversely, if the plane curve is singular, then Ω is *not* locally free of rank one. For example, consider the plane curve $\operatorname{Spec} A$ where $A = \mathbb{C}[x,y]/(y^2 - x^3)$, so

$$\Omega_{A/\mathbb{C}} = (A\ dx \oplus A\ dy)/(2y\ dy - 3x^2\ dx).$$

Then the fiber of $\Omega_{A/\mathbb{C}}$ over the origin (computed by setting $x = y = 0$) is rank 2, as it is generated by dx and dy, with no relation.

Implicit in the above discussion is the following exercise, showing that Ω can be computed using the Jacobian matrix.

23.2.E. IMPORTANT BUT EASY EXERCISE (JACOBIAN DESCRIPTION OF $\Omega_{A/B}$). Suppose $A = B[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then $\Omega_{A/B} = \{\oplus_i A dx_i\}/\{df_j = 0\}$ maybe interpreted as the cokernel of the Jacobian matrix (13.1.4.1) $J : A^{\oplus r} \to A^{\oplus n}$.

23.2.8. Example: localization. If S is a multiplicative subset of B, and $A = S^{-1}B$, then $\Omega_{A/B} = 0$. Reason: by the quotient rule (Exercise 23.2.B), if $a = b/s$, then $da = (s\ db - b\ ds)/s^2 = 0$. If $A = B_f$, this is intuitively believable; then Spec A is an open subset of Spec B, so there should be no vertical (co)tangent vectors.

23.2.F. IMPORTANT EXERCISE (FIELD EXTENSIONS). This notion of relative differentials is interesting even for finite field extensions. In other words, even when you map a reduced point to a reduced point, there is interesting differential information going on.
(a) Suppose K/k is a separable algebraic extension. Show that $\Omega_{K/k} = 0$. Do not assume that K/k is a finite extension! (Hint: for any $\alpha \in K$, there is a polynomial such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.)
(b) Suppose k is a field of characteristic p, $K = k(t^p)$, $L = k(t)$. Compute $\Omega_{K/L}$.
(c) Compute $\Omega_{k(t)/k}$.
(d) If K/k is **separably generated by** $t_1, \dots, t_n \in K$ (i.e. $t_1, \dots, t_n$ form a transcendence basis, and $K/k(t_1, \dots, t_n)$ is algebraic and separable), show that $\Omega_{K/k}$ is a free K-module (i.e. vector space) with basis $dt_1, \dots, dt_n$.

We now delve a little deeper, and discuss two useful and geometrically motivated exact sequences.

23.2.9. Theorem (relative cotangent sequence, affine version). — *Suppose* $C \to B \to A$ *are ring homomorphisms. Then there is a natural exact sequence of* A*-modules*

$$A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to \Omega_{A/B} \to 0.$$

The proof will be quite straightforward algebraically, but the statement comes fundamentally from geometry, and that is how best to remember it. Figure 23.2 is a sketch of a map $X \xrightarrow{f} Y$. Here X should be interpreted as Spec A, Y as Spec B, and Spec C is a point. (If you would like a picture with a higher-dimensional Spec C, just "take the product of Figure 23.2 with a curve".) In the Figure, Y is "smooth", and X is "smooth over Y" — roughly, all fibers are smooth. p is a point of X. Then the tangent space of the fiber of f at p is certainly a subspace of the tangent space of the total space of X at p. The cokernel is naturally the pullback of the tangent space of Y at $f(p)$. This short exact sequence for each p should be part of a short exact sequence of sheaves

$$0 \to \mathcal{T}_{X/Y} \to \mathcal{T}_{X/Z} \to f^*\mathcal{T}_{Y/Z} \to 0$$

on X. Dualizing this yields

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

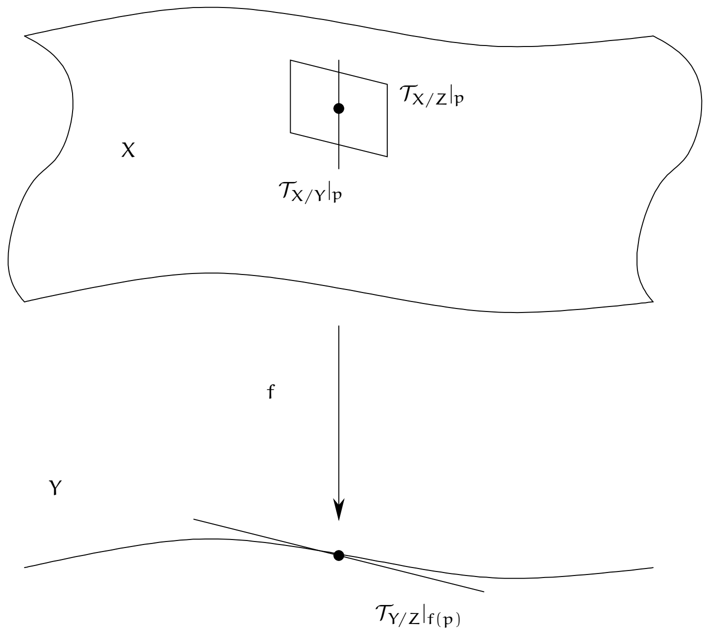


FIGURE 23.2. A sketch of the geometry behind the relative cotangent sequence

This is precisely the statement of Theorem 23.2.9, except we also have left-exactness. This discrepancy is because the statement of the theorem is more general; we will see in Theorem 26.3.1 that in the "smooth" case, we indeed have left-exactness.

23.2.10. *Unimportant aside.* As always, whenever you see something right-exact, you should suspect that there should be some sort of (co)homology theory so that this is the end of a long exact sequence. This is indeed the case, and this exact sequence involves *André-Quillen homology* (see [**E**, p. 386] for more). You should expect that the next term to the left should be the first homology corresponding to A/B, and in particular shouldn't involve C. So if you already suspect that you have exactness on the left in the case where A/B and B/C are "smooth" (whatever that means), and the intuition of Figure 23.2 applies, then you should expect further that all that is necessary is that A/B be "smooth", and that this would imply that the first André-Quillen homology should be zero. Even though you wouldn't precisely know what all the words meant, you would be completely correct! You would also be developing a vague inkling about the *cotangent complex*.

Proof of the relative cotangent sequence (affine version) 23.2.9.

First, note that surjectivity of $\Omega_{A/C} \to \Omega_{A/B}$ is clear, as this map is given by $da \mapsto da$ (where $a \in A$).

Next, the composition over the middle term is clearly 0, as this composition is given by $db \mapsto db \mapsto 0$.

Finally, we wish to identify $\Omega_{A/B}$ as the cokernel of $A \otimes_B \Omega_{B/C} \to \Omega_{A/C}$. Now $\Omega_{A/B}$ is exactly the same as $\Omega_{A/C}$, except we have extra relations: $db = 0$ for $b \in B$. These are precisely the images of $1 \otimes db$ on the left. $\square$

23.2.11. Theorem (conormal exact sequence, affine version). — *Suppose* B *is a* C*-algebra,* I *is an ideal of* B*, and* $A = B/I$*. Then there is a natural exact sequence of* A*-modules*

$$I/I^2 \xrightarrow{\delta: i \mapsto 1 \otimes di} A \otimes_B \Omega_{B/C} \xrightarrow{a \otimes db \mapsto a\, db} \Omega_{A/C} \longrightarrow 0.$$

Before getting to the proof, some discussion may be helpful. First, the map δ needs to be rigorously defined. It is the map $1 \otimes d : B/I \otimes_B I \to B/I \otimes_B \Omega_{B/C}$.

As with the relative cotangent sequence (Theorem 23.2.9), the conormal exact sequence is fundamentally about geometry. To motivate it, consider the sketch of Figure 23.3. In the sketch, everything is "smooth", X is one-dimensional, Y is two-dimensional, j is the inclusion $j : X \hookrightarrow Y$, and Z is a point. Then at a point $p \in X$, the tangent space $\mathcal{T}_X|_p$ clearly injects into the tangent space of $j(p)$ in Y, and the cokernel is the normal vector space to X in Y at p. This should give an exact sequence of bundles on X:

$$0 \to \mathcal{T}_X \to j^*\mathcal{T}_Y \to \mathcal{N}_{X/Y} \to 0.$$

dualizing this should give

$$0 \to \mathcal{N}^\vee_{X/Y} \to j^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0.$$

This is precisely what appears in the statement of the Theorem, except (i) the exact sequence in algebraic geometry is not necessary exact on the left, and (ii) we see I/I^2 instead of $\mathcal{N}^\vee_{\operatorname{Spec} A/\operatorname{Spec} B}$.

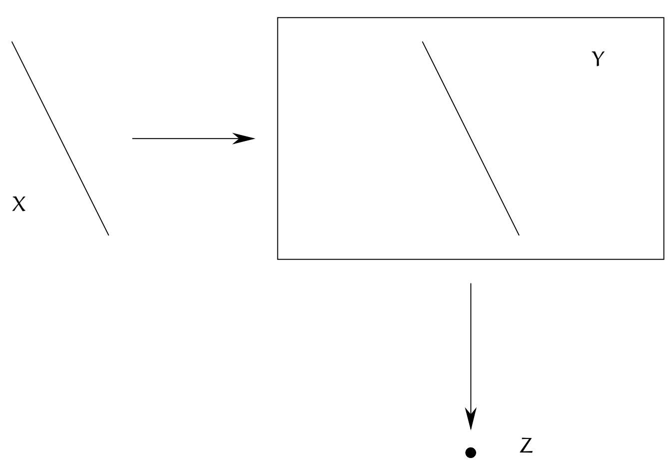


FIGURE 23.3. A sketch of the geometry behind the conormal exact sequence

23.2.12. We resolve the first issue (i) by expecting that the sequence of Theorem 23.2.11 is exact on the left in appropriately "smooth" situations, and this is indeed the case (see Theorem 25.9.8). (If you enjoyed Remark 23.2.10, you might correctly guess several things. The next term on the left should be the André-Quillen homology of A/C, so we should only need that A/C is smooth, and B should be irrelevant. Also, if $A = B/I$, then we should expect that I/I^2 is the first André-Quillen homology of A/B.)

23.2.13. *Conormal modules and conormal sheaves.* We resolve the second issue (ii) by *declaring* I/I^2 to be the **conormal module**, and indeed we will soon see the obvious analogue as the *conormal sheaf*.

Here is some geometric intuition as to why we might want to call (the sheaf associated to) I/I^2 the conormal sheaf, which will likely confuse you, but may offer some enlightenment. First, if $\operatorname{Spec} A$ is a closed point of $\operatorname{Spec} B$, we expect the conormal space to be precisely the cotangent space. And indeed if $A = B/\mathfrak{m}$, the Zariski cotangent space is $\mathfrak{m}/\mathfrak{m}^2$. (We made this subtle connection in §13.1.) In particular, at some point you will develop a sense of why the conormal (=cotangent) space to the origin in $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ is naturally the space of linear forms $\alpha x + \beta y$. But then consider the z-axis in $\operatorname{Spec} k[x, y, z] = \mathbb{A}^3_k$, cut out by $I = (x, y)$. Elements of I/I^2 may be written as $\alpha(z)x + \beta(z)y$, where $\alpha(z)$ and $\beta(z)$ are polynomial. This reasonably should be the conormal space to the z-axis: as z varies, the coefficients of x and y vary. More generally, the same idea suggests that the conormal module/sheaf to any coordinate k-plane inside n-space corresponds to I/I^2. Now consider a k-dimensional (smooth or differential real) manifold X inside an n-dimensional manifold Y, with the classical topology. We can apply the same construction: if $\mathcal{I}$ is the ideal sheaf of X in Y, then $\mathcal{I}/\mathcal{I}^2$ can be identified with the conormal sheaf (essentially the conormal vector bundle), because analytically locally $X \hookrightarrow Y$ can be identified with $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$. For this reason, you might hope that in algebraic geometry, if $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} B$ is an inclusion of something "smooth" in something "smooth", I/I^2 should be the conormal module (or, after applying $\sim$, the conormal sheaf). Motivated by this, we define the conormal module as I/I^2 *always*, and then notice that it has good properties (such as Theorem 23.2.11), but take care to learn what unexpected behavior it might have when we are not in the "smooth" situation, by working out examples such as that of §23.2.7.

23.2.14. *Definition.* Suppose $i : X \hookrightarrow Y$ is a closed immersion of schemes cut out by ideal sheaf $\mathcal{I}$. Define the **conormal sheaf for a closed immersion** by $\mathcal{I}/\mathcal{I}^2$, denoted by $\mathcal{N}^\vee_{X/Y}$. Note that $\mathcal{N}^\vee_{X/Y}$ is a quasicoherent sheaf on X. (The product of quasicoherent ideal sheaves was defined in Exercise 15.3.D.)

Define the **normal sheaf** as its dual $\mathcal{N}_{X/Y} := \mathcal{H}om(\mathcal{N}^\vee_{X/Y}, \mathcal{O}_X)$. This is imperfect notation, because it suggests that the dual of $\mathcal{N}$ is always $\mathcal{N}^\vee$. This is not always true, as for A-modules, the natural morphism from a module to its double-dual is not always an isomorphism. (Modules for which this is true are called **reflexive**, but we won't use this notion.)

23.2.G. EXERCISE. Define the **conormal sheaf** $\mathcal{N}_{X/Y}$ (and hence the normal sheaf) *for a locally closed immersion* $i : X \hookrightarrow Y$ of schemes, a quasicoherent sheaf on X.

23.2.H. EXERCISE: NORMAL BUNDLES TO EFFECTIVE CARTIER DIVISORS. Suppose $D \subset X$ is an effective Cartier divisor (§9.1.2). Show that the conormal sheaf $\mathcal{N}^\vee_{D/X}$ is $\mathcal{O}(-D)|_D$ (and in particular is an invertible sheaf), and hence that the normal sheaf is $\mathcal{O}(D)|_D$. It may be surprising that the normal sheaf should be locally free if $X \cong \mathbb{A}^2$ and D is the union of the two axes (and more generally if X is nonsingular but D is singular), because you may be used to thinking that a "tubular neighborhood" being isomorphic to the normal bundle.

23.2.15. *Proof of Theorem 23.2.11.* The composition

$$I/I^2 \xrightarrow{\delta : i \mapsto 1 \otimes di} A \otimes_B \Omega_{B/C} \xrightarrow{a \otimes db \mapsto a\, db} \Omega_{A/C}$$

is clearly zero: for $i \in I$, $i = 0$ in A, so $di = 0$ in $\Omega_{A/C}$.

We need to identify the cokernel of $\delta : I/I^2 \to A \otimes_B \Omega_{B/C}$ with $\Omega_{A/C}$. Consider $A \otimes_B \Omega_{B/C}$. As an A-module, it is generated by db (where $b \in B$), subject to three relations: $dc = 0$ for $c \in \phi(C)$ (where $\phi : C \to B$ describes B as a C-algebra), additivity, and the Leibniz rule. Given any relation *in* B, d of that relation is 0.

Now $\Omega_{A/C}$ is defined similarly, except there are more relations *in* A; these are precisely the elements of $I \subset B$. Thus we obtain $\Omega_{A/C}$ by starting out with $A \otimes_B \Omega_{B/C}$, and adding the additional relations di where $i \in I$. But this is precisely the image of δ! $\square$

23.2.16. Second definition: universal property. Here is a second definition that is important philosophically, by universal property. Of course, it is a characterization rather than a definition: by universal property nonsense, it shows that if the module exists (with the d map), then it is unique up to unique isomorphism, and then one still has to construct it to make sure that it exists.

Suppose A is a B-algebra, and M is a A-module. A **B-linear derivation of A into M** is a map $d : A \to M$ of B-modules (*not necessarily a map of A-modules*) satisfying the Leibniz rule: $d(fg) = f\,dg + g\,df$. As an example, suppose $B = k$, and $A = k[x]$, and $M = A$. Then d/dx is a k-linear derivation. As a second example, if $B = k$, $A = k[x]$, and $M = k$, then $(d/dx)|_0$ (the operator "evaluate the derivative at 0") is a k-linear derivation.

A third example is $d : A \to \Omega_{A/B}$, and indeed $d : A \to \Omega_{A/B}$ is the *universal* B-*linear derivation of* A. Precisely, the map $d : A \to \Omega_{A/B}$ is defined by the following universal property: any other B-linear derivation $d' : A \to M$ factors uniquely through d:

$$\begin{tikzcd} A \arrow[rr, "d'"] \arrow[dr, "d"'] & & M \\ & \Omega_{A/B} \arrow[ur, "f"'] & \end{tikzcd}$$

Here f is a map of A-modules. (Note again that d and d′ are not necessarily maps of A-modules — they are only B-linear.) By universal property nonsense, if it exists, it is unique up to unique isomorphism. The map $d : A \to \Omega_{A/B}$ clearly satisfies this universal property, essentially by definition.

The next result gives more evidence that this deserves to be called the (relative) cotangent bundle.

23.2.17. Proposition. — *Suppose* B *is a* k*-algebra, and* $\mathfrak{m} \subset B$ *is a maximal ideal with residue field* k*. Then there is a isomorphism of* k*-vector spaces* $\delta : \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k$ *(where the* k *on the right is a* B*-module via the isomorphism* $k \cong B/\mathfrak{m}$*).*

Proof. We instead show an isomorphism of dual vector spaces

$$\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k).$$

By adjunction, we have a canonical isomorphism

$$\begin{aligned}\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k) &= \mathrm{Hom}_B(\Omega_{B/k} \otimes_B k, k)\\ &= \mathrm{Hom}_B(\Omega_{B/k}, \mathrm{Hom}_B(k, k))\\ &= \mathrm{Hom}_B(\Omega_{B/k}, \mathrm{Hom}_k(k, k))\\ &= \mathrm{Hom}_B(\Omega_{B/k}, k),\end{aligned}$$

where in the right argument of $\mathrm{Hom}_B(\Omega_{B/k}, k)$, k is a B-module via its manifestation as $B/\mathfrak{m}$. By the universal property of $\Omega_{B/k}$ (§23.2.16), $\mathrm{Hom}_B(\Omega_{B/k}, k)$ corresponds to the k-derivations of B into $B/\mathfrak{m} \cong k$. By Exercise 13.1.A, these are precisely the elements of $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. (That exercise assumed that B was a local ring, but the solution doesn't use that hypothesis.) $\square$

You can verify that this δ is the one appearing in the conormal exact sequence, Theorem 23.2.11, with $I = \mathfrak{m}$ and $A = C = k$. In fact from the conormal exact sequence, we can immediately see that δ is a surjection, as $\Omega_{k/k} = 0$.

23.2.18. *Remark.* Proposition 23.2.17, in combination with the Jacobian exercise 23.2.E above, gives a second proof of Exercise 13.1.E, the Jacobian method for computing the Zariski tangent space at a k-valued point of a finite type k-scheme.

Depending on how your brain works, you may prefer using the first (constructive) or second (universal property) definition to do the next two exercises.

23.2.I. EXERCISE. (a) *(pullback of differentials)* If

$$\begin{tikzcd} A' & A \arrow[l] \\ B' \arrow[u] & B \arrow[l] \arrow[u] \end{tikzcd}$$

is a commutative diagram, describe a natural homomorphism of A'-modules $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$. An important special case is $B = B'$.
(b) *(differentials behave well with respect to base extension, affine case)* If furthermore the above diagram is a tensor diagram (i.e. $A' \cong B' \otimes_B A$, so the diagram is "co-Cartesian") then show that $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$ is an isomorphism.

23.2.J. EXERCISE: LOCALIZATION (STRONGER FORM). If S is a multiplicative set of A, show that there is a natural isomorphism $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$. (Again, this should be believable from the intuitive picture of "vertical cotangent vectors".) If T is a multiplicative set of B, show that there is a natural isomorphism $\Omega_{S^{-1}A/T^{-1}B} \cong S^{-1}\Omega_{A/B}$ where S is the multiplicative set of A that is the image of the multiplicative set $T \subset B$.

23.2.19. Third definition: global. We now want to globalize this definition for an arbitrary morphism of schemes $f : X \to Y$. We could do this "affine by affine"; we just need to make sure that the above notion behaves well with respect to "change of affine sets". Thus a relative differential on X would be the data of, for every affine $U \subset X$, a differential of the form $\sum a_i \, db_i$, and on the intersection of two affine open sets $U \cap U'$, with representatives $\sum a_i \, db_i$ on U and $\sum a'_i \, db'_i$ on the second, an equality on the overlap. Instead, we take a different tack. I will give the (seemingly unintuitive) definition, then tell you how to think about it, and then get back to the definition.

Suppose $f : X \to Y$ be any morphism of schemes. Recall that $\delta : X \to X \times_Y X$ is a locally closed immersion (Proposition 11.1.3). *Define* the **relative cotangent sheaf** $\Omega_{X/Y}$ as the conormal sheaf $\mathcal{N}^{\vee}_{X, X\times_Y X}$ (see §23.2.13 — and if $X \to Y$ is separated you needn't even worry about Exercise 23.2.G). (Now is also as good a time as any to define the **relative tangent sheaf** $\mathcal{T}_{X/Y}$ as the dual $\mathcal{H}om(\Omega_{X/Y}, \mathcal{O}_X)$ to the relative cotangent sheaf. If we are working in the category of k-schemes, then $\Omega_{X/k}$ and $\mathcal{T}_{X/k}$ are often called the **cotangent sheaf** and **tangent sheaf** of X respectively.)

We now define $d : \mathcal{O}_X \to \Omega_{X/Y}$. Let $\pi_1, \pi_2 : X \times_Y X \to X$ be the two projections. Then define $d : \mathcal{O}_X \to \Omega_{X/Y}$ on the open set U as follows: $df = \pi_2^* f - \pi_1^* f$. (*Warning:* this is not a morphism of quasicoherent sheaves on X, although it *is* $\mathcal{O}_Y$-linear in the only possible meaning of that phrase.) We will soon see that this is indeed a derivation of the sheaf $\mathcal{O}_X$ (in the only possible meaning of the phrase), and at the same time see that our new notion of differentials agrees with our old definition on affine open sets, and hence globalizes the definition. Note that for any open subset $U \subset Y$, d induces a map

$$\Gamma(U, \mathcal{O}_X) \to \Gamma(U, \Omega_{X/Y}), \tag{23.2.19.1}$$

which we also call d, and interpret as "taking the derivative".

23.2.20. *Motivation.* Before connecting this to our other definitions, let me try to convince you that this is a reasonable definition to make. (This discussion is informal and rigorous.) Say for example that Y is a point, and X is something smooth. Then the tangent bundle on to $X \times X$ is $T_X \oplus T_X$: $T_{X\times X} = T_X \oplus T_X$. Restrict this to the diagonal Δ, and look at the normal bundle exact sequence:

$$0 \to T_\Delta \to T_{X\times X}|_\Delta \to N_{\Delta/X} \to 0.$$

Now the left morphism sends v to (v, v), so the cokernel can be interpreted as $(v, -v)$. Thus $N_{\Delta/X}$ is isomorphic to T_X. Thus we can turn this on its head: we know how to find the normal bundle (or more precisely the conormal sheaf), and we can use this to define the tangent bundle (or more precisely the cotangent sheaf). (Experts may want to ponder the above paragraph when Y is more general, but where $X \to Y$ is "nice". You may wish to think in the category of manifolds, and let $X \to Y$ be a submersion.)

23.2.21. *Testing this out in the affine case.* Let's now see how this works for the special case $\operatorname{Spec} A \to \operatorname{Spec} B$. Then the diagonal $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A \otimes_B A$ corresponds to the ideal I of $A \otimes_B A$ that is the cokernel of the ring map

$$f : \sum x_i \otimes y_i \to \sum x_i y_i.$$

23.2.22. The ideal I of $A \otimes_B A$ is generated by the elements of the form $1 \otimes a - a \otimes 1$. Reason: if $f(\sum x_i \otimes y_i) = 0$, i.e. $\sum x_i y_i = 0$, then

$$\sum x_i \otimes y_i = \sum (x_i \otimes y_i - x_i y_i \otimes 1) = \sum x_i (1 \otimes y_i - y_i \otimes 1).$$

The derivation is $d : A \to A \otimes_B A$, $a \mapsto 1 \otimes a - a \otimes 1$ (taken modulo I^2). (We shouldn't really call this "d" until we have verified that it agrees with our earlier definition, but we irresponsibly will anyway.)

Let's check that d is indeed a derivation. Two of the three axioms (see §23.2.16) are immediate: d is linear, and vanishes on elements of b. So we check the Leibniz rule:

$$\begin{aligned} d(aa') - a\, da' - a'\, da &= 1 \otimes aa' - aa' \otimes 1 - a \otimes a' + aa' \otimes 1 - a' \otimes a + a'a \otimes 1 \\ &= -a \otimes a' - a' \otimes a + a'a \otimes 1 + 1 \otimes aa' \\ &= (1 \otimes a - a \otimes 1)(1 \otimes a' - a' \otimes 1) \\ &\in I^2. \end{aligned}$$

Thus by the universal property of $\Omega_{A/B}$, we get a natural morphism $\Omega_{A/B} \to I/I^2$ of A-modules.

23.2.23. Theorem. — *The natural morphism $f : \Omega_{A/B} \to I/I^2$ induced by the universal property of $\Omega_{A/B}$ is an isomorphism.*

Proof. We will show this as follows. (i) We will show that f is surjective, and (ii) we will describe $g : I/I^2 \to \Omega_{A/B}$ such that $g \circ f : \Omega_{A/B} \to \Omega_{A/B}$ is the identity (showing that f is injective).

(i) The map f sends da to $1 \otimes a - a \otimes 1$, and such elements generate I (§23.2.22), so f is surjective.

(ii) Define $g : I/I^2 \to \Omega_{A/B}$ by $x \otimes y \mapsto x\, dy$. We need to check that this is well-defined, i.e. that elements of I^2 are sent to 0, i.e. we need that

$$\left(\sum x_i \otimes y_i\right)\left(\sum x'_j \otimes y'_j\right) = \sum_{i,j} x_i x'_j \otimes y_i y'_j \mapsto 0$$

where $\sum_i x_i y_i = \sum x'_j y'_j = 0$. But by the Leibniz rule,

$$\begin{aligned} \sum_{i,j} x_i x'_j\, d(y_i y'_j) &= \sum_{i,j} x_i x'_j y_i\, dy'_j + \sum_{i,j} x_i x'_j y'_j\, dy_i \\ &= \left(\sum_i x_i y_i\right)\left(\sum_j x'_j\, dy'_j\right) + \left(\sum_i x_i\, dy_i\right)\left(\sum_j x'_j y'_j\right) \\ &= 0. \end{aligned}$$

Then $f \circ g$ is indeed the identity, as

$$da \xmapsto{g} 1 \otimes a - a \otimes 1 \xmapsto{f} 1\, da - a\, d1 = da$$

as desired. $\square$

We can now use our understanding of how Ω works on affine open sets to generalize previous statements to non-affine settings.

23.2.K. EXERCISE. If $U \subset X$ is an open subset, show that the map (23.2.19.1) is a derivation.

23.2.L. EXERCISE. Suppose $f : X \to Y$ is locally of finite type, and Y (and hence X) is locally Noetherian. Show that $\Omega_{X/Y}$ is a coherent sheaf on X. (Feel free to weaken the Noetherian hypotheses for weaker conclusions.)

The relative cotangent exact sequence and the conormal exact sequence for schemes now directly follow.

23.2.24. Theorem. — *(Relative cotangent exact sequence) Suppose* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms of schemes. Then there is an exact sequence of quasicoherent sheaves on* X

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

(Conormal exact sequence) Suppose $f : X \to Y$ *is a morphism of schemes, and* $Z \hookrightarrow X$ *is a closed subscheme of* X*, with ideal sheaf* $\mathcal{I}$*. Then there is an exact sequence of sheaves on* Z*:*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

Proof. Both can be checked affine locally, and the affine cases are Theorems 23.2.9 and 23.2.11 respectively. □

(As described in §23.2.12, we expect the conormal exact sequence to be exact on the left in appropriately "smooth" situations, and this is indeed the case, see Theorem 25.9.8.)

Similarly, the sheaf of relative differentials pull back, and behave well under base change.

23.2.25. Theorem (pullback of differentials). —
(a) If

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a commutative diagram of schemes, there is a natural homomorphism of quasicoherent sheaves on X' $g^*\Omega_{X/Y} \to \Omega_{X'/Y'}$*. An important special case is* $Y = Y'$*.*
(b) (Ω *behaves well under base change) If furthermore the above diagram is a tensor diagram (i.e.* $X' \cong X \otimes_Y Y'$*) then* $g^*\Omega_{X/Y} \to \Omega_{X'/Y'}$ *is an isomorphism.*

This follows immediately from Exercise 23.2.I.

As a particular case of part (b), the fiber of the sheaf of relative differentials is indeed the sheaf of differentials of the fiber. *Thus this notion indeed glues together the differentials on each fiber.*

23.3 Examples

23.3.1. Geometric genus. A nonsingular projective curve C (over a field k) has **geometric genus** $h^0(C, \Omega_{C/k})$. (This will be generalized to higher dimension in §23.4.3.) This is always finite, as $\Omega_{C/k}$ is coherent (Exercise 23.2.L), and coherent sheaves on projective k-schemes have finite-dimensional spaces of sections (Theorem 20.1.3(a)). (The geometric genus is also called the *first algebraic de Rham cohomology group*, in analogy with de Rham cohomology in the differentiable setting.)

Sadly, this isn't really a new invariant. We will see in Exercise 23.3.C that this agrees with our earlier definition of genus, i.e. $h^0(C, \Omega_{C/k}) = h^1(C, \mathcal{O}_C)$.

23.3.2. The projective line. As an important first example, consider $\mathbb{P}^1_k$, with the usual projective coordinates x_0 and x_1. As usual, the first patch corresponds to $x_0 \neq 0$, and is of the form $\operatorname{Spec} k[x_{1/0}]$ where $x_{1/0} = x_1/x_0$. The second patch corresponds to $x_1 \neq 0$, and is of the form $\operatorname{Spec} k[x_{0/1}]$ where $x_{0/1} = x_0/x_1$.

Both patches are isomorphic to $\mathbb{A}^1_k$, and $\Omega_{\mathbb{A}^1_k} = \mathcal{O}_{\mathbb{A}^1_k}$. (More precisely, $\Omega_{k[x]/k} = k[x]\, dx$.) Thus $\Omega_{\mathbb{P}^1_k}$ is an invertible sheaf (a line bundle). The invertible sheaves on $\mathbb{P}^1_k$ are of the form $\mathcal{O}(m)$. So which invertible sheaf is $\Omega_{\mathbb{P}^1/k}$?

Let's take a section, $dx_{1/0}$ on the first patch. It has no zeros or poles there, so let's check what happens on the other patch. As $x_{1/0} = 1/x_{0/1}$, we have $dx_{1/0} = -(1/x_{0/1}^2)\ dx_{0/1}$. Thus this section has a double pole where $x_{0/1} = 0$. Hence $\Omega_{\mathbb{P}^1_k/k} \cong \mathcal{O}(-2)$.

Note that the above argument works equally well if k were replaced by $\mathbb{Z}$: our theory of Weil divisors and line bundles of Chapter 15 applies ($\mathbb{P}^1_{\mathbb{Z}}$ is factorial), so the previous argument essentially without change shows that $\Omega_{\mathbb{P}^1_{\mathbb{Z}}/\mathbb{Z}} \cong \mathcal{O}(-2)$. And because Ω behaves well with respect to base change (Exercise 23.2.25(b)), and any scheme maps to $\operatorname{Spec} \mathbb{Z}$, this implies that $\Omega_{\mathbb{P}^1_B/B} \cong \mathcal{O}_{\mathbb{P}^1_B}(-2)$ for *any* base scheme B.

(Also, as promised in §20.4.6, this shows that $\Omega_{\mathbb{P}^1/k}$ is the dualizing sheaf for $\mathbb{P}^1_k$; see also §20.4.7. But given that we haven't yet proved Serre duality, this isn't so meaningful.)

23.3.3. Hyperelliptic curves.

Throughout this discussion of hyperelliptic curves, we suppose that $k = \overline{k}$ and $\operatorname{char} k \neq 2$, so we may apply the discussion of §21.4. Consider a double cover $f : C \to \mathbb{P}^1_k$ by a nonsingular curve C, branched over $2g+2$ distinct points. We will use the explicit coordinate description of hyperelliptic curves of (21.4.2.1). By Exercise 21.4.1, C has genus g.

23.3.A. EXERCISE: DIFFERENTIALS ON HYPERELLIPTIC CURVES. What is the degree of the invertible sheaf $\Omega_{C/k}$? (Hint: let x be a coordinate on one of the coordinate patches of $\mathbb{P}^1_k$. Consider $f^* dx$ on C, and count poles and zeros. Use the explicit coordinates of §21.4. You should find that $f^* dx$ has $2g+2$ zeros and 4 poles, for a total of $2g-2$.) Doing this exercise will set you up well for the Riemann-Hurwitz formula, §23.5.

23.3.B. EXERCISE ("THE FIRST ALGEBRAIC DE RHAM COHOMOLOGY GROUP OF A HYPERELLIPTIC CURVE"). Show that $h^0(C, \Omega_{C/k}) = g$ as follows.

(a) Show that $\frac{dx}{y}$ is a (regular) differential on $\operatorname{Spec} k[x]/(y - f(x))$ (i.e. an element of $\Omega_{(k[x]/(y-f(x)))/k}$).

(b) Suppose $x^i(dx)/y$ extends to a global differential ω_i on C (i.e. with no poles).
(c) Show that the ω_i ($0 \leq i < g$) are linearly independent differentials. (Hint: Show that the valuation of ω_i at the origin is i. If $\omega := \sum_{j=i}^{g-1} a_j \omega_j$ is a nontrivial linear combination, with $a_j \in k$, and $a_i \neq 0$, show that the valuation of ω at the origin is i, and hence $\omega \neq 0$.)
$\star$ (d) Show that the ω_i form a basis for the differentials.

23.3.C. $\star$ EXERCISE (TOWARD SERRE DUALITY).
(a) Show that $h^1(C, \Omega_{C/k}) = 1$. (In the course of doing this, you might interpret a generator of $H^1(C, \Omega_{C/k})$ as $x^{-1}\,dx$. In particular, the pullback map $H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1/k}) \to H^1(C, \Omega_{C/k})$ is an isomorphism.)
(b) Describe a natural perfect pairing

$$H^0(C, \Omega_{C/k}) \times H^1(C, \mathcal{O}_C) \to H^1(C, \Omega_{C/k}).$$

In terms of our explicit coordinates, you might interpret it as follows. Recall from the proof of the hyperelliptic Riemann-Hurwitz formula (Theorem 21.4.1) that $H^1(C, \mathcal{O}_C)$ can be interpreted as

$$\langle \frac{y}{x}, \frac{y}{x^2}, \dots, \frac{y}{x^g} \rangle.$$

Then the pairing

$$\langle \frac{dx}{y}, \dots, x^{g-1}\frac{dx}{y} \rangle \times \langle \frac{y}{x}, \dots, \frac{y}{x^g} \rangle \to \langle x^{-1}\,dx \rangle$$

is basically "multiply and read off the $x^{-1}\,dx$ term". Or in fancier informal terms: "multiply and take the residue".

23.3.4. Another random facts about curves (used in the proof of Riemann-Hurwitz, §23.5).

23.3.D. EXERCISE. Suppose A is a discrete valuation ring over the algebraically closed field k, with residue field k, and uniformizer t. Show that the differentials are free of rank one, generated by dt: $\Omega_{A/k} = A\ dt$. Hint: by Exercise 13.2.F, $\Omega_{\operatorname{Spec} A/k}$ is locally free of rank 1. By endowing any generator with valuation 0, endow each differential with a non-negative valuation ν. We wish to show that $\nu(dt) = 0$. Suppose $\nu(dt) > 0$. Show that there is some $u \in A$ with $\nu(du) = 0$. Then $u = u' + tu''$, where $u' \in k$ and $u'' \in A$, from which $du = t\ du'' + u''\ dt$. Obtain a contradiction from this.

23.3.5. Projective space and the Euler exact sequence.
We next examine the differentials of projective space $\mathbb{P}^n_k$, or more generally $\mathbb{P}^n_A$ where A is an arbitrary ring. As projective space is covered by affine open sets of the form $\mathbb{A}^n$, on which the differentials form a rank n locally free sheaf, $\Omega_{\mathbb{P}^n_A/A}$ is also a rank n locally free sheaf.

23.3.6. Theorem (the Euler exact sequence). — *The sheaf of differentials $\Omega_{\mathbb{P}^n_A/A}$ satisfies the following exact sequence*

$$0 \to \Omega_{\mathbb{P}^n_A/A} \to \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n_A} \to 0.$$

This is handy, because you can get a hold of $\Omega_{\mathbb{P}^n_A/A}$ in a concrete way. See Exercise 23.4.H for an application. By dualizing this exact sequence, we have (at least if A is Noetherian, by Exercise 14.7.B) an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n_A} \to \mathcal{O}_{\mathbb{P}^n_A}(1)^{\oplus(n+1)} \to \mathcal{T}_{\mathbb{P}^n_A/A} \to 0$.

$\star$ *Proof of Theorem 23.3.6.* (What's really going on in this proof is that we consider those differentials on $\mathbb{A}^{n+1}_A \setminus \{0\}$ that are pullbacks of differentials on $\mathbb{P}^n_A$.)

We first describe a map $\phi : \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}$, and later identify the kernel with $\Omega_{X/Y}$. The map is given by

$$\phi : (s_0, s_1, \dots, s_n) \mapsto x_0 s_0 + x_1 s_1 + \cdots + x_n s_n.$$

You should think of this as a "degree 1" map, as each x_i has degree 1.

23.3.E. EASY EXERCISE. Show that ϕ is surjective, by checking on the open set $D(x_i)$. (There is a one-line solution.)

Now we must identify the kernel of this map with differentials, and we can do this on each $D(x_i)$ (so long as we do it in a way that works simultaneously for each open set). So we consider the open set U_0, where $x_0 \neq 0$, and we have coordinates $x_{j/0} = x_j/x_0$ $(1 \leq j \leq n)$. Given a differential

$$f_1(x_{1/0}, \dots, x_{n/0})\ dx_{1/0} + \cdots + f_n(x_{1/0}, \dots, x_{n/0})\ dx_{n/0}$$

we must produce $n+1$ sections of $\mathcal{O}(-1)$. As motivation, let me just look at the first term, and pretend that the projective coordinates are actual coordinates.

$$\begin{aligned} f_1\ dx_{1/0} &= f_1\ d(x_1/x_0) \\ &= f_1 \frac{x_0\ dx_1 - x_1\ dx_0}{x_0^2} \\ &= -\frac{x_1}{x_0^2} f_1\ dx_0 + \frac{f_1}{x_0}\ dx_1 \end{aligned}$$

Note that x_0 times the "coefficient of dx_0" plus x_1 times the "coefficient of dx_1" is 0, and also both coefficients are of homogeneous degree -1. Motivated by this, we take:

$$f_1\ dx_{1/0} + \cdots + f_n\ dx_{n/0} \mapsto \left(-\frac{x_1}{x_0^2} f_1 - \cdots - \frac{x_n}{x_0^2} f_n, \frac{f_1}{x_0}, \frac{f_2}{x_0}, \cdots, \frac{f_n}{x_0} \right) \tag{23.3.6.1}$$

Note that over U_0, this indeed gives an injection of $\Omega_{\mathbb{P}^n_A}$ to $\mathcal{O}(-1)^{\oplus(n+1)}$ that surjects onto the kernel of $\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_X$ (if $(g_0, \dots, g_n)$ is in the kernel, take $f_i = x_0 g_i$ for $i > 0$).

Let's make sure this construction, applied to two different coordinate patches (say U_0 and U_1) gives the same answer. (This verification is best ignored on a first reading.) Note that

$$\begin{aligned} f_1\ dx_{1/0} + f_2\ dx_{2/0} + \cdots &= f_1\ d\frac{1}{x_{0/1}} + f_2\ d\frac{x_{2/1}}{x_{0/1}} + \cdots \\ &= -\frac{f_1}{x_{0/1}^2} dx_{0/1} + \frac{f_2}{x_{0/1}} dx_{2/1} - \frac{f_2 x_{2/1}}{x_{0/1}^2} dx_{0/1} + \cdots \\ &= -\frac{f_1 + f_2 x_{2/1} + \cdots}{x_{0/1}^2} dx_{0/1} + \frac{f_2 x_1}{x_0} dx_{2/1} + \cdots . \end{aligned}$$

Under this map, the $dx_{2/1}$ term goes to the second factor (where the factors are indexed 0 through n) in $\mathcal{O}(-1)^{\oplus(n+1)}$, and yields f_2/x_0 as desired (and similarly for $dx_{j/1}$ for $j > 2$). Also, the $dx_{0/1}$ term goes to the "zero" factor, and yields

$$\left(\sum_{j=1}^{n} f_i(x_i/x_1)/(x_0/x_1)^2\right)/x_1 = f_i x_i/x_0^2$$

as desired. Finally, the "first" factor must be correct because the sum over i of x_i times the ith factor is 0. $\square$

Generalizations of the Euler exact sequence are quite useful. We won't use them later, so no proofs will be given. Note that the argument applies without change if Spec A is replaced by an arbitrary base scheme. The Euler exact sequence further generalizes in a number of ways. As a first step, suppose $\mathcal{V}$ is a rank $n+1$ locally free sheaf (or vector bundle) on a scheme X. Then $\Omega_{\mathbb{P}\mathcal{V}/X}$ sits in an Euler exact sequence:

$$0 \to \Omega_{\mathbb{P}\mathcal{V}/X} \to \mathcal{O}(-1) \otimes \mathcal{V}^\vee \to \mathcal{O}_X \to 0$$

If $\pi : \mathbb{P}\mathcal{V} \to X$, the map $\mathcal{O}(-1) \otimes \mathcal{V}^\vee \to \mathcal{O}_X$ is induced by $\mathcal{V}^\vee \otimes \pi_*\mathcal{O}(1) \cong (\mathcal{V}^\vee \otimes \mathcal{V}) \otimes \mathcal{O}_X \to \mathcal{O}_X$, where $\mathcal{V}^\vee \otimes \mathcal{V} \to \mathcal{O}_X$ is the trace map (§14.7.1).

This may not look very useful, but we have already seen it in the case of $\mathbb{P}^1$-bundles over curves, in Exercise 22.2.J, where the normal bundle to a section was identified in this way.

23.3.7. ⋆⋆ *Generalization to the Grassmannian.* For another generalization, fix a base field k, and let $G(m, n+1)$ be the space of sub-vector spaces of dimension m in an $(n+1)$-dimensional vector space V (the Grassmannian, §17.6). Over $G(m, n+1)$ we have a short exact sequence of locally free sheaves

$$0 \to \mathcal{S} \to V \otimes \mathcal{O}_{G(m,n+1)} \to \mathcal{Q} \to 0$$

where $V \otimes \mathcal{O}_{G(m,n+1)}$ is the "trivial bundle whose fibers are V" (do you understand what that means?), and $\mathcal{S}$ is the "universal subbundle" (such that over a point $[V' \subset V]$ of the Grassmannian $G(m, n+1)$, $\mathcal{S}|_{[V' \subset V]}$ is V, if you can make that precise). Then

$$\Omega_{G(m,n+1)/k} \cong \mathcal{H}om(\mathcal{Q}, \mathcal{S}). \tag{23.3.7.1}$$

23.3.F. EXERCISE. Recall that in the case of projective space, i.e. $m = 1$, $\mathcal{S} = \mathcal{O}(-1)$ (Exercise 18.1.H). Verify (23.3.7.1) in this case using the Euler exact sequence (Theorem 23.3.6).

23.3.G. EXERCISE. Prove (23.3.7.1), and explain how it generalizes 22.2.I. (The hint to Exercise 22.2.I may help.)

This Grassmannian fact generalizes further to Grassmannian bundles.

23.4 Nonsingularity and k-smoothness revisited

In this section, we examine the relation between differentials and nonsingularity, and define smoothness over a field. We construct birational invariants of nonsingular varieties over algebraically closed fields (such as the geometric genus), motivate the notion of an unramified morphism, show that varieties are "mostly nonsingular", and get a first glimpse of Hodge theory.

23.4.1. Definition. Suppose k is a field. Since §13.2.4, we have used an awkward definition of k-smoothness, and we finally rectify this. A k-scheme X is **k-smooth of dimension n** or **smooth of dimension n over k** if it is locally of finite type, pure dimension n, and $\Omega_{X/k}$ is locally free of rank n. The dimension n is often omitted, but one might possibly want to call something smooth if it is the (scheme-theoretic) disjoint union of things smooth of various dimensions.

23.4.A. EXERCISE. Verify that this definition indeed is equivalent to the one given in §13.2.4.

As a consequence of our better definition, we see that smoothness can be checked on any affine cover by using the Jacobian criterion on each affine open set in the cover.

We recall that we have shown in §13.3.10 that if k is perfect (e.g. if $\operatorname{char} k = 0$), then a finite type k-scheme is smooth if and only if it is nonsingular at closed points; this was quite easy in the case when $k = \overline{k}$ (Exercise 13.2.F). Recall that it is also true that for *any* k, a smooth k-scheme is nonsingular at its closed points (mentioned but not proved in §13.2.5), but finite type k-schemes can be regular without being smooth (if k is not perfect, see the example in §13.2.5).

23.4.2. The geometric genus, and other birational invariants from i-forms $\Omega^i_{X/Y}$.

Suppose X is a projective scheme over k. Then for each i, $h^i(X, \Omega_{X/k})$ is an invariant of X, which can be useful. The first useful fact is that it, and related invariants, are *birational invariants* if X is smooth, as shown in the following exercise. We first define the **sheaf of (relative) i-forms** $\Omega^i_{X/Y} := \wedge^i \Omega_{X/Y}$. Sections of $\Omega^i_{X/Y}$ (over some open set) are called **(relative) i-forms** (over that open set).

23.4.B. EXERCISE ($h^0(X, \Omega^i_{X/k})$ ARE BIRATIONAL INVARIANTS). Suppose X and X' are birational projective smooth k-varieties. Show (for each i) that $H^0(X, \Omega^i_{X/k}) \cong H^0(X', \Omega^i_{X'/k})$. Hint: fix a birational map $\phi : X \dashrightarrow X'$. By Exercise 17.5.B, the complement of the domain of definition U of ϕ is codimension at least 2. By pulling back i-forms from X' to U, we get a map $\phi^* : H^0(X', \Omega^i_{X'/k}) \to H^0(U, \Omega^i_{X/k})$. Use Hartogs' theorem 12.3.10 and the fact that Ω^i is locally free to show the map extends to a map $\phi^* : H^0(X', \Omega^i_{X'/k}) \to H^0(X, \Omega^i_{X/k})$. If $\psi : X' \dashrightarrow X$ is the inverse rational map, we similarly get a map $\psi^* : H^0(X, \Omega^i_{X/k}) \to H^0(X', \Omega^i_{X'/k})$. Show that ϕ^* and ψ^* are inverse by showing that each composition is the identity on a dense open subset of X or X'.

23.4.3. *The geometric genus.* If X is a dimension n smooth projective (or even proper) k-variety, the birational invariant $h^0(X, \det \Omega_{X/k}) = h^0(X, \Omega^n_{X/k})$ has particular importance. It is called the **geometric genus**, and is often denoted $p_g(X)$. We saw this in the case of curves in §23.3.1. If X is an irreducible variety that is *not* smooth or projective, the phrase geometric genus refers to $h^0(X', \Omega^n_{X'/k})$ for some projective smooth X' *birational* to X. (By Exercise 23.4.B, this is independent of X'.) For example, if X is an irreducible reduced projective curve over k, the geometric genus is the geometric genus of the normalization of X. (But in higher dimension, it is not obvious if there exists such an X'. It is a nontrivial fact that this is true in characteristic 0 — Hironaka's resolution of singularities — and it is not yet known in positive characteristic in full generality; see Remark 19.4.6.)

It is a miracle that for a complex curve this is the same as the topological genus and the arithmetic genus. We will connect the geometric genus to the topological genus in our discussion of the Riemann-Hurwitz formula soon (Exercise 23.5.H). We will begin the connection of geometric genus to arithmetic genus via the continuing miracle of Serre duality very soon (Exercise 23.4.D).

23.4.C. UNIMPORTANT EXERCISE. The j**th plurigenus** of a smooth projective k-variety is $h^0(X, (\det \Omega_{X/k})^{\otimes j})$. Show that the jth plurigenus is a birational invariant. (We won't use this notion further.)

23.4.4. *Further Serre duality miracle: Ω^n is dualizing (for smooth k-varieties).* It is a further miracle of Serre duality that for an n-dimensional smooth k-variety X, the sheaf of "algebraic volume forms" is (isomorphic) to the dualizing sheaf $\mathcal{K}_{X/k}$:

$$\det \Omega_{X/k} = \Omega^n_{X/k} \cong \mathcal{K}_X. \tag{23.4.4.1}$$

We will prove this in §27.6.

23.4.D. EASY EXERCISE. Assuming Serre duality, and the miracle (23.4.4.1), show that the geometric genus of a smooth projective curve over $k = \overline{k}$ equals its arithmetic genus.

23.4.5. Unramified morphisms.

Suppose $\pi : X \to Y$ is a morphism of schemes. The support of the quasicoherent sheaf $\Omega_\pi = \Omega_{X/Y}$ is called the **ramification locus**, and the image of its support, $\pi_* \operatorname{Supp} \Omega_{X/Y}$, is called the **branch locus**. If $\Omega_\pi = 0$, we say that π is **formally unramified**, and if π is also furthermore of finite presentation, we say π is **unramified**. (Noetherian readers will happily ignore the difference.) We will discuss unramifiedness at length in Chapter 26.

23.4.E. EASY EXERCISE. (a) Show that locally finitely presented locally closed immersions are unramified.
(b) Show that the condition of $\pi : X \to Y$ being unramified is local on X and on Y.
(c) *(localization is unramified)* Show that if S is a multiplicative subset of the ring B, then $\operatorname{Spec} S^{-1}B \to \operatorname{Spec} B$ is formally unramified. (Thus for example by (b), if η is the generic point of an integral scheme Y, $\operatorname{Spec} \mathcal{O}_{Y,\eta} \to Y$ is formally unramified.)
(d) Show that finite separable field extensions (or more correctly, the corresponding map of schemes) are unramified.
(e) Show that the property of being unramified is preserved under composition and base change.

23.4.F. EXERCISE. Suppose $\pi : X \to Y$ is a morphism of varieties over k. Use the conormal exact sequence (Theorem 23.2.11) and Proposition 23.2.17 relating Ω to the Zariski tangent space to show the following.

(a) Suppose that $\dim X = \dim Y = n$, and π is unramified. Show that if Y is k-smooth, then X is k-smooth.
(b) Suppose $\dim X = m > \dim Y = n$, Y is k-smooth, and the fibers of π over closed points are smooth of dimension $m - n$. Show that X is k-smooth.

23.4.6. *Arithmetic side remark: the different and discriminant.* If B is the ring of integers in a number field (§10.6.1), the **different ideal** of B is the annihilator of $\Omega_{B/\mathbb{Z}}$. It measures the failure of $\operatorname{Spec} B \to \operatorname{Spec} \mathbb{Z}$ to be unramified, and is a scheme-theoretic version of the ramification locus. The **discriminant ideal** can be interpreted as the ideal of $\mathbb{Z}$ corresponding to effective divisor on $\operatorname{Spec} \mathbb{Z}$ that is the "push forward" (not defined here, but defined as you might expect) of the divisor corresponding to the different. It is a scheme-theoretic version of the branch locus. If B/A is an extension of rings of integers of number fields, the **relative different ideal** (of B) and **relative discriminant ideal** (of A) are defined similarly. (We won't use these ideas.)

23.4.7. Generic smoothness.

We can now verify something your intuition may already have told you. In positive characteristic, this is a hard theorem, in that it uses a result from commutative algebra that we have not proved.

23.4.8. Theorem (generic smoothness of varieties). — *If X is an integral variety over $k = \overline{k}$, there is an dense open subset U of X such that U is smooth.*

Hence, by Fact 13.3.8, U is nonsingular. Theorem 26.4.1 will generalize this to smooth *morphisms*, at the expense of restricting to characteristic 0.

Proof. The $n = 0$ case is immediate, so we assume $n > 0$.

We will show that the rank at the generic point is n. Then by upper semicontinuity of the rank of a coherent sheaf (Exercise 14.7.I), it must be n in an open neighborhood of the generic point, and we are done.

We thus have to check that if K is the fraction field of a dimension n integral finite-type k-scheme, i.e. (by Theorem 12.2.1) if K/k is a transcendence degree n extension, then $\Omega_{K/k}$ is an n-dimensional vector space. But every extension of transcendence degree $n > 1$ is separably generated: we can find n algebraically independent elements of K over k, say $x_1, \dots, x_n$, such that $K/k(x_1, \dots, x_n)$ is separable. (In characteristic 0, this is automatic from transcendence theory, see Exercise 12.2.A, as all finite extensions are separable. But it even holds in positive characteristic, see [**M-CA**, p. 194 Cor.].) Then $\Omega_{k[x_1,\dots,x_n]/k}$ is generated by dx_1, $\dots$, dx_n (by Exercise 23.2.F(d)). $\square$

23.4.9. $\star$ **Aside: Infinitesimal deformations and automorphisms.**

It is beyond the scope of these notes to make this precise, but if X is a variety, $H^0(X, \mathcal{T}_X)$ parametrizes infinitesimal automorphisms of X, and $H^1(X, \mathcal{T}_X)$ parametrizes infinitesimal deformations. As an example if $X = \mathbb{P}^1$ (over a field), $\mathcal{T}_{\mathbb{P}^1} \cong \mathcal{O}(2)$ (§23.3.2), so $h^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}) = 3$, which is precisely the dimension of the automorphism group of $\mathbb{P}^1$ (Exercise 17.4.B).

23.4.G. EXERCISE. Compute $h^0(\mathbb{P}^n_k, \mathcal{T}_{\mathbb{P}^n_k})$ using the Euler exact sequence (Theorem 23.3.6). Compare this to the dimension of the automorphism group of $\mathbb{P}^n_k$ (Exercise 17.4.B).

23.4.H. EXERCISE. Show that $H^1(\mathbb{P}^n_A, \mathcal{T}_{\mathbb{P}^n_A}) = 0$. Thus projective space can't deform, and is "rigid".)

23.4.I. EXERCISE. Assuming Serre duality, and the miracle (23.4.4.1), compute $h^i(C, \mathcal{T})$ for a genus g projective nonsingular geometrically irreducible curve over k, for $i = 0$ and 1. You should notice that $h^1(C, \mathcal{T})$ for genus 0, 1, and $g > 1$ is 0, 1, and $3g - 3$ respectively; after doing this, re-read §21.7.1.

23.4.10. $\star$ **A first glimpse of Hodge theory.**

The invariant $h^j(X, \Omega^i_{X/k})$ is called the **Hodge number** $h^{i,j}(X)$. By Exercise 23.4.B, $h^{i,0}$ are birational invariants. We will soon see (in Exercise 23.4.M) that this isn't true for all $h^{i,j}$.

23.4.J. EXERCISE. Suppose X is a nonsingular projective variety over $k = \overline{k}$. Assuming Serre duality, and the miracle (23.4.4.1), show that Hodge numbers satisfy the symmetry $h^{p,q} = h^{n-p,n-q}$.

23.4.K. EXERCISE (THE HODGE NUMBERS OF PROJECTIVE SPACE). Show that $h^{p,q}(\mathbb{P}^n_k) = 1$ if $0 \leq p = q \leq n$ and $h^{p,q}(\mathbb{P}^n_k) = 0$ otherwise. Hint: use the Euler exact sequence (Theorem 23.3.6) and apply Exercise 14.5.F.

23.4.11. *Remark: the Hodge diamond.* Over $k = \mathbb{C}$, further miracles occur. If X is an irreducible nonsingular projective complex variety, then it turns out that there is a direct sum decomposition

$$H^m(X, \mathbb{C}) = \oplus_{i+j=m} H^j(X, \Omega^i_{X/\mathbb{C}}), \tag{23.4.11.1}$$

from which $h^m(X, \mathbb{C}) = \sum_{i+j=m} h^{i,j}$, so the Hodge numbers (purely algebraic objects) yield the Betti numbers (a priori topological information). Moreover, complex conjugation interchanges $H^j(X, \Omega^i_{X/\mathbb{C}})$ with $H^i(X, \Omega^i_{X/\mathbb{C}})$, from which

$$h^{i,j} = h^{j,i}. \tag{23.4.11.2}$$

This additional symmetry holds in characteristic 0 in general, but can fail in positive characteristic. This is the beginning of the vast and fruitful subject of Hodge theory.

If we write the Hodge numbers in a diamond, with $h^{i,j}$ the ith entry in the $(i+j)$th row, then the diamond has the two symmetries coming from Serre duality and complex conjugation. For example, the Hodge diamond of an irreducible nonsingular projective complex surface will be of the following form:

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

where p_g is the geometric genus of the surface, and $q = h^{0,1} = h^{1,0} = h^{2,1} = h^{1,2}$ is called the **irregularity** of the surface. As another example, by Exercise 23.4.K, the Hodge diamond of $\mathbb{P}^n$ is all 0 except for 1's down the vertical axis of symmetry.

You won't need the unproved statements (23.4.11.1) or (23.4.11.2) to solve the following problems.

23.4.L. EXERCISE. Assuming the Serre duality miracle 23.4.4.1, show that the Hodge diamond of a projective nonsingular geometrically irreducible genus g curve over a field k is the following.

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

23.4.M. EXERCISE. Show that the Hodge diamond of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is the following.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

By comparing your answer to the Hodge diamond of $\mathbb{P}^2_k$ (Exercise 23.4.K), show that $h^{1,1}$ is not a birational invariant.

Notice that in both cases, $h^{1,1}$ is the Picard number ρ (defined in §20.4.11). In general, $\rho \leq h^{1,1}$.

23.5 The Riemann-Hurwitz Formula

The Riemann-Hurwitz formula generalizes our calculation of the genus g of a double cover of $\mathbb{P}^1$ branched at $2g+2$ points, Theorem 21.4.1, to higher degree covers, and to higher genus target curves.

23.5.1. *Definition.* A finite morphism between integral schemes $f : X \to Y$ is said to be **separable** if it is dominant, and the induced extension of function fields $K(X)/K(Y)$ is a separable extension. (Similarly, a generically finite morphism is **generically separable** if it is dominant, and the induced extension of function fields is a separable extension. We won't use this notion.) Note that finite morphisms of integrable schemes are automatically separable in characteristic 0.

23.5.2. Proposition. — *If $f : X \to Y$ is a finite separable morphism of nonsingular integral varieties, then the relative cotangent sequence (Theorem 23.2.24) is exact on the left as well:*

$$0 \longrightarrow f^*\Omega_{Y/k} \xrightarrow{\phi} \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0. \tag{23.5.2.1}$$

Proof. We must check that ϕ is injective. Now $\Omega_{Y/k}$ is an invertible sheaf on Y, so $f^*\Omega_{Y/k}$ is an invertible sheaf on X. We come to a clever point: an invertible sheaf on an integral scheme (such as $f^*\Omega_{Y/k}$) is torsion-free (any section over any open set is non-zero at the generic point), so if a subsheaf of it (such as $\ker \phi$) is nonzero, it is nonzero at the generic point. Thus to show the injectivity of ϕ, we need only check that ϕ is an inclusion at the generic point. We thus tensor with $\mathcal{O}_\eta$ where η is the generic point of X. This is an exact functor (localization is exact, Exercise 2.6.F), and $\mathcal{O}_\eta \otimes \Omega_{X/Y} = 0$ (as $K(X)/K(Y)$ is a separable extension by hypothesis, and Ω for separable field extensions is 0 by Exercise 23.2.F(a)). Also, $\mathcal{O}_\eta \otimes f^*\Omega_{Y/k}$ and $\mathcal{O}_\eta \otimes \Omega_{X/k}$ are both one-dimensional $\mathcal{O}_\eta$-vector spaces (they are the stalks of invertible sheaves at the generic point). Thus by considering

$$\mathcal{O}_\eta \otimes f^*\Omega_{Y/k} \to \mathcal{O}_\eta \otimes \Omega_{X/k} \to \mathcal{O}_\eta \otimes \Omega_{X/Y} \to 0$$

(which is $\mathcal{O}_\eta \to \mathcal{O}_\eta \to 0 \to 0$) we see that $\mathcal{O}_\eta \otimes f^*\Omega_{Y/k} \to \mathcal{O}_\eta \otimes \Omega_{X/k}$ is injective, and thus that $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is injective. $\square$

People not confined to characteristic 0 should note what goes wrong for non-separable morphisms. For example, suppose k is a field of characteristic p, and consider the map $f : \mathbb{A}^1_k = \operatorname{Spec} k[t] \to \mathbb{A}^1_k = \operatorname{Spec} k[u]$ given by $u = t^p$. Then Ω_f is the trivial invertible sheaf generated by dt. As another (similar but different) example, if $K = k(x)$ and $K' = K(x^p)$, then the inclusion $K' \hookrightarrow K$ induces $f : \operatorname{Spec} K[t] \to \operatorname{Spec} K'[t]$. Once again, Ω_f is an invertible sheaf, generated by dx (which in this case is pulled back from $\Omega_{K/K'}$ on $\operatorname{Spec} K$). In both of these cases, we have maps from one affine line to another, and there are vertical tangent vectors.

23.5.A. EXERCISE. If X and Y are dimension n, and $f : X \to Y$ is separable, show that the ramification locus is pure codimension 1, and has a natural interpretation as an effective divisor, as follows. Interpret ϕ as an $n \times n$ Jacobian matrix (13.1.4.1) in appropriate local coordinates, and hence interpret the locus where ϕ is not an isomorphism as (locally) the vanishing scheme of the determinant of an $n \times n$ matrix. Hence the branch locus is also pure codimension 1. (This is a special case of Zariski's theorem on *purity of (dimension of) the branch locus*.) Hence we use the terms **ramification divisor** and **branch divisor**.

Suppose now that X and Y are dimension 1. (We will discuss higher-dimensional consequences in §23.5.7.) Then the ramification locus is a finite set (ramification *points*) of X, and the branch locus is a finite set (branch *points*) of Y. Now assume that $k = \overline{k}$. We examine $\Omega_{X/Y}$ near a point $x \in X$.

As motivation for what we will see, we note that in complex geometry, nonconstant maps from (complex) curves to curves may be written in appropriate local coordinates as $x \mapsto x^m = y$, from which we see that dy pulls back to $mx^{m-1}\,dx$, so $\Omega_{X/Y}$ locally looks like functions times dx modulo multiples of $mx^{m-1}\,dx$.

Consider now our map $\pi : X \to Y$, and fix $x \in X$, and $y = \pi(x)$. Because the construction of Ω behaves well under base change (Theorem 23.2.25(b)), we may replace Y with Spec of the local ring $\mathcal{O}_{Y,y}$ at y, i.e. we may assume $Y = \operatorname{Spec} B$, where B is a discrete valuation ring (as Y is a nonsingular curve), with residue field k corresponding to y. Then as π is finite, X is affine too. Similarly, as the construction of Ω behaves well with respect to localization (Exercise 23.2.8), we may replace X by $\operatorname{Spec} \mathcal{O}_{X,x}$, and thus assume $X = \operatorname{Spec} A$, where A is a discrete valuation ring, and π corresponds to $B \to A$, inducing an isomorphism of residue fields (with k).

Suppose their uniformizers are s and t respectively, with $t \mapsto us^n$ where u is a unit of A. Recall that the differentials of a discrete valuation ring over k are generated by the d of the uniformizer (Exercise 23.3.D). Then

$$dt = d(us^n) = uns^{n-1}\, ds + s^n\, du.$$

This differential on $\operatorname{Spec} A$ vanishes to order at least $n-1$, and precisely $n-1$ if n doesn't divide the characteristic. The former case is called **tame** ramification, and the latter is called **wild** ramification. We call this order the **ramification order** at this point of X.

23.5.B. EXERCISE. Show that the degree of $\Omega_{X/Y}$ at x is precisely the ramification order of π at x.

23.5.C. EXERCISE: INTERPRETING THE RAMIFICATION DIVISOR IN TERMS OF NUMBER OF PREIMAGES. Suppose all the ramification above $y \in Y$ is tame (which is always true in characteristic 0). Show that the degree of the branch divisor at y is $\deg \pi - |\pi^{-1}(y)|$. Thus the multiplicity of the branch divisor counts the extent to which the number of preimages is less than the degree (see Figure 23.4).

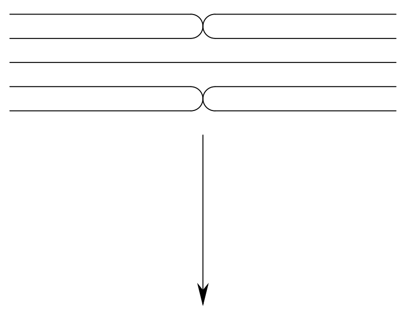


FIGURE 23.4. An example where the branch divisor appears with multiplicity 2 (see Exercise 23.5.C)

23.5.3. Theorem (the Riemann-Hurwitz formula). — *Suppose $\pi : X \to Y$ is a finite separable morphism of projective nonsingular curves. Let $n = \deg f$, and let R be the ramification divisor. Then*

$$2g(X) - 2 = n(2g(Y) - 2) + \deg R.$$

23.5.D. EXERCISE. Prove the Riemann-Hurwitz formula. Hint: Apply the fact that degree is additive in exact sequences (Exercise 20.4.K) to (23.5.2.1). Recall that degrees of line bundles pull back well under finite morphisms of integral projective curves, Exercise 20.4.F. Note that a torsion sheaf on a curve (such as Ω_π) is supported in dimension 0, so $\chi(\Omega_\pi) = h^0(\Omega_\pi)$. Show that the degree of R as a divisor is the same as its degree in the sense of h^0.

Here are some applications of the Riemann-Hurwitz formula.

23.5.4. *Example.* The degree of R is always even: any cover of a curve must be branched over an even number of points (counted with appropriate multiplicity).

23.5.E. EASY EXERCISE. Show that there is no nonconstant map from a smooth projective irreducible genus 2 curve to a smooth projective irreducible genus 3 curve. (Hint: $\deg R \geq 0$.)

23.5.5. *Example.* If $k = \overline{k}$, the only connected unbranched finite separable cover of $\mathbb{P}^1_k$ is the isomorphism, for the following reason. Suppose X is connected and $X \to \mathbb{P}^1_k$ is unramified. Then X is a curve, and nonsingular by Exercise 23.4.F(a). Applying the Riemann-Hurwitz theorem, using that the ramification divisor is 0, we have $2 - 2g_C = 2d$ with $d \geq 1$ and $g_c \geq 0$, from which $d = 1$ and $g_C = 0$.

23.5.F. EXERCISE. Show that if $k = \overline{k}$ has characteristic 0, the only connected unbranched cover of $\mathbb{A}^1_k$ is itself. (Aside: in characteristic p, this needn't hold; $\operatorname{Spec} k[x, y]/(y^p - x^p - y) \to \operatorname{Spec} k[x]$ is such a map. You can show this yourself, using Eisenstein's criterion to show irreducibility of the source. Once the theory of the algebraic fundamental group is developed, this translates to: "$\mathbb{A}^1$ is not simply connected in characteristic p." This cover is an example of an *Artin-Schreier cover*. Fun fact: the group $\mathbb{Z}/p$ acts on this cover via the map $y \mapsto y + 1$.)

23.5.G. UNIMPORTANT EXERCISE. Extend Example 23.5.5 and Exercise 23.5.F, by removing the $k = \overline{k}$ hypothesis, and changing "connected" to "geometrically connected".

23.5.6. *Example: Lüroth's theorem.* Continuing the notation of Theorem 23.5.3, suppose $g(X) = 0$. Then from the Riemann-Hurwitz formula (23.5.2.1), $g(Y) = 0$. (Otherwise, if $g(Y)$ were at least 1, then the right side of the Riemann-Hurwitz formula would be non-negative, and thus couldn't be -2, which is the left side. This has a nonobvious algebraic consequence, by our identification of covers of curves with field extensions (Theorem 18.4.3): all subfields of $k(x)$ containing k are of the form $k(y)$ where $y = f(x)$. (It turns out that the hypotheses $\operatorname{char} k = 0$ and $k = \overline{k}$ are not necessary.)

23.5.H. $\star$ EXERCISE (GEOMETRIC GENUS EQUALS TOPOLOGICAL GENUS). This exercise is intended for those with some complex background, who know that the Riemann-Hurwitz formula holds in the complex analytic category. Suppose C is an irreducible nonsingular projective complex curve. Show that there is an algebraic nonconstant map $\pi : C \to \mathbb{P}^1_{\mathbb{C}}$. Describe the corresponding map of Riemann surfaces. Use the previous exercise to show that the algebraic notion of genus (as computed using the branched cover π) agrees with the topological notion of genus (using the same branched cover). (Recall that assuming the Serre duality miracle 23.4.4.1, we know that the geometric genus equals the arithmetic genus, Exercise 23.4.D.)

23.5.I. UNIMPORTANT EXERCISE (CF. §23.5.7, ESPECIALLY EXERCISE 23.5.L. Suppose $\pi : X \to Y$ is a dominant morphism of nonsingular curves, and R is the ramification divisor of π. Show that $\Omega_X(-R) \cong \pi^* \Omega_Y$. (This exercise is geometrically pleasant, but we won't use it.)

23.5.7. Higher-dimensional applications of Exercise 23.5.A.

We now obtain some higher-dimensional consequences of the explicit Exercise 23.5.A. We begin with something (literally) small but fun. Suppose $\pi : X \to Y$ is a surjective k-morphism from a smooth k-scheme that contracts a subset of codimension greater than 1. More precisely, suppose π is an isomorphism over an open subset of Y, from an open subset U of X whose complement has codimension greater than 1. Then by Exercise 23.5.A, Y *cannot* be smooth. (*Small resolutions*, defined in Exercise 19.4.N, are examples of such π. In particular, you can find an example there.)

23.5.8. *Change of the canonical line bundle under blow-ups.*

As motivation, consider $\pi : \mathrm{Bl}_{(0,0)} \mathbb{A}^2 \to \mathbb{A}^2$ (defined in Exercise 10.2.M — you needn't have read Chapter 19 on blowing up to understand this). Let $X = \mathrm{Bl}_{(0,0)} \mathbb{A}^2$ and $Y = \mathbb{A}^2$ for convenience. We use Exercise 23.5.A to relate $\pi^* \mathcal{K}_Y$ with $\mathcal{K}_X$.

We pick a generator for $\mathcal{K}_Y$ near $(0,0)$: $dx \wedge dy$. (This is in fact a generator for $\mathcal{K}_Y$ everywhere on $\mathbb{A}^2$, but for the sake of generalization, we point out that all that matters is that is a generator at $(0,0)$, and hence *near* $(0,0)$ by geometric Nakayama, Exercise 14.7.D.) When we pull it back to X, we can interpret it as a section of $\mathcal{K}_X$, which will generate $\mathcal{K}_X$ away from the exceptional divisor E, but may contain E with some multiplicity μ. Recall that X can be interpreted as the data of a point in $\mathbb{A}^2$ as well as the choice of a line through the origin. We consider the open subset U where the line is not vertical, and thus can be written as $y = mx$. Here we have natural coordinates: $U = \operatorname{Spec} k[x, y, m]/(y - mx)$, which we can interpret as $\operatorname{Spec} k[x, m]$. The exceptional divisor E meets U, at $x = 0$ (in the coordinates on U), so we can calculate μ on this open set. Pulling back $dx \wedge dy$ to U, we get

$$dx \wedge dy = dx \wedge d(xm) = m(dx \wedge dx) + x(dx \wedge dm) = x(dx \wedge dm)$$

as $dx \wedge dx = 0$. Thus $\pi^* dx \wedge dy$ vanishes to order 1 along e.

23.5.J. EXERCISE. Explain how this determines an isomorphism $\mathcal{K}_X \cong (\pi^* \mathcal{K}_Y)(E)$.

23.5.K. EXERCISE. Repeat the above calculation in dimension n. Show that the exceptional divisor appears with multiplicity $(n-1)$.

23.5.L. ⋆ EXERCISE (FOR THOSE WHO HAVE READ CHAPTER 19 ON BLOWING UP).
(a) Suppose X is a surface over k, and p is a smooth k-valued point, and let $\pi : Y \to X$ be the blow-up of X at p. Show that $\mathcal{K}_X \cong (\pi^* \mathcal{K}_Y)(E)$. Hint: to find a generator of $\mathcal{K}_X$ near p, choose generators $\overline{x}$ and $\overline{y}$ of $\mathfrak{m}/\mathfrak{m}^2$ (where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$), and lift them to elements of $\mathcal{O}_{X,p}$. Why does $dx \wedge dy$ generate $\mathcal{K}_X$ at p?
(b) Repeat part (a) in arbitrary dimension (following Exercise 23.5.K).

23.5.M. ⋆ EXERCISE (FOR THOSE WHO HAVE READ CHAPTER 19). We work over an algebraically closed field k. Suppose Z is a smooth m-dimensional (closed) subvariety of a smooth n-dimensional variety X, and let $\pi : Y \to X$ be the blow-up of X along Z. Show that $\mathcal{K}_Y \cong (\pi^* \mathcal{K}_X)((n-m-1)E)$. (You will need Theorem 13.3.5, which shows that $Z \hookrightarrow X$ is a local complete intersection. This is where $k = \overline{k}$ is needed. As noted in Remark 13.3.6, we can remove this assumption, at the cost of invoking unproved Fact 13.3.1 that regular local rings are integral domains.)

Part VI

More

CHAPTER 24

Derived functors

In this chapter, we discuss derived functors, introduced by Grothendieck in his celebrated "Tôhoku article" **[Gr]**, and their applications to sheaves. For quasicoherent sheaves on quasicompact separated schemes, derived functor cohomology will agree with Čech cohomology (§24.5). Čech cohomology will suffice for most of our purposes, and is quite down to earth and computable, but derived functor cohomology is worth seeing. First, it will apply much more generally in algebraic geometry (e.g. étale cohomology) and elsewhere, although this is beyond the scope of these notes. Second, it will easily provide us with some useful notions, such as the Ext functors and the Leray spectral sequence. But derived functors can be intimidating the first time you see them, so feel free to just skim the main results, and to return to them later. I was tempted to make this chapter a "starred" optional section, but if I did, I would be ostracized from the algebraic geometry community.

24.1 The Tor functors

We begin with a warm-up: the case of Tor. This is a hands-on example, but if you understand it well, you will understand derived functors in general. Tor will be useful to prove facts about flatness, which we will discuss in §25.3. Tor is short for "torsion" (see Remark 25.3.1).

If you have never seen this notion before, you may want to just remember its properties. But I will to prove everything anyway — it is surprisingly easy.

The idea behind Tor is as follows. Whenever we see a right-exact functor, we always hope that it is the end of a long-exact sequence. Informally, given a short exact sequence

(24.1.0.1) $$0 \to N' \to N \to N'' \to 0,$$

we hope $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$ will extend to a long exact sequence
(24.1.0.2)

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Tor}_i^A(M,N') \arrow[r] & \mathrm{Tor}_i^A(M,N) \arrow[r] & \mathrm{Tor}_i^A(M,N'') \arrow[r] & \cdots \\
\arrow[r] & \mathrm{Tor}_1^A(M,N') \arrow[r] & \mathrm{Tor}_1^A(M,N) \arrow[r] & \mathrm{Tor}_1^A(M,N'') & \\
\arrow[r] & M \otimes_A N' \arrow[r] & M \otimes_A N \arrow[r] & M \otimes_A N'' \arrow[r] & 0.
\end{tikzcd}$$

More precisely, we are hoping for *covariant functors* $\mathrm{Tor}_i^A(\cdot, N)$ from A-modules to A-modules (covariance giving 2/3 of the morphisms in (24.1.0.2)), with $\mathrm{Tor}_0^A(M,N) \equiv M \otimes_A N$, and natural "connecting" homomorphism $\delta : \mathrm{Tor}_{i+1}^A(M,N'') \to \mathrm{Tor}_i^A(M,N')$ for every short exact sequence (24.1.0.1) giving the long exact sequence (24.1.0.2). ("Natural" means: given a morphism of short exact sequences, the natural square you would write down involving the δ-morphism must commute.)

It turns out to be not too hard to make this work, and this will also motivate derived functors. Let's now define $\mathrm{Tor}_i^A(M,N)$.

Take any resolution $\mathcal{R}$ of N by free modules:

$$\cdots \longrightarrow A^{\oplus n_2} \longrightarrow A^{\oplus n_1} \longrightarrow A^{\oplus n_0} \longrightarrow N \longrightarrow 0.$$

More precisely, build this resolution from right to left. Start by choosing generators of N as an A-module, giving us $A^{\oplus n_0} \to N \to 0$. Then choose generators of the kernel, and so on. Note that we are not requiring the n_i to be finite (although we could, if N is a finitely-generated module and A is Noetherian). Truncate the resolution, by stripping off the last term N (replacing $\to N \to 0$ with $\to 0$). Then tensor with M (which does not preserve exactness). Note that $M \otimes (\mathbb{A}^{\oplus n_i}) = M^{\otimes n}$, as tensoring with M commutes with arbitrary direct sums — you can check this by hand. Let $\mathrm{Tor}_i^A(M,N)_{\mathcal{R}}$ be the homology of this complex at the ith stage ($i \geq 0$). The subscript $\mathcal{R}$ reminds us that our construction depends on the resolution, although we will soon see that it is independent of $\mathcal{R}$.

We make some quick observations.

• $\mathrm{Tor}_0^A(M,N)_{\mathcal{R}} \cong M \otimes_A N$, canonically. Reason: as tensoring is right exact, and $A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0$ is exact, we have that $M^{\oplus n_1} \to M^{\oplus n_0} \to M \otimes_A N \to 0$ is exact, and hence that the homology of the truncated complex $M^{\oplus n_1} \to M^{\oplus n_0} \to 0$ is $M \otimes_A N$.

• If $M \otimes \cdot$ is exact (i.e. M is *flat*, §2.6.10), then $\mathrm{Tor}_i^A(M,N)_{\mathcal{R}} = 0$ for all $i > 0$. (This characterizes flatness, see Exercise 24.1.D.)

Now given two modules N and N' and resolutions $\mathcal{R}$ and $\mathcal{R}'$ of N and N', we can "lift" any morphism $N \to N'$ to a morphism of the two resolutions:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{\oplus n_i} \arrow[r] \arrow[d] & \cdots \arrow[r] & A^{\oplus n_1} \arrow[r] \arrow[d] & A^{\oplus n_0} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A^{\oplus n'_i} \arrow[r] & \cdots \arrow[r] & A^{\oplus n'_1} \arrow[r] & A^{\oplus n'_0} \arrow[r] & N' \arrow[r] & 0
\end{tikzcd}$$

Here we use the freeness of $A^{\oplus n_i}$: if $a_1, \ldots, a_{n_i}$ are generators of $A^{\oplus n_i}$, to lift the map $b : A^{\oplus n_i} \to A^{\oplus n'_{i-1}}$ to $c : A^{\oplus n_i} \to A^{\oplus n'_i}$, we arbitrarily lift $b(a_i)$ from $A^{\oplus n'_{i-1}}$ to $A^{\oplus n'_i}$, and declare this to be $c(a_i)$.

Denote the choice of lifts by $\mathcal{R} \to \mathcal{R}'$. Now truncate both complexes (remove column $N \to N'$) and tensor with M. Maps of complexes induce maps of homology (Exercise 2.6.D), so we have described maps (a priori depending on $\mathcal{R} \to \mathcal{R}'$)

$$\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \to \mathrm{Tor}_i^A(M, N')_{\mathcal{R}'}.$$

We say two maps of complexes $f, g : C_\bullet \to C'_\bullet$ are **homotopic** if there is a sequence of maps $w : C_i \to C'_{i+1}$ such that $f - g = dw + wd$.

24.1.A. EXERCISE. Show that two homotopic maps give the same map on homology.

24.1.B. CRUCIAL EXERCISE. Show that any two lifts $\mathcal{R} \to \mathcal{R}'$ are homotopic.

We now pull these observations together.

(1) We get a covariant functor $\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \to \mathrm{Tor}_i^A(M, N')_{\mathcal{R}'}$, independent of the lift $\mathcal{R} \to \mathcal{R}'$.
(2) Hence for any two resolutions $\mathcal{R}$ and $\mathcal{R}'$ of an A-module N, we get a canonical isomorphism $\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \cong \mathrm{Tor}_i^A(M, N)_{\mathcal{R}'}$. Here's why. Choose lifts $\mathcal{R} \to \mathcal{R}'$ and $\mathcal{R}' \to \mathcal{R}$. The composition $\mathcal{R} \to \mathcal{R}' \to \mathcal{R}$ is homotopic to the identity (as it is a lift of the identity map $N \to N$). Thus if $f_{\mathcal{R} \to \mathcal{R}'} : \mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \to \mathrm{Tor}_i^1(M, N)_{\mathcal{R}'}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, and similarly $f_{\mathcal{R}' \to \mathcal{R}}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, then $f_{\mathcal{R}' \to \mathcal{R}} \circ f_{\mathcal{R} \to \mathcal{R}'}$ is the identity, and similarly $f_{\mathcal{R} \to \mathcal{R}'} \circ f_{\mathcal{R}' \to \mathcal{R}}$ is the identity.
(3) Hence the covariant functor Tor_i^A doesn't depend on the choice of resolution.

24.1.1. *Remark.* Note that if N is a free module, then $\mathrm{Tor}_i^A(M, N) = 0$ for all M and all $i > 0$, as N has the trivial resolution $0 \to N \to N \to 0$ (it is "its own resolution").

Finally, we get long exact sequences:

24.1.2. Proposition. — *For any short exact sequence* (24.1.0.1) *we get a long exact sequence of* Tor*'s* (24.1.0.2).

Proof. Given a short exact sequence (24.1.0.1), choose resolutions of N' and N''. Then use these to get a resolution for N as follows (see (24.1.2.1)).

(24.1.2.1)
$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{\oplus n_1'} \arrow[r] \arrow[d] & A^{\oplus n_0'} \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A^{\oplus(n_1'+n_1'')} \arrow[r] \arrow[d] & A^{\oplus(n_0'+n_0'')} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A^{\oplus n_1''} \arrow[r] \arrow[d] & A^{\oplus n_0''} \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

The map $A^{\oplus(n_{i+1}'+n_{i+1}'')} \to A^{\oplus(n_i'+n_i'')}$ is the composition $A^{\oplus n_{i+1}'} \to A^{\oplus n_i'} \hookrightarrow A^{\oplus(n_i'+n_i'')}$ along with a lift of $A^{\oplus n_{i+1}''} \to A^{\oplus n_i''}$ to $A^{\oplus(n_i'+n_i'')}$ ensuring that the middle row is a *complex*.

24.1.C. EXERCISE. Verify that it is possible choose such a lift of $A^{\oplus n_{i+1}''} \to A^{\oplus n_i''}$ to $A^{\oplus(n_i'+n_i'')}$.

Hence (24.1.2.1) is *exact* (not just a complex), using the long exact sequence in cohomology (Theorem 2.6.5), and the fact that the top and bottom rows are exact. Thus the middle row is a resolution, and (24.1.2.1) is a short exact sequence of resolutions. It may be helpful to notice that the columns other than the "N-column" are all "direct sum exact sequences", and the horizontal maps in the middle row are "block upper triangular".

Then truncate (removing the right column $0 \to N' \to N \to N'' \to 0$), tensor with M (obtaining a short exact sequence of complexes) and take cohomology, yielding the desired long exact sequence. $\square$

24.1.D. EXERCISE. Show that the following are equivalent conditions on an A-module M.

(i) M is flat.
(ii) $\mathrm{Tor}_i^A(M, N) = 0$ for all $i > 0$ and all A-modules N.
(iii) $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N.

Caution: Given that free modules are immediately seen to be flat, you might think that Exercise 24.1.D implies Remark 24.1.1. This would follow if we knew that $\mathrm{Tor}_i^A(M, N) \cong \mathrm{Tor}_i^A(N, M)$, which is clear for $i = 0$ (as $\otimes$ is symmetric), but we won't know this about Tor_i when $i > 0$ until Exercise 24.3.A.

24.1.E. EXERCISE. Show that the connecting homomorphism δ constructed above is independent of all of choices (of resolutions, etc.). Try to do this with as little annoyance as possible. (Possible hint: given two sets of choices used to build

(24.1.2.1), build a map — a three-dimensional diagram — from one version of (24.1.2.1) to the other version.)

24.1.F. UNIMPORTANT EXERCISE. Show that $\mathrm{Tor}_i^A(M, \cdot)$ is an *additive* functor (Definition 2.6.1). (We won't use this later, so feel free to skip it.)

We have thus established the foundations of Tor.

24.2 Derived functors in general

24.2.1. Projective resolutions. We used very little about free modules in the above construction of Tor — in fact we used only that free modules are **projective**, i.e. those modules P such that for any surjection $M \twoheadrightarrow N$, it is possible to lift any morphism $P \to N$ to $P \to M$:

(24.2.1.1)
$$\begin{tikzcd} P \arrow[d, dashed, "\text{exists}"'] \arrow[dr] & \\ M \arrow[r, two heads] & N \end{tikzcd}$$

Equivalently, $\mathrm{Hom}(P, \cdot)$ is an exact functor (recall that $\mathrm{Hom}(Q, \cdot)$ is always left-exact for any Q). More generally, the same idea yields the definition of a **projective object in any abelian category**. Hence by following through our entire argument with projective modules replacing free modules throughout, (i) we can compute $\mathrm{Tor}_i^A(M, N)$ by taking any projective resolution of N, and (ii) $\mathrm{Tor}_i^A(M, N) = 0$ for any projective A-module N.

24.2.A. EXERCISE. Show that projective modules are flat. (Hint: Exercise 24.1.D).

24.2.B. UNIMPORTANT EXERCISE WE WON'T USE (BUT GOOD PRACTICE). Show that an object P is projective if and only if every short exact sequence $0 \to A \to B \to P \to 0$ splits. Hence show that an A-module M is projective if and only if M is a direct summand of a free module.

24.2.2. Definition: Derived functors.

The above description was low-tech, but immediately generalizes drastically. All we are using is that $M \otimes_A \cdot$ is a right-exact functor, and that for any A-module N, we can find a surjection $P \twoheadrightarrow N$ from a projective module. In general, if F is *any* right-exact covariant functor from the category of A-modules to any abelian category, this construction will define a sequence of functors $L_i F$ such that $L_0 F = F$ and the $L_i F$'s give a long-exact sequence. We can make this more general still. We say that an abelian category **has enough projectives** if for any object N there is a surjection onto it from a projective object. Then if F is any right-exact covariant functor from an abelian category with enough projectives to any abelian category, then we can define the *left-derived functors* to F, denoted $L_i F$ ($i \geq 0$). You should reread §24.1 and see that throughout we only use the fact we have a projective resolution (repeatedly lifting maps as in (24.2.1.1)), as well as the fact that F sends products to products (a consequence of additivity of the functor, see Remark 2.6.2) to show that F applied to (24.1.2.1) preserves the exactness of the columns.

24.2.C. EXERCISE. The notion of an **injective object** in an abelian category is dual to the notion of a projective object.
(a) State precisely the definition of an injective object.
(b) Define derived functors for (i) covariant left-exact functors (these are called **right-derived functors**), (ii) contravariant left-exact functors (also called **right-derived functors**), and (iii) contravariant right-exact functors (these are called **left-derived functors**), making explicit the necessary assumptions of the category having enough injectives or projectives.

24.2.3. *Notation.* If F is a right-exact functor, its (left-)derived functors are denoted $L_i F$ ($i \geq 0$, with $L_0 F = F$). If F is a left-exact functor, its (right-) derived functors are denoted $R^i F$. The i is a superscript, to indicate that the long exact sequence is "ascending in i".

24.2.4. The Ext **functors.**

24.2.D. EASY EXERCISE (AND DEFINITION): Ext FUNCTORS FOR A-MODULES, FIRST VERSION. As $\operatorname{Hom}(\cdot, N)$ is a contravariant left-exact functor in Mod_A, which has enough projectives, define $\operatorname{Ext}^i_A(M, N)$ as the ith left derived functor of $\operatorname{Hom}(\cdot, N)$, applied to M. State the corresponding long exact sequence for Ext-modules.

24.2.E. EASY EXERCISE (AND DEFINITION): Ext FUNCTORS FOR A-MODULES, SECOND VERSION. The category Mod_A has enough injectives (see §24.2.5). As $\operatorname{Hom}(M, \cdot)$ is a covariant left-exact functor in Mod_A, define $\operatorname{Ext}^i_A(M, N)$ as the ith right derived functor of $\operatorname{Hom}(M, \cdot)$, applied to N. State the corresponding long exact sequence for Ext-modules.

We seem to have a problem with the previous two exercises: we have defined $\operatorname{Ext}^i(M, N)$ twice, and we have two different long exact sequences! Fortunately, these two definitions agree (see Exercise 24.3.B).

24.2.5. $\star$ **The category of A-modules has enough injectives.** We will need the fact that Mod_A has enough injectives, but the details of the proof won't come up again, so feel free to skip this discussion.

24.2.F. EXERCISE. Suppose Q is an A-module, such that for every ideal $I \subset A$, every homomorphism $I \to Q$ extends to $A \to Q$. Show that Q is an injective A-module. Hint: suppose $N \subset M$ is an inclusion of A-modules, and we are given $\beta : N \to Q$. We wish to show that β extends to $M \to Q$. Use the axiom of choice to show that among those A-modules N' with $N \subset N' \subset M$, such that β extends to N', there is a maximal one. If this N' is not M, give an extension of β to $N' + Am$, where $m \in M \setminus N'$, obtaining a contradiction.

24.2.G. EASY EXERCISE (USING THE AXIOM OF CHOICE, IN THE GUISE OF ZORN'S LEMMA). Show that a $\mathbb{Z}$-module (i.e. abelian group) Q is injective if and only if it is **divisible** (i.e. for every $q \in Q$ and $n \in \mathbb{Z}^{\neq 0}$, there is $q' \in Q$ with $nq' = q$). Hence show that any quotient of an injective $\mathbb{Z}$-module is also injective.

24.2.H. EXERCISE. Show that the category of $\mathbb{Z}$-modules $Mod_{\mathbb{Z}} = Ab$ has enough injectives. (Hint: if M is a $\mathbb{Z}$-module, then write it as the quotient of a free $\mathbb{Z}$-module F by some K. Show that M is contained in the divisible group $(F \otimes_{\mathbb{Z}} \mathbb{Q})/K$.)

24.2.I. EXERCISE. Suppose Q is an injective $\mathbb{Z}$-module, and A is a ring. Show that $\mathrm{Hom}_{\mathbb{Z}}(A, Q)$ is an injective A-module. Hint: First describe the A-module structure on $\mathrm{Hom}_{\mathbb{Z}}(A, Q)$. You will only use the fact that $\mathbb{Z}$ is a ring, and that A is an algebra over that ring.

24.2.J. EXERCISE. Show that Mod_A has enough injectives. Hint: suppose M is an A-module. By Exercise 24.2.H, we can find an inclusion *of $\mathbb{Z}$-modules* $M \hookrightarrow Q$ where Q is an injective *$\mathbb{Z}$-module*. Describe a sequence of inclusions of A-modules

$$M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, Q).$$

(The A-module structure on $\mathrm{Hom}_{\mathbb{Z}}(A, M)$ is via the A-action on the left argument A, not via the A-action on the right argument M.) The right term is injective by the previous Exercise 24.2.I.

24.3 Fun with spectral sequences and derived functors

A number of useful facts can be easily proved using spectral sequences. By doing these exercises, you will lose any fear of spectral sequence arguments in similar situations, as you will realize they are all the same.

Before you read this section, you should read §2.7 on spectral sequences.

24.3.1. Symmetry of Tor**.**

24.3.A. EXERCISE (SYMMETRY OF Tor). Show that there is an isomorphism $\mathrm{Tor}_i^A(M, N) \cong \mathrm{Tor}_i^A(N, M)$. (Hint: take a free resolution of M and a free resolution of N. Take their "product" to somehow produce a double complex. Use both orientations of the obvious spectral sequence and see what you get.)

On a related note:

24.3.B. EXERCISE. Show that the two definitions of $\mathrm{Ext}^i(M, N)$ given in Exercises 24.2.D and 24.2.E agree.

24.3.2. Derived functors can be computed using acyclic resolutions. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a right-exact additive functor of abelian categories, and that $\mathcal{A}$ has enough projectives. (In other words, the hypotheses ensure the existence of left-derived functors of F. Analogous facts will hold with the other types of derived functors, Exercise 24.2.C(b).) We say that $A \in \mathcal{A}$ is F**-acyclic** (or just **acyclic** if the F is clear from context) if $L_i F\, A = 0$ for $i > 0$.

The following exercise is a good opportunity to learn a useful trick (Hint 24.3.3).

24.3.C. EXERCISE. Show that you can also compute the derived functors of an objects B of $\mathcal{A}$ using **acyclic resolutions**, i.e. by taking a resolution

$$\cdots \to A_2 \to A_1 \to A_0 \to B \to 0$$

by F-acyclic objects A_i, truncating, applying F, and taking homology. Hence $\mathrm{Tor}_i(M, N)$ can be computed with a flat resolution of M or N.

24.3.3. *Hint for Exercise 24.3.C (and a useful trick: building a "double complex resolution of a complex").* Show that you can construct a double complex

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & P_{2,1} \arrow[r] \arrow[d] & P_{1,1} \arrow[r] \arrow[d] & P_{0,1} \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_{2,0} \arrow[r] \arrow[d] & P_{1,0} \arrow[r] \arrow[d] & P_{0,0} \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A_2 \arrow[r] & A_1 \arrow[r] & A_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

where the rows and columns are exact and the $P_?$'s are projective. Do this by constructing the $P_?$'s inductively from the bottom right. Remove the bottom row, and the right-most nonzero column, and then apply F, to obtain a new double complex. Use a spectral sequence argument to show that (i) the double complex has homology equal to $L_i F\, B$, and (ii) the homology of the double complex agrees with the construction given in the statement of the exercise.

24.3.4. The Grothendieck composition-of-functors spectral sequence. Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are abelian categories; $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are a left-exact additive covariant functors; and $\mathcal{A}$ and $\mathcal{B}$ have enough injective. Thus right derived functors of F, G, and $G \circ F$ exist. A reasonable question (especially in concrete circumstances) is: how are they related? (Essentially the same discussion will apply to different variants of derived functors.)

24.3.D. EXERCISE. If F sends injective elements of $\mathcal{A}$ to G-acyclic elements of $\mathcal{B}$, then for each $A \in \mathcal{A}$, show that there is a spectral sequence with $E^2_{p,q} = R^q G(R^p F(A))$ converging to $R^{p+q}(G \circ F)(A)$. (Hint: This is simpler than it looks. Just follow your nose, and use the construction of Hint 24.3.3.)

We will soon see the Leray spectral sequence as an application of the Grothendieck (composition-of-functors) spectral seqeunce (Exercise 24.4.E).

24.4 ⋆ Derived functor cohomology of $\mathcal{O}$-modules

We wish to apply the machinery of derived functors to define cohomology of quasicoherent sheaves on a scheme X. Sadly, this category $QCoh_X$ usually doesn't have enough injectives! Fortunately, the larger category $Mod_{\mathcal{O}_X}$ does.

24.4.1. Theorem. — *Suppose $(X, \mathcal{O}_X)$ is a ringed space. Then the category of $\mathcal{O}_X$-modules $Mod_{\mathcal{O}_X}$ has enough injectives.*

As a side benefit (of use to others more than us), taking $\mathcal{O}_X = \mathbb{Z}$, we see that the category of sheaves of abelian groups on a fixed topological space have enough injectives.

We prove Theorem 24.4.1 in a series of exercises. Suppose $\mathcal{F}$ is an $\mathcal{O}_X$-module. We will exhibit an injection $\mathcal{F} \hookrightarrow \mathcal{Q}'$ into an injective $\mathcal{O}_X$-module. For each $x \in X$, choose an inclusion $\mathcal{F}_x \hookrightarrow Q_x$ into an injective $\mathcal{O}_{X,x}$-module (possible as the category of $\mathcal{O}_{X,x}$-modules has enough injectives, Exercise 24.2.J).

24.4.A. EXERCISE (PUSHFORWARD OF INJECTIVES ARE INJECTIVE). Suppose $\pi : X \to Y$ is a morphism of ringed spaces, and suppose $\mathcal{Q}$ is an injective $\mathcal{O}_X$-module. Show that $\pi_* \mathcal{Q}$ is an injective $\mathcal{O}_Y$-module. Hint: use the fact that π_* is a right-adjoint (of π^*).

24.4.B. EXERCISE. By considering the inclusion $x \hookrightarrow X$ and using the previous exercise, show that the skyscraper sheaf $\mathcal{Q}_x := i_{x,*} Q_x$, with module Q_x at point x, is an injective $\mathcal{O}_X$-module.

24.4.C. EASY EXERCISE. Show the direct product (possibly infinite) of injective objects in an abelian category is also injective.

By the previous two exercises, $\mathcal{Q}' := \prod_{x \in X} \mathcal{Q}_x$ is an injective $\mathcal{O}_X$-module.

24.4.D. EASY EXERCISE. By considering stalks, show that the natural map $\mathcal{F} \to \mathcal{Q}'$ is an injection.

This completes the proof of Theorem 24.4.1. $\square$

We can now make a number of definitions.

24.4.2. Definitions. If $(X, \mathcal{O}_X)$ is a ringed space, and $\mathcal{F}$ is an $\mathcal{O}_X$-module, define $H^i(X, \mathcal{F})$ as $R^i\Gamma(X, \mathcal{F})$. If furthermore $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of ringed spaces, we have derived pushforwards $R^i\pi_* : Mod_{\mathcal{O}_X} \to Mod_{\mathcal{O}_Y}$.

We have defined these notions earlier in special cases, for quasicoherent sheaves on separated quasicompact schemes (Chapter 20). We will soon (§24.5) show that they agree. Thus the derived functor definition applies much more generally than our Čech definition. But it is worthwhile to note that almost everything we use will come out of the Čech definition. A notable exception is the following.

24.4.E. EXERCISE: THE LERAY SPECTRAL SEQUENCE. Suppose $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces. Show that for any $\mathcal{O}_X$-module $\mathcal{F}$, there is a spectral sequence with E_2 term given by $H^p(Y, R^q\pi_*\mathcal{F})$ abutting to $H^{p+q}(X, \mathcal{F})$. Hint: Use the Grothendieck (or composition-of-functors) spectral sequence (Exercise 24.3.D) and the fact that the pushforward of an injective $\mathcal{O}$-module is an injective $\mathcal{O}$-module (Exercise 24.4.A).

Your argument will extend without change to a composition of derived pushforwards for

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y) \xrightarrow{g} (Z, \mathcal{O}_Z).$$

24.5 ⋆ Čech cohomology and derived functor cohomology agree

We next prove that Čech cohomology and derived functor cohomology agree, where the former is defined.

24.5.1. Theorem. — *Suppose X is a quasicompact separated scheme, and $\mathcal{F}$ is a quasicoherent sheaf. Then the Čech cohomology of $\mathcal{F}$ agrees with the derived functor cohomology of $\mathcal{F}$.*

This statement is not as precise as it should be. We would want to know that this isomorphism is functorial in $\mathcal{F}$, and that it respects long exact sequences (so the connecting homomorphism defined for Čech cohomology agrees with that for derived functor cohomology). There is also an important extension to higher pushforwards. We leave these issues for the end of this section, §24.5.5

The central idea (albeit with a twist) is a spectral sequence argument in the same style as those of §24.3, and uses two "cohomology-vanishing" ingredients, one for each orientation of the spectral sequence.

(A) If $(X, \mathcal{O}_X)$ is a ringed space, $\mathcal{Q}$ is an injective $\mathcal{O}_X$-module, and $X = \cup_i U_i$ is a finite open cover, then $\mathcal{Q}$ has no ith Čech cohomology with respect to this cover for $i > 0$.

(B) If X is an affine scheme, and $\mathcal{F}$ is a quasicoherent sheaf on X, then $R^i\Gamma\mathcal{F} = 0$ for $i > 0$.

(Translation: **(A)** says that building blocks of derived functor cohomology have no Čech cohomology, and **(B)** says that building blocks of Čech cohomology have no derived functor cohomology.)

We will also need the following fact, which will also be useful in our proof of Serre duality.

24.5.A. EXERCISE. Suppose X is a topological space, $\mathcal{Q}$ is an injective sheaf on X, and $i : U \hookrightarrow X$ is an open subset. Show that $\mathcal{Q}|_U$ is injective on U. Hint: similar to Exercise 24.4.A, use the fact that i^{-1} is a right-adjoint. (Exercise 3.6.G showed that $(i_!, i^{-1})$ is an adjoint pair.)

24.5.2. *Proof of Theorem 24.5.1, assuming* **(A)** *and* **(B)**. As in the facts proved in §24.3, we take the only approach that is reasonable: we choose an injective resolution $\mathcal{F} \to \mathcal{Q}_\bullet$ of $\mathcal{F}$, and a Čech cover of X, mix these two types of information in a double complex, and toss it into our spectral sequence machine (§2.7). More precisely, choose a finite affine open cover $X = \cup_i U_i$ and an injective resolution

$$0 \to \mathcal{F} \to \mathcal{Q}_0 \to \mathcal{Q}_1 \to \cdots .$$

Consider the double complex

(24.5.2.1)
$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \oplus_i \mathcal{Q}_2(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_2(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_2(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \oplus_i \mathcal{Q}_1(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_1(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_1(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \oplus_i \mathcal{Q}_0(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_0(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_0(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

We take this as the E_0 term in a spectral sequence. First, let's use the filtration corresponding to choosing the rightward arrow. As higher Čech cohomology of injective $\mathcal{O}$-modules is 0 (assumption **(A)**), we get 0's everywhere except in "column 0", where we get $\mathcal{Q}_i(X)$ in row i:

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 & \mathcal{Q}_2(X) \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \cdots \\
0 \arrow[u] & \mathcal{Q}_1(X) \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \cdots \\
0 \arrow[u] & \mathcal{Q}_0(X) \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

Then we take cohomology in the vertical direction, and we get derived functor cohomology of $\mathcal{F}$ on X on the E_2 page:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 & R^2\Gamma(X,\mathcal{F}) & 0 & 0 & \cdots \\
0 & R^1\Gamma(X,\mathcal{F}) & 0 & 0 & \cdots \\
0 & \Gamma(X,\mathcal{F}) \arrow[uul] & 0 \arrow[uul] & 0 \arrow[uul] & \cdots \\
 & 0 \arrow[uul] & 0 \arrow[uul] & 0 \arrow[uul] &
\end{tikzcd}$$

We then start over on the E_0 page, and this time use the filtration corresponding to choosing the upward arrow first. By Proposition 24.5.A, $I|_{U_J}$ is injective on U_J, so we are computing the derived functor cohomology of $\mathcal{F}$ on U_J. Then the higher derived functor cohomology is 0 (assumption **(B)**), so all entries are 0 except possibly on row 0. Thus the E_1 term is:

(24.5.2.2)

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
0 \arrow[r] & \oplus_i \Gamma(U_i,\mathcal{F}) \arrow[r] & \oplus_{i,j}\Gamma(U_{ij},\mathcal{F}) \arrow[r] & \oplus_{i,j,k}\Gamma(U_{ijk},\mathcal{F}) \arrow[r] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Row 0 is precisely the Čech complex of $\mathcal{F}$, so the spectral sequence converges at the E_2 term, yielding the Čech cohomology. Since one orientation yields derived functor cohomology and one yields Čech cohomology, we are done. $\square$

So it remains to show **(A)** and **(B)**.

24.5.3. Ingredient (A): injectives have no Čech cohomology.

We make an intermediate definition that is independently important. A sheaf $\mathcal{F}$ on a topological space is **flasque** (also sometimes called *flabby*) if all restriction maps are surjective, i.e. if $\mathrm{res}_{U\subset V} : \mathcal{F}(V) \to \mathcal{F}(U)$ is surjective for all $U \to V$.

24.5.B. EXERCISE. Suppose $X = \cup_j U_j$ is a finite cover of X by open sets, and $\mathcal{F}$ is a flasque sheaf on X. Show that the Čech complex for $\mathcal{F}$ with respect to $\cup_j U_j$ has no cohomology in positive degree, i.e. that it is exact except in degree

0 (where it has cohomology $\mathcal{F}(X)$), by the sheaf axioms. Hint: use induction on j. Consider the short exact sequence of complexes (20.2.4.2) (see also (20.2.3.1)). The corresponding long exact sequence will immediately give the desired result for $i > 1$, and flasqueness will be used for $i = 1$.

24.5.C. EXERCISE. Suppose $(X, \mathcal{O}_X)$ is a ringed space, and $\mathcal{Q}$ is an injective $\mathcal{O}_X$-module. Show that $\mathcal{Q}$ is flasque. Hint: Show that the functor $\mathrm{Hom}_{\mathcal{O}_V}(\cdot, \mathcal{Q})$ is exact, then apply it to the inclusion of $\mathcal{O}_V$-modules $i_! \mathcal{O}_U \hookrightarrow \mathcal{O}_V$ (see Exercise 3.6.G(c)).

This is all we need for our algebro-geometric applications, but to show you how general this machinery is, we give some other applications, one serious, and one more entertaining.

24.5.D. EXERCISE. (a) Suppose X is a topological space, so X can be thought of as a locally ringed space with structure sheaf $\mathcal{O}_X = \underline{\mathbb{Z}}$. Suppose that X has a finite cover by contractible open sets U_i such that any intersection of the U_i is also contractible. Show that the derived functor cohomology of $\mathcal{O}_X$ agrees with the Čech cohomology of $\mathbb{Z}$ with respect to this cover. (Here $\mathbb{Z}$ can be replaced by any abelian group.)
(b) Under reasonable hypotheses on X, this computes simplicial cohomology. Use this to compute the cohomology of the circle S^1.

24.5.E. EXERCISE (PERVERSE PROOF OF INCLUSION-EXCLUSION THROUGH COHOMOLOGY OF SHEAVES). The inclusion-exclusion principle is (equivalent to) the following: suppose that X is a finite set, and U_i $(1 \le i \le n)$ are finite sets covering X. As usual, define $U_I = \cap_{i \in I} U_i$ for $I \subset \{1, \dots, n\}$. Then

$$|X| = \sum |U_i| - \sum_{|I|=2} |U_{|I|}| + \sum_{|I|=3} |U_{|I|}| - \sum_{|I|=4} |U_{|I|}| + \cdots .$$

Prove this by endowing X with the discrete topology, showing that the constant sheaf $\underline{\mathbb{Q}}$ is flasque, considering the Čech complex computing $H^i(X, \underline{\mathbb{Q}})$ using the cover U_i, and using Exercise 2.6.B.

24.5.4. Ingredient (B): quasicoherent sheaves on affine schemes have no derived functor cohomology.

The following argument is a version of a great explanation of Martin Olsson.

We show the following statement by induction on k. Suppose X is an affine scheme, and $\mathcal{F}$ is a quasicoherent sheaf on X. Then $R^i\Gamma(X, \mathcal{F}) = 0$ for $0 < i \le k$. The result is vacuously true for $k = 0$; so suppose we know the result for all $0 < k' < k$. Suppose $\alpha \in R^k\Gamma(X, \mathcal{F})$. We wish to show that $\alpha = 0$. Choose an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q}_0 \xrightarrow{d_0} \mathcal{Q}_1 \xrightarrow{d_1} \cdots .$$

Then α has a representative α' in $\mathcal{Q}_k(X)$, such that $d\alpha' = 0$. Because the injective resolution is exact, α' is locally a boundary. In other words, in the neighborhood of any point $x \in X$, there is an open set V_x such that $\alpha|_{V_x} = d\alpha'$ for some $\alpha' \in \mathcal{Q}_{k-1}(V_x)$. By shrinking V_x if necessary, we can assume V_x is affine. By the quasicompactness of X, we can choose a finite number of the V_x's that cover X. Rename these U_i, so we have an affine cover X. Consider the Čech cover of X with respect to *this* affine cover (*not* the affine cover you might have thought we would use — that of X by itself — but instead an affine cover tailored to our particular α). Consider the double complex (24.5.2.1), as the E_0 term in a spectral sequence.

First choose the filtration corresponding to considering the rightward arrows first. As in the argument in §24.5.2, the spectral sequence converges at E_2, where we get 0 everywhere, except that the derived functor cohomology appears in the 0th column.

Next, start over again, choosing the upward filtration. On the E_1 page, row 0 is the Čech complex, as in (24.5.2.2). All the rows between 1 and $k-1$ are 0 by our inductive hypothesis, but we don't yet know much about the higher rows. Because we are interested in the kth derived functor, we focus on the kth antidiagonal ($E_\bullet^{p,k-p}$). The only possibly nonzero terms in this antidiagonal are $E_1^{k,0}$ and $E_1^{0,k}$. We look first at the term on the bottom row $E_1^{k,0} = \prod_{|I|=k} \Gamma(U_I, \mathcal{F})$, which is part of the Čech complex:

$$\cdots \to \prod_{|I|=k-1} \Gamma(U_I, \mathcal{F}) \to \prod_{|I|=k} \Gamma(U_I, \mathcal{F}) \to \prod_{|I|=k+1} \Gamma(U_I, \mathcal{F}) \to \cdots .$$

But we have already verified that the Čech cohomology of a quasicoherent sheaf on an affine scheme vanishes, so this term vanishes by the E_2 page (i.e. $E_i^{k,0} = 0$ for $i \geq 2$).

So the only term of interest in the kth antidiagonal of E_1 is $E_1^{0,k}$, which is the homology of

$$\prod_i \mathcal{Q}_{k-1}(U_i) \to \prod_i \mathcal{Q}_k(U_i) \to \prod_i \mathcal{Q}_{k+1}(U_i), \tag{24.5.4.1}$$

which is $\prod_i R^k\Gamma(U_i, \mathcal{F})$ (using the fact that the $\mathcal{Q}_j|_{U_i}$ are injective on U_i, and they can be used to compute $R^k(\Gamma(U_i, \mathcal{F}))$. So $E_2^{0,k}$ is the homology of

$$0 \to \prod_i R^k\Gamma(U_i, \mathcal{F}) \to \prod_{i,j} R^k\Gamma(U_{ij}, \mathcal{F})$$

and thereafter all differentials to and from the $E_\bullet^{0,k}$ terms will be 0, as the sources and targets of those arrows will be 0. Consider now our lift of α' of our original class $\alpha \in R^k\Gamma(X, \mathcal{F})$. Its image in the homology of (24.5.4.1) is *zero* — this was how we chose our cover U_i to begin with! Thus $\alpha = 0$ as desired, completing our proof. □

24.5.F. EXERCISE. The proof is not quite complete. We have a class $\alpha \in R^k\Gamma(X, \mathcal{F})$, and we have interpreted $R^k\Gamma(X, \mathcal{F})$ as

$$\ker \left(\prod_i R^k\Gamma(U_i, \mathcal{F}) \to \prod_{i,j} R^k\Gamma(U_{ij}, \mathcal{F}) \right).$$

We have two maps $R^k\Gamma(X, \mathcal{F}) \to R^k\Gamma(U_i, \mathcal{F})$, one coming from the natural restriction (under which we can see that the image of α is zero), and one coming from the actual spectral sequence machinery. Verify that they are the same map. (Possible hint: with the filtration used, the $E_\infty^{0,k}$ term is indeed the quotient of the homology of the double complex, so the map goes the right way.)

24.5.5. Tying up loose ends.

24.5.G. IMPORTANT EXERCISE. State and prove the generalization of Theorem 24.5.1 to higher pushforwards $R^i\pi_*$, where $\pi : X \to Y$ is a quasicompact separated morphism of schemes.

24.5.H. EXERCISE. Show that the isomorphism of Theorem 24.5.1 is functorial in $\mathcal{F}$, i.e. given a morphism $\mathcal{F} \to \mathcal{G}$, the diagram

$$\begin{tikzcd} H^i(X, \mathcal{F}) \arrow[r, leftrightarrow] \arrow[d] & R^i\Gamma(X, \mathcal{F}) \arrow[d] \\ H^i(X, \mathcal{G}) \arrow[r, leftrightarrow] & R^i\Gamma(X, \mathcal{G}) \end{tikzcd}$$

commutes, where the horizontal arrows are the isomorphisms of Theorem 24.5.1, and the vertical arrows come from functoriality of H^i and $R^i\Gamma$. (Hint: "spectral sequences are functorial in E_0", which is clear from the construction, although we haven't said it explicitly.)

24.5.I. EXERCISE. Show that the isomorphisms of Theorem 24.5.1 induce isomorphisms of long exact sequences.

CHAPTER 25

Flatness

The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers. — David Mumford [**M-Red**, III.10]

It is a riddle, wrapped in a mystery, inside an enigma; but perhaps there is a key. — Winston Churchill, 1939 (before the theory of flatness was developed)

25.1 Introduction

We come next to the important concept of flatness. We could have discussed flatness at length as soon as we had discussed quasicoherent sheaves and morphisms. But it is an unexpected idea, and the algebra and geometry are not obviously connected, so we have left it for relatively late. The translation of the french word "plat" that best describes this notion is "phat", but unfortunately that word had not yet been coined when flatness first made its appearance.

Serre has stated that he introduced flatness (in his landmark "GAGA" paper [**S-GAGA**])purely for reasons of algebra in GAGA, and that it was Grothendieck who recognized its geometric significance.

A flat morphism $\pi : X \to Y$ is the right notion of a "nice", or "nicely varying" family over Y. For example, if π is a projective flat family over a connected base (translation: $\pi : X \to Y$ is a projective flat morphism, with Y connected), we will see that various numerical invariants of fibers are constant, including the dimension (§25.5.4), and numbers interpretable in terms of an Euler characteristic (see §25.6):

(a) Hilbert polynomial of the fiber (Corollary 25.6.2),
(b) degree of the fiber in projective space (Exercise 25.6.A(a)),
(c) genus (Exercise 25.6.A(b)),
(d) degree of a line bundle if the fiber is a curve (Corollary 25.6.3), and
(e) intersections of divisors and line bundles on the fiber (§25.6.4).

One might think that the right hypothesis might be smoothness (to be defined properly in Chapter 26), or more generally some sort of equisingularity, but we only need something weaker. And this is a good thing: branched covers are not fibrations in any traditional sense, yet they still behave well — the double cover $\mathbb{A}^1 \to \mathbb{A}^1$ given by $y \mapsto x^2$ has constant degree 2 (§18.4.8). Another key example is that of a family of smooth curves degenerating to a nodal curve (Figure 25.1). One can prove things about nonsingular curves by first proving them about a nodal degeneration, and then showing that the result behaves well in flat families. Degeneration techniques such as this are ubiquitous in algebraic geometry.

Given the cohomological nature of the constancy of Euler characteristic result, you should not be surprised that the hypothesis needed (flatness) is cohomological

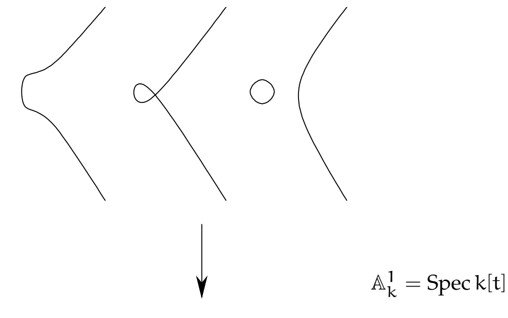


FIGURE 25.1. A flat family of smooth curves degenerating to a nodal curve: $y^2 = x^3 - tx^2$.

in nature — it can be characterized by vanishing of Tor (Exercise 24.1.D), which we use to great effect in §25.3.

Because the algebra of flatness seems so unrelated to the geometry, it can be nonintuitive. We will necessarily begin with algebraic foundations, but you should focus on the following points: methods of showing things are flat (both general criteria and explicit examples), and classification of flat modules over particular kinds of rings. You should try every exercise dealing with explicit examples such as these.

But flatness is important for other reasons too. As a start: as this the right notion of a "nice family", it allows us to define the notion of moduli space. For example, the **Hilbert scheme** of $\mathbb{P}^n$ "parametrizes closed subschemes of $\mathbb{P}^n$". Maps from a scheme B to the Hilbert scheme correspond to (finitely presented) closed subschemes of $\mathbb{P}^n_B$ flat over B. By universal property nonsense, this defines the Hilbert scheme up to unique isomorphism (although we of course must show that it exists, which takes some time — **[M-CAS]** gives an excellent exposition). The moduli space of projective smooth curves is defined by the universal property that maps to the moduli space correspond to projective flat (finitely presented) families whose geometric fibers are smooth curves. (Sadly, this moduli space does not exist...) On a related note, flatness is central in deformation theory: it is key to understanding how schemes (and other geometric objects, such as vector bundles) can deform (cf. §23.4.9). Finally, the notion of Galois descent generalizes to "(faithfully) flat descent", which allows us to "glue" in more exotic Grothendieck topologies in the same way we do in the Zariski topology (or more classical topologies); but this is beyond the scope of our current discussion.

25.1.1. Structure of the chapter.

Flatness has many aspects of different flavors, and it is easy to loose track of the forest for the trees. Here is an outline of the chapter, to keep you focused.

- In §25.2, we discuss some of the easiest facts, which are algebraic in nature.

- Then §s:torfun and §25.4 should be read together. The first uses Tor to understand flatness, and the second uses these insights to develop ideal-theoretic criteria for flatness.
- §25.5 is relatively free-standing, and deals with topological aspects of flatness (such as the fact that flat morphisms are open in good situations).
- §25.6—s:mumfordrocks deal with how flatness interacts with cohomology of quasicoherent sheaves. §25.6 is surprisingly easy given its utility. §25.7 is intended to introduce you to powerful cohomology and base change results. Proofs are given in the starred section §25.8.
- Finally, some harder facts are discussed in the final sections of the chapter.

You should focus on what flatness implies, but also on explicit criteria for flatness in different situation, such as for integral domains (Observation 25.2.2), principal ideal domains (Exercise 25.4.A, discrete valuation rings (Exercise 25.4.B), the dual numbers (Exercise 25.4.C, and local rings (Theorem 25.4.2).

25.2 Easy facts

Many facts about flatness are easy or immediate, although a number are tricky. I will try to make clear which is which, to help you remember the easy facts and the key ideas of proofs of the harder facts. We will pick the low-hanging fruit first.

We recall the definition of a *flat* A*-module* (§2.6.10). If $M \in Mod_A$, $M \otimes_A \cdot$ is right-exact. We say that M **is a flat** A**-module** (or *flat over* A or A*-flat*) if $M \otimes_A \cdot$ is an exact functor. We say that a *ring homomorphism* $B \to A$ *is* **flat** if A is flat as a B-module. (In particular, the algebra structure of A is irrelevant.)

25.2.1. *Two key examples.*

(i) Free modules A-modules are clearly flat, even in the infinite rank case. More generally, projective modules are flat (Exercise 24.2.A).

(ii) Localizations are flat: Suppose S is a multiplicative subset of B. Then $B \to S^{-1}B$ is a flat ring morphism (Exercise 2.6.F(a)).

25.2.A. EASY EXERCISE: FIRST EXAMPLES.
(a) (trick question) Classify flat modules over a field k.
(b) Show that $A[x_1, \dots, x_n]$ is a flat A-module.
(c) Show that the ring homomorphism $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$, is flat. (This will help us understand Example 10.3.3 better, see §25.4.5.)

We make some quick but important observations.

25.2.2. *Important Observation: Flat modules over integral domains are torsion-free.* If x is a non-zerodivisor of A, and M is a flat A-module, then $M \xrightarrow{\times x} M$ is injective. (Reason: apply the exact functor $M \otimes_A$ to the exact sequence $0 \longrightarrow A \xrightarrow{\times x} A$.) In particular, if A is an integral domain, then flat modules over A are torsion-free (defined in §14.5.4). This observation gives an easy way of recognizing when a module is *not* flat. We will use it many times.

25.2.B. EXERCISE: ANOTHER EXAMPLE. Show that a finitely generated module over a discrete valuation ring is flat if and only if it is torsion-free if and only if it is free. Hint: Remark 13.4.16 classifies finitely generated modules over a discrete valuation ring. (Exercise 25.4.A sheds more light on flatness over a discrete valuation ring. Proposition 14.7.3 is also related.)

25.2.C. EXERCISE (FLATNESS IS PRESERVED BY CHANGE OF BASE RING). Show that if M flat A-module, $A \to B$ is a homomorphism, then $M \otimes_A B$ is a flat B-module. Hint: $(M \otimes_A B) \otimes_B \cdot \cong M \otimes_B (B \otimes_A \cdot)$.

25.2.D. EXERCISE (TRANSITIVITY OF FLATNESS). Show that if A is a flat B-algebra, and M is A-flat, then M is also B-flat. (The same hint as in the previous exercise applies.)

25.2.3. Proposition (flatness is a stalk/prime-local property). — *An A-module M is flat if and only if $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module for all primes $\mathfrak{p}$.*

Proof. Suppose first that M is a flat A-module. Given any exact sequence of $A_\mathfrak{p}$-modules

$$0 \to N' \to N \to N'' \to 0, \tag{25.2.3.1}$$

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact too. But $M \otimes_A N$ is canonically isomorphic to $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N$ (do you see why?), so $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module.

Suppose next that $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module for all $\mathfrak{p}$. Given any short exact sequence (25.2.3.1), tensoring with M yields

$$0 \longrightarrow K \longrightarrow M \otimes_A N' \longrightarrow M \otimes_A N \longrightarrow M \otimes_A N'' \longrightarrow 0 \tag{25.2.3.2}$$

where K is the kernel of $M \otimes_A N' \to M \otimes_A N$. We wish to show that $K = 0$. It suffices to show that $K_\mathfrak{p} = 0$ for all prime $\mathfrak{p} \subset A$ (see the comment after Exercise 5.3.F). Given any $\mathfrak{p}$, localizing (25.2.3.1) at $\mathfrak{p}$ and tensoring with the exact $A_\mathfrak{p}$-module $M_\mathfrak{p}$ yields

$$0 \longrightarrow M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N'_\mathfrak{p} \longrightarrow M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \longrightarrow M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N''_\mathfrak{p} \longrightarrow 0. \tag{25.2.3.3}$$

But localizing (25.2.3.2) at $\mathfrak{p}$ and using the isomorphisms $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \cong (M \otimes_A N')_{A_\mathfrak{p}}$, we obtain the exact sequence

$$0 \longrightarrow K_\mathfrak{p} \longrightarrow M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N'_\mathfrak{p} \longrightarrow M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \longrightarrow M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N''_\mathfrak{p} \longrightarrow 0,$$

which is the same as the exact sequence (25.2.3.3) except for the $K_\mathfrak{p}$. Hence $K_\mathfrak{p} = 0$ as desired. $\square$

25.2.4. Flatness for schemes.

Motivated by Proposition 25.2.3, the extension of the notion of flatness to schemes is straightforward.

25.2.5. *Definition: flat quasicoherent sheaves.* We say that a quasicoherent sheaf $\mathcal{F}$ on a scheme X is **flat at** $x \in X$ if $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module. We say that a quasicoherent sheaf $\mathcal{F}$ on a scheme X is **flat** (over X) if it is flat at all $x \in X$. In light of Proposition 25.2.3, we can check this notion on affine open cover of X.

25.2.6. *Definition: flat morphism.* Similarly, we say that a morphism of schemes $\pi : X \to Y$ is **flat at** $x \in X$ if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,\pi(x)}$-module. We say that a morphism of schemes $\pi : X \to Y$ is **flat** if it is flat at all $x \in X$. We can check flatness (affine-)locally on the source and target.

We can combine these two definitions into a single fancy definition.

25.2.7. *Definition: flat quasicoherent sheaf over a base.* Suppose $\pi : X \to Y$ is a morphism of schemes, and $\mathcal{F}$ is a quasicoherent sheaf on X. We say that $\mathcal{F}$ **is flat (over** Y**) at** $x \in X$ if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,\pi(x)}$-module. We say that $\mathcal{F}$ **is flat (over** Y**)** if it is flat at all $x \in X$.

Definitions 25.2.5 and 25.2.6 correspond to the cases $X = Y$ and $\mathcal{F} = \mathcal{O}_X$ respectively. (Definition 25.2.7 applies without change to the category of ringed spaces, but we won't use this.)

25.2.E. EASY EXERCISE (REALITY CHECK). Show that open immersions are flat.

Our results about flatness over rings above carry over easily to schemes.

25.2.F. EXERCISE. Show that a map of rings $B \to A$ is flat if and only if the corresponding morphism of schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ is flat. More generally, if $B \to A$ is a map of rings, and M is a B-module, show that M is A-flat if and only if $\tilde{M}$ is flat over Spec A.

25.2.G. EASY EXERCISE (EXAMPLES AND REALITY CHECKS).
(a) If X is a scheme, and x is a point, show that the natural morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ is flat. (Hint: localization is flat, §25.2.1.)
(b) Show that $\mathbb{A}^n_A \to \operatorname{Spec} A$ is flat.
(c) If $\mathcal{F}$ is a locally free sheaf on a scheme X, show that $\mathbb{P}\mathcal{F} \to X$ (Definition 18.2.2) is flat.

25.2.H. EXERCISE (TRANSITIVITY OF FLATNESS). Suppose $\pi : X \to Y$ and $\mathcal{F}$ is a quasicoherent sheaf on X, flat over Y. Suppose also that $\psi : Y \to Z$ is a flat morphism. Show that $\mathcal{F}$ is flat over Z.

25.2.I. EXERCISE (FLATNESS IS PRESERVED BY BASE CHANGE). Suppose $\pi : X \to Y$ is a morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X, flat over Y. If $\rho : Y' \to Y$ is any morphism, and $\rho' : X \times_Y Y' \to X$ is the induced morphism, show that $(\rho')^*\mathcal{F}$ is flat over Y'.

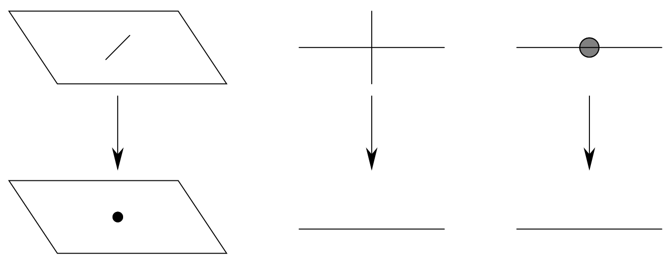
The following exercise is very useful for visualizing flatness and non-flatness (see for example Figure 25.2).

25.2.J. FLAT MAPS SEND ASSOCIATED POINTS TO ASSOCIATED POINTS. Suppose $\pi : X \to Y$ is a flat morphism of locally Noetherian schemes. Show that any associated point of X must map to an associated point of Y. Hint: suppose $\pi^\sharp : (B, \mathfrak{n}) \to (A, \mathfrak{m})$ is a local homomorphism of local Noetherian rings. Suppose $\mathfrak{n}$ is not an associated prime of B. Show that there is an element $f \in B$ that does is not in any associated prime of B (perhaps using Exercise 12.3.D), and hence is a non-zerodivisor. Show that $\pi^\sharp f \in \mathfrak{m}$ is a non-zerodivisor of A using Observation 25.2.2, and thus show that $\mathfrak{m}$ is not an associated prime of A.

25.2.K. EXERCISE. Use Exercise 25.2.J to that the following morphisms are not flat (see Figure 25.2):

(a) $\operatorname{Spec} k[x,y]/(xy) \to \operatorname{Spec} k[x]$,
(b) $\operatorname{Spec} k[x,y]/(y^2, xy) \to \operatorname{Spec} k[x]$,
(c) $\operatorname{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$.

Hint for (c): first pull back to a line through the origin to obtain a something akin to (a). (This foreshadows the statement and proof Proposition 25.5.5, which says that for flat morphisms "there is no jumping of fiber dimension".)



(a) (b) (c)

FIGURE 25.2. Morphisms that are not flat (Exercise 25.2.K) **[Figure to be updated to reflect ordering in Exercise 25.2.K later]**

25.2.8. Theorem (cohomology commutes with flat base change). — *Suppose*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is a fiber diagram, and f (and thus f') is quasicompact and separated (so higher pushforwards of quasicoherent sheaves by f and f' exist, as described in §20.7). Suppose also that g is flat, and $\mathcal{F}$ is a quasicoherent sheaf on X. Then the natural morphisms $g^*(R^i f_* \mathcal{F}) \to R^i f'_*(g'^* \mathcal{F})$ are isomorphisms.

25.2.L. EXERCISE. Prove Theorem 25.2.8. (Hint: Exercise 20.7.B(b) is the special case where f is affine. Extend it to the quasicompact and separated case using the same idea as the proof of Theorem 17.2.1, which was actually proved in Exercise 14.3.I, using Exercise 14.3.E. Your proof of the case $i = 0$ will only need a quasiseparated hypothesis in place of the separated hypothesis.)

A useful special case is where Y' is the generic point of a component of Y. In other words, in light of Exercise 25.2.G(a), the stalk of the higher pushforward of $\mathcal{F}$ at the generic point is the cohomology of $\mathcal{F}$ on the fiber over the generic point. This is a first example of something important: understanding cohomology of (quasicoherent sheaves on) fibers in terms of higher pushforwards. (We would certainly hope that higher pushforwards would tell us something about higher cohomology of fibers, but this is certainly not a priori clear!)

In comparison to this result, which shows that cohomology of *any* quasicoherent sheaf commutes with *flat* base change, §25.6–25.8 deal with when and how cohomology of a *flat* quasicoherent sheaf commutes with *any* base change.

25.3 Flatness through Tor

We defined the Tor functor in §24.1: $\mathrm{Tor}_i^A(M, N)$ is obtained by taking a free resolution of N, removing the N, tensoring it with M, and taking homology. Exercise 24.1.D characterized flatness in terms of Tor: M is A-flat if $\mathrm{Tor}_1^A(M, N) = 0$ for all N. In this section, we reap the easier benefits of this characterization, recalling key properties of Tor when needed. In the next section, we work harder to extract more from Tor.

It is sometimes possible to compute Tor from its definition, as shown in the following exercise that we will use repeatedly.

25.3.A. EXERCISE. If x is not a zerodivisor, show that

$$\mathrm{Tor}_i^A(M, A/x) = \begin{cases} M/xM & \text{if } i = 0; \\ (M : x) & \text{if } i = 1 \text{ (recall } (M : x) = \{m \in M \;:\; xm = 0\}); \\ 0 & \text{if } i > 1. \end{cases}$$

(the elements of M annihilated by multiplication by x). Hint: use the resolution

$$0 \longrightarrow A \xrightarrow{\times x} A \longrightarrow A/x \longrightarrow 0$$

of A/x.

25.3.1. *Remark.* As a corollary of Exercise 25.3.A, we see again that flat modules are torsion-free (Observation 25.2.2). Also, Exercise 25.3.A gives the reason for the notation Tor — it is short for *torsion*.

25.3.B. EXERCISE. If B is A-flat, use the FHHF theorem (Exercise 2.6.H(c)) to give an isomorphism $B \otimes \mathrm{Tor}_i^A(M, N) \cong \mathrm{Tor}_i^B(B \otimes M, B \otimes N)$.

Recall that the Tor functor is symmetric in its entries (there is an isomorphism $\mathrm{Tor}_i^A(M, N) \leftrightarrow \mathrm{Tor}_i^A(N, M)$, Exercise 24.3.A). This gives us a quick but very useful result.

25.3.C. EASY EXERCISE. If $0 \to N' \to N \to N'' \to 0$ is exact, and N'' is flat (e.g. free), show that $0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$ is exact.

We would have cared about this result long before learning about Tor, so it gives some motivation for learning about Tor. (Can you give a direct proof, by a diagram chase?)

25.3.D. EXERCISE. If $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence of flat A-modules, show that it remains flat upon tensoring with any other A-module. (Hint: as always, break the exact sequence into short exact sequences.)

25.3.E. EXERCISE (IMPORTANT CONSEQUENCE OF EXERCISE 25.3.C). Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of quasicoherent sheaves on a scheme Y, and $\mathcal{F}''$ is flat (e.g. locally free). Show that if $\pi : X \to Y$ is any morphism of schemes, the pulled back sequence $0 \to \pi^*\mathcal{F}' \to \pi^*\mathcal{F} \to \pi^*\mathcal{F}'' \to 0$ remains exact.

25.3.F. EXERCISE (CF. EXERCISE 14.5.B FOR THE ANALOGOUS FACTS ABOUT VECTOR BUNDLES). Suppose $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules.
(a) If M and M'' are both flat, show that M' is too. (Hint: Recall the long exact sequence for Tor, Proposition 24.1.2. Also, use that N is flat if and only if $\mathrm{Tor}_i(N, N') = 0$ for all $i > 0$ and all N', Exercise 24.1.D.)
(b) If M' and M'' are both flat, show that M is too. (Same hint.)
(c) If M' and M are both flat, show that M'' need not be flat.

25.3.G. EASY EXERCISE. If $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence, and M_i is flat for $i > 0$, show that M_0 is flat too. (Hint: as always, break the exact sequence into short exact sequences.)

We will use the Exercises 25.3.D and 25.3.G later this chapter.

25.4 Ideal-theoretic criteria for flatness

The following theorem will allow us to classify flat modules over a number of rings. It is a refined version of Exercise 24.1.D, that M is a flat A-module if and only if $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N.

25.4.1. Theorem (ideal-theoretic criterion for flatness). — *M is flat if and only if $\mathrm{Tor}_1^A(M, A/I) = 0$ for all ideals I.*

Proof. By Exercise 24.1.D, we need only show that $\mathrm{Tor}_1^A(M, A/I) = 0$ for all I implies $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N, and hence that M is flat.

We first prove that $\mathrm{Tor}_1^A(M, N) = 0$ for all *finitely generated* modules N, by induction on the number n of generators $a_1, \dots, a_n$ of N. The base case (if $n = 1$, so $N \cong A/(a_1)$) is our assumption. If $n > 1$, then $Aa_n \cong A/(a_n)$ is a submodule of N, and the quotient Q is generated by the images of $a_1, \dots, a_{n-1}$, so the result follows by considering the Tor_1 portion of the Tor long exact sequence for

$$0 \to A/(a_1) \to N \to Q \to 0.$$

We deal with the case of general N by abstract nonsense. Notice that N is the union of its finitely-generated submodules $\{N_\alpha\}$. In fancy language, this union is a filtered colimit — any two finitely generated submodules are contained in a finitely generated submodule (specifically, the submodule they generate). Filtered colimits of modules commute with cohomology (Exercise 2.6.L), so $\mathrm{Tor}_1(M, N)$ is the colimit over α of $\mathrm{Tor}_1(M, N_\alpha) = 0$, and is thus 0. $\square$

We use Theorem 25.4.1 to get explicit characterizations of flat modules over three (types of) rings: principal ideal domains, dual numbers, and Noetherian local rings.

Recall Observation 25.2.2, that flatness over an integral domain implies torsion-free. The converse is true for principal ideal domains:

25.4.A. EXERCISE (FLAT = TORSION-FREE FOR A PID). Show that a module over a principal ideal domain is flat if and only if it is torsion-free.

25.4.B. EXERCISE (FLATNESS OVER A DVR). Suppose M is a module over a discrete valuation ring A with uniformizer t. Show that M is flat if and only if t is not a zerodivisor on M, i.e. $(M : t) = 0$. (See Exercise 25.2.B for the case of finitely generated modules.) This yields a simple geometric interpretation of flatness over a nonsingular curve, which we discuss in §25.4.4.

25.4.C. EXERCISE (FLATNESS OVER THE DUAL NUMBERS). Show that M is flat over $k[t]/(t^2)$ if and only if the "multiplication by t" map $M/tM \to tM$ is an isomorphism. Hint: $k[t]/(t^2)$ has only three ideals. (This fact is important in deformation theory and elsewhere.)

25.4.2. Important Theorem (flat = free = projective for finitely presented modules over local rings). — *Suppose $(A, \mathfrak{m})$ is a local ring (not necessarily Noetherian), and M is a finitely presented A-module. Then M is flat if and only if it is free if and only if it is projective.*

Warning: modules over local rings can be flat without being free: let A be a discrete valuation ring, and consider $A \to K(A)$, which like all localizations is flat §25.2.1. Do you see why it is not free?

Proof. For any ring, free modules are projective (§24.2.1), and projective modules are flat (Exercise 24.2.A), so we need only show that flat modules are free for a local ring.

Now $M/\mathfrak{m}M$ is a finite-dimensional vector space over the field $A/\mathfrak{m}$. Choose a basis of $M/\mathfrak{m}M$, and lift it to elements $m_1, \ldots, m_n \in M$. Consider $A^{\oplus n} \to M$ given by $e_i \mapsto m_i$. We will show this is an isomorphism. It is surjective by Nakayama's lemma (see Exercise 8.2.H): the image is all of M modulo the maximal ideal, hence is everything. As M is finitely presented, by Exercise 14.6.A ("finitely presented implies always finitely presented"), the kernel K is finitely generated. Tensor $0 \to K \to A^{\oplus n} \to M \to 0$ with $A/\mathfrak{m}$. As M is flat, the result is still exact (Exercise 25.3.C):

$$0 \to K/\mathfrak{m}K \to (A/\mathfrak{m})^{\oplus n} \to M/\mathfrak{m}M \to 0.$$

But $(A/\mathfrak{m})^{\oplus n} \to M/\mathfrak{m}M$ is an isomorphism by construction, so $K/\mathfrak{m}K = 0$. As K is finitely generated, $K = 0$ by Nakayama's Lemma 8.2.9. $\square$

Here is an immediate corollary — really just a geometric interpretation.

25.4.3. Corollary. — *Suppose $\mathcal{F}$ is a coherent sheaf on a locally Noetherian scheme X. Then $\mathcal{F}$ is flat over X if and only if it is locally free.*

Proof. Local-freeness of a finite type sheaf can be checked at the stalks, Exercise 14.7.E. (This exercise required Noetherian hypotheses.) □

25.4.D. ⋆ EXERCISE (UNIMPORTANT FOR US, BUT AN INTERESTING VARIANT OF THEOREM 25.4.2). Suppose A is a ring (not necessarily local), and M is a finitely presented A module. Show that M is flat if and only if it is projective. (Hint: show that M is projective if and only if $M_{\mathfrak{m}}$ is free for every maximal ideal $\mathfrak{m}$. The harder direction of this implication uses the fact that $\operatorname{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) = \operatorname{Hom}_A(M, N)_{\mathfrak{m}}$, which follows from Exercise 2.6.G.)

25.4.E. EXERCISE. Make precise and prove the following statement: "finite flat morphisms have locally constant degree". (You may want to glance at §18.4.4 to make this precise. We will revisit that example in §25.4.5.)

25.4.F. EXERCISE. Prove the following useful criterion for flatness: Suppose $X \to Y$ is a finite morphism, and Y is integral and locally Noetherian. Then f is flat if and only if $f_*\mathcal{O}_X$ is locally free, if and only if the rank of $f_*\mathcal{O}_X$ is constant ($\dim_{\kappa(y)}(f_*\mathcal{O}_X)_y \otimes K(y)$ is constant). Partial hint: Exercise 14.7.J.

25.4.G. EXERCISE. Show that the normalization of the node (see Figure 8.4) is not flat. (Hint: use Exercise 25.4.F.)

This exercise can be strengthened to show that nontrivial normalizations are *never* flat.

25.4.4. Flat families over nonsingular curves, and flat limits.
Exercise 25.4.B gives an easy geometric criterion for when morphisms to nonsingular curves are flat.

25.4.H. EXERCISE (CRITERION FOR FLATNESS OVER A NONSINGULAR CURVE). Suppose $\pi : X \to Y$ is a morphism from a locally Noetherian scheme to a nonsingular (locally Noetherian) curve. (The local Noetherian hypothesis on X is so we can discuss its associated points.) Show that π is flat if and only if all associated points of X map to a generic point of Y. (This is a partial converse to Exercise 25.2.J, that flat maps always send associated points to associated points.)

For example, a nonconstant map from an integral (locally Noetherian) scheme to a nonsingular curve must be flat. Exercise 25.4.G (and the comment after it) shows that the nonsingular condition is necessary.

25.4.5. *Revisiting the degree of a projective morphism from a curve to a nonsingular curve.* As hinted in Remark 18.4.10, we can now better understand why nonconstant projective morphisms from a curve to a nonsingular curve have a well-defined degree, which can be determined by taking the preimage of any point (§18.4.4). (Example 10.3.3 was particularly enlightening.) This is because such maps are flat by Exercise 25.4.H, and then the degree is constant by Exercise 25.4.E (see also Exercise 25.4.F).

Also, Exercise 25.4.F now yields a new proof of Proposition 18.4.5.

25.4.6. *Flat limits.* Here is an important consequence of Exercise 25.4.H, which we can informally state as: we can take flat limits over one-parameter families. More precisely: suppose A is a discrete valuation ring, and let 0 be the closed point of $\operatorname{Spec} A$ and η the generic point. Suppose X is a locally Noetherian scheme over A, and Y is a closed subscheme of $X|_\eta$. Let Y' be the scheme-theoretic closure of Y in X. Then Y' is flat over A. Similarly, suppose Z is a one-dimensional Noetherian scheme, 0 is a nonsingular point of Z, and $\pi : X \to Z$ is a morphism from a locally Noetherian scheme to Z. If Y is a closed subscheme of $\pi^{-1}(Z - \{0\})$, and Y' is the scheme-theoretic closure of Y in X, then Y' is flat over Z. In both cases, the closure $Y'|_0$ is often called the **flat limit** of Y. (Feel free to weaken the Noetherian hypotheses on X.)

25.4.I. EXERCISE. Suppose (with the language of the previous paragraph) that A is a discrete valuation ring, X is a locally Noetherian A-scheme, and Y is a closed subscheme of the generic fiber $X|_\eta$. Show that there is only one closed subscheme Y' of X such that $Y'|_\eta = Y$, and Y' is flat over A.

25.4.J. EXERCISE (AN EXPLICIT FLAT LIMIT). Let $X = \mathbb{A}^3 \times \mathbb{A}^1 \to Y = \mathbb{A}^1$ over a field k, where the coordinates on $\mathbb{A}^3$ are x, y, and z, and the coordinates on $\mathbb{A}^1$ are t. Define X away from $t = 0$ as the union of the two lines $y = z = 0$ (the x-axis) and $x = z - t = 0$ (the y-axis translated by t). Find the flat limit at $t = 0$. (Hints: (i) it is *not* the union of the two axes, although it includes this union. The flat limit is nonreduced at the node, and the "fuzz" points out of the plane they are contained in. (ii) $(y, z)(x, z) \neq (xy, z)$. (iii) Once you have a candidate flat limit, be sure to check that it *is* the flat limit. (iv) If you get stuck, read Example 25.4.7 below.)

Consider a projective version of the previous example, where two lines in $\mathbb{P}^3$ degenerate to meet. The limit consists of two lines meeting at a node, with some nonreduced structure at the node. Before the two lines come together, their space of global sections is two-dimensional. When they come together, it is not immediately obvious that their flat limit also has two-dimensional space of global sections as well. The reduced version (the union of the two lines meeting at a point) has a one-dimensional space of global sections, but the effect of the nonreduced structure on the space of global sections may not be immediately clear. However, we will see that "cohomology groups can only jump up in flat limits", as a consequence (indeed the main moral) of the Semicontinuity Theorem 25.7.1.

25.4.7. $\star$ *Example of variation of cohomology groups in flat families.* We can use a variant of Exercise 25.4.J to see an example of a cohomology group actually jumping. We work over an algebraically closed field to avoid distractions. Before we get down to explicit algebra, here is the general idea. Consider a twisted cubic C in $\mathbb{P}^3$. A projection pr_p from a random point $p \in \mathbb{P}^3$ will take C to a nodal plane cubic. Picture this projection "dynamically", by choosing coordinates so p is at $[1; 0; 0; 0]$, and considering the map $\phi_t : [w; x; y; z] \mapsto [w; tx; ty; tz]$; ϕ_1 is the identity on $\mathbb{P}^3$, ϕ_t is an automorphism of $\mathbb{P}^3$ for $t \neq 0$, and ϕ_0 is the projection. The limit of $\phi_t(C)$ as $t \to 0$ will be a nodal cubic, with nonreduced structure at the node "analytically the same" as what we saw when two lines came together (Exercise 25.4.J).

Let's now see this in practice. Rather than working directly with the twisted cubic, we use another example where we saw a similar picture. Consider the nodal (affine) plane cubic $y^2 = x^3 + x^2$. Its normalization (see Figure 8.4, Example (3) of §8.3.6, Exercise 10.6.E, ...) was obtained by adding an extra variable m corresponding to y/x (which can be interpreted as blowing up the origin, see §19.4.3). We use the variable m rather than t (used in §8.3.6) in order to reserve t for the parameter for the flat family.

We picture the nodal cubic C as lying in the xy-plane in 3-space $\mathbb{A}^3 = \operatorname{Spec} k[x, y, m]$, and the normalization $\tilde{C}$ projecting to it, with $m = y/x$. What are the equations for $\tilde{C}$? Clearly, they include the equations $y^2 = x^3 + x^2$ and $y = mx$, but these are not enough — the m-axis (i.e. $x = y = 0$) is also in $V(y^2 - x^3 - x^2, y - mx)$. A little thought (and the algebra we have seen earlier in this example) will make clear that we have a third equation $m^2 = (x+1)$, which along with $y = mx$ implies $y^2 = x^2 + x^3$. *Now* we have enough equations: $k[x, y, m]/(m^2 - (x+1), y - mx)$ is an integral domain, as it is clearly isomorphic to $k[m]$. Indeed, you should recognize this as the algebra appearing in Exercise 10.6.E.

Next, we want to formalize our intuition of the dynamic projection to the xy-plane of $\tilde{C} \subset \mathbb{A}^3$. We picture it as follows. Given a point (x, y, m) at time 1, at time t we want it to be at (x, y, mt). At time $t = 1$, we "start with" $\tilde{C}$, and at time $t = 0$ we have (set-theoretically) C. Thus *at time* $t \neq 0$, the curve $\tilde{C}$ is sent to the curve cut out by equations

$$k[x, y, m]/(m^2 - t(x + 1), ty - mx).$$

The family over $\operatorname{Spec} k[t, t^{-1}]$ is thus

$$k[x, y, m, t, t^{-1}]/(m^2 - t(x + 1), ty - mx).$$

Notice that we have inverted t because we are so far dealing only with nonzero t. For $t \neq 0$, this is certainly a "nice" family, and so surely flat. Let's make sure this is true.

25.4.K. EXERCISE. Check this, as painlessly as possible! Hint: by a clever change of coordinates, show that the family is constant "over $\operatorname{Spec} k[t, t^{-1}]$", and hence pulled back (in some way you must figure out) via $k[t, t^{-1}] \to k$ from

$$\operatorname{Spec} k[X, Y, M]/(M^2 - (X + 1), Y - MX) \to \operatorname{Spec} k,$$

which is flat by Trick Question 25.2.A(a).

We now figure out the flat limit of this family over $t = 0$, in $\operatorname{Spec} k[x, y, m, t] \to \mathbb{A}^1 = \operatorname{Spec} k[t]$. We first hope that our flat family is given by the equations we have already written down:

$$\operatorname{Spec} k[x, y, m, t]/(m^2 - t(x + 1), ty - mx).$$

But this is *not* flat over $\mathbb{A}^1 = \operatorname{Spec} k[t]$, as the fiber dimension jumps (§25.5.4): substituting $t = 0$ into the equations (obtaining the fiber over $0 \in \mathbb{A}^1$), we find $\operatorname{Spec} k[x, y, m]/(m^2, mx)$. This is set-theoretically the xy-plane ($m = 0$), which of course has dimension 2. Notice for later reference that this "false limit" is scheme-theoretically the xy-plane, *with some nonreduced structure along the* y*-axis*. (This may remind you of Figure 6.1.)

So we are missing at least one equation. One clue as to what equation is missing: the equation $y^2 = x^3 + x^2$ clearly holds for $t \neq 0$, and does *not* hold for our naive attempt at a limit scheme $m^2 = mx = 0$. So we put this equation back in, and have a second hope for describing the flat family over $\mathbb{A}^1$:

$$\operatorname{Spec} k[x, y, m, t]/(m^2 - t(x + 1), ty - mx, y^2 - x^3 - x^3) \to \operatorname{Spec} k[t].$$

Let $A = k[x, y, m, t]/(m^2 - t(x+1), ty - mx, y^2 - x^3 - x^3)$ for convenience. The morphism $\operatorname{Spec} A \to \mathbb{A}^1$ *is* flat at $t = 0$. How can we show it? We could hope to show that A is an integral domain, and thus invoke Exercise 25.4.H. Instead we use Exercise 25.4.A, and show that t is not a zerodivisor on A. We do this by giving a "normal form" for elements of A.

25.4.L. EXERCISE. Show that each element of A can be written uniquely as a polynomial in x, y, m, and t such that no monomial in it is divisible by m^2, mx, or y^2. Then show that t is not a zerodivisor on A, and conclude that $\operatorname{Spec} A \to \mathbb{A}^1$ is indeed flat.

25.4.M. EXERCISE. Thus the flat limit when $t = 0$ is given by

$$\operatorname{Spec} k[x, y, m]/(m^2, mx, y^2 - x^2 - x^3).$$

Show that the flat limit is nonreduced, and the "nonreducedness has length 1 and supported at the origin". More precisely, if $X = \operatorname{Spec} A/(t)$, show that $\mathcal{I}_{X^{\mathrm{red}}}$ is a skyscraper sheaf, with value k, supported at the origin. Sketch this flat limit X.

25.4.8. Note that we have a nonzero global function on X, given by m, which is supported at the origin (i.e. 0 away from the origin).

We now use this example to get a projective example with interesting behaviour. We take the projective completion of this example, to get a family of cubic curves in $\mathbb{P}^3$ degenerating to a nodal cubic C with a nonreduced point.

25.4.N. EXERCISE. Do this: describe this family (in $\mathbb{P}^3 \times \mathbb{A}^1$) precisely.

Take the long exact sequence corresponding to

$$0 \longrightarrow \mathcal{I}_{C^{\mathrm{red}}} \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C^{\mathrm{red}}} \longrightarrow 0,$$

to get

$$H^1(C, \mathcal{I}_{C^{\mathrm{red}}}) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^1(C, \mathcal{O}_{C^{\mathrm{red}}}) \longrightarrow$$

$$H^0(C, \mathcal{I}_{C^{\mathrm{red}}}) \xrightarrow{\alpha} H^0(C, \mathcal{O}_C) \longrightarrow H^0(C, \mathcal{O}_{C^{\mathrm{red}}}) \longrightarrow 0$$

We have $H^1(C, \mathcal{I}_{C^{\mathrm{red}}}) = 0$ as $\mathcal{I}_{C^{\mathrm{red}}}$ is supported in dimension 0 (by dimensional vanishing, Theorem 20.2.6). Also, $H^i(C^{\mathrm{red}}, \mathcal{O}_{C^{\mathrm{red}}}) = H^i(C, \mathcal{O}_{C^{\mathrm{red}}})$ (property **(v)** of cohomology, see §20.1). The (reduced) nodal cubic C^{red} has $h^0(\mathcal{O}) = 1$ (Exercise 20.1.C) and $h^1(\mathcal{O}) = 1$ (cubic plane curves have genus 1, (20.5.3.1)). Also, $h^0(C, \mathcal{I}_{C^{\mathrm{red}}}) = 1$ as observed above. Finally, α is not 0, as there exists a nonzero function on C vanishing on C^{red} (§25.4.8 — convince yourself that this function extends from the affine patch $\operatorname{Spec} A$ to the projective completion).

Using the long exact sequence, we conclude $h^0(C, \mathcal{O}_C) = 2$ and $h^1(C, \mathcal{O}_C) = 1$. Thus in this example we see that $(h^0(\mathcal{O}), h^1(\mathcal{O})) = (1, 0)$ for the general member of the family (twisted cubics are isomorphic to $\mathbb{P}^1$), and the special member (the flat limit) has $(h^0(\mathcal{O}), h^1(\mathcal{O})) = (2, 1)$. Notice that both cohomology groups have jumped, yet the Euler characteristic has remained the same. The first behavior, as stated after Exercise 25.4.J, is an example of the Semicontinuity Theorem 25.7.1.

The second, constancy of Euler characteristics in flat families, is what we turn to next. (It is no coincidence that the example had a singular limit, see §25.7.2.)

25.5 Topological aspects of flatness

We now discuss some topological aspects and consequences of flatness, that boil down to the Going-Down theorem for flat morphisms (§25.5.2), which in turn comes from faithful flatness. Because dimension in algebraic geometry is a topological notion, we will show that dimensions of fibers behave well in flat families (§25.5.4).

25.5.1. Faithful flatness. The notion of faithful flatness is handy for many reasons, but we will just give some initial uses. A B-module M is **faithfully flat** if for all complexes of B-modules

$$N' \to N \to N'', \tag{25.5.1.1}$$

(25.5.1.1) is exact if and only if (25.5.1.1)$\otimes_B M$ is exact. A B**-algebra** A **is faithfully flat** if it is faithfully flat as a B-module. (More generally, if A is a B-algebra, and M is an A-module, then M is faithfully flat over B if it is faithfully flat as a B-module.)

25.5.A. EXERCISE. Suppose M is a flat A-module. Show that the following are equivalent.

(a) M is faithfully flat;
(b) for all prime ideals $\mathfrak{p} \subset A$, $M \otimes_A \kappa(\mathfrak{p})$ is nonzero (i.e. $\operatorname{Supp} M = \operatorname{Spec} A$);
(c) for all maximal ideals $\mathfrak{m} \subset A$, $M \otimes_A \kappa(\mathfrak{m}) = M/\mathfrak{m}M$ is nonzero.

Suppose $\pi : X \to Y$ is a morphism of schemes, and $\mathcal{F}$ is a quasicoherent sheaf on X. We say that $\mathcal{F}$ is **faithfully flat** over Y if it is flat over Y, and $\operatorname{Supp} \mathcal{F} \to Y$ is surjective. We say that π is **faithfully flat** if it is flat and surjective (or equivalently, if $\mathcal{O}_X$ is faithfully flat over Y).

25.5.B. EXERCISE (CF. 25.5.A). Suppose $B \to A$ is a ring homomorphism and M is an A-module. Show that M is faithfully flat over B if and only if $\tilde{M}$ is faithfully flat over $\operatorname{Spec} B$. Show that A is faithfully flat over B if and only if $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat.

Faithful flatness is preserved by base change, as both surjectivity and flatness are (Exercises 10.4.D and 25.2.I respectively).

25.5.C. EXERCISE. Suppose $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is flat.
(a) Show that π is faithfully flat if and only if every *closed* point $x \in \operatorname{Spec} B$ is in the image of π. (Hint: Exercise 25.5.A(c).)
(b) Hence show that a flat homomorphism of local rings (Definition 7.3.1) is faithfully flat.

25.5.2. Going-Down for flat morphisms. A consequence of Exercise 25.5.C is the following useful result, whose statement makes no use of faithful flatness.

25.5.D. EXERCISE (GOING-DOWN THEOREM FOR FLAT MORPHISMS).
(a) Suppose that $B \to A$ is a flat morphism of rings, corresponding to a map

$\pi : \operatorname{Spec} A \to \operatorname{Spec} B$. Suppose $\mathfrak{q} \subset \mathfrak{q}'$ are prime ideals of B, and $\mathfrak{p}'$ is a prime ideal of A with $\pi([\mathfrak{p}']) = \mathfrak{q}'$. Show that there exists a prime $\mathfrak{p} \subset \mathfrak{p}'$ of A with $\pi([\mathfrak{p}]) = \mathfrak{p}'$. Hint: show that $B_{\mathfrak{q}'} \to A_{\mathfrak{p}'}$ is a flat local ring homomorphism, and hence faithfully flat by the Exercise 25.5.C(b).
(b) Part (a) gives a geometric consequence of flatnes. Draw a picture illustrating this.
(c) Recall the Going-Up Theorem, described in §8.2.4. State the Going-Down Theorem for flat morphisms in a way parallel to Exercise 8.2.F, and prove it.

25.5.E. IMPORTANT EXERCISE: FLAT MORPHISMS ARE OPEN IN REASONABLE SITUATIONS. Suppose $\pi : X \to Y$ is locally of finite type and flat, and Y (and hence X) is locally Noetherian. Show that π is an open map (i.e. sends open sets to open sets). Hint: reduce to showing that $\pi(X)$ is open. Reduce to the case where X is affine. Use Chevalley's Theorem 8.4.2 to show that $\pi(X)$ is constructible. Use the Going-Doing Theorem for flat morphisms, Exercise 25.5.D, to show that $\pi(X)$ is closed under specialization. Conclude using Exercise 8.4.B.

25.5.3. *Follow-ups to Exercise 25.5.E.* In quite reasonable circumstances, flat morphisms are *not* open: witness $\operatorname{Spec} k(t) \to \operatorname{Spec} k[t]$ (flat by Example 25.2.1(b)). But you can weaken the hypotheses of "locally of finite type" and "locally Noetherian" to just "locally finitely presented" [**EGA**, IV_2.2.4.6] — as with the similar generalization in Exercise 8.4.G of Chevalley's Theorem 8.4.2, use the fact that any such morphisms is "locally" pulled back from a Noetherian situation. We won't use this, and hence omit the details.

25.5.4. Dimensions of fibers of flat morphisms.

25.5.5. Proposition. — *Suppose $\pi : X \to Y$ is a flat morphism of locally Noetherian schemes, with $p \in X$ and $q \in Y$ such that $\pi(p) = q$. Then*

$$\operatorname{codim}_X p = \operatorname{codim}_Y q + \operatorname{codim}_{\pi^{-1} q} p.$$

Informal translation: the dimension of the fibers is the difference of the dimensions of X and Y (at least locally).

Compare this to Exercise 12.3.H, which stated that without the flatness hypothesis, we would only have inequality.

25.5.F. EXERCISE. Prove Proposition 25.5.5 as follows. Given a chain of irreducible closed subsets in Y containing $\overline{q}$, and a chain of irreducible closed subsets in $\pi^{-1} q \subset X$ containing $\overline{p}$, construct a chain of irreducible closed subsets in X containing $\overline{p}$, using the Going-Down Theorem for flat morphisms (Exercise 25.5.D).

As a consequence of Proposition 25.5.5, if $\pi : X \to Y$ is a flat map of irreducible varieties, then the fibers of π all have pure dimension $\dim X - \dim Y$. (Warning: $\operatorname{Spec} k[t]/(t) \to \operatorname{Spec} k[t]/(t^2)$ does not exhibit dimensional jumping of fibers, and also sends associated points to associated points, cf. Exercise 25.2.J, but is not flat.) This leads us to the following useful definition.

25.5.6. *Definition.* If a morphism $\pi : X \to Y$ is flat morphism of locally Noetherian schemes, and all fibers of π have pure dimension n, we say that π is **flat of relative dimension** n.

25.5.G. EXERCISE. Suppose $\pi : X \to Y$ is a flat morphism of locally Noetherian schemes, and Y is pure dimensional. Show that the following are equivalent.

(a) The scheme X has pure dimension $\dim Y + n$.
(b) The morphism π is flat of relative dimension n.

25.5.H. EXERCISE. Suppose $f : X \to Y$ and $g : Y \to Z$ are flat morphisms of locally Noetherian schemes, of relative dimension m and n respectively. Show that $g \circ f$ is flat of relative dimension $m + n$. Hint: use Exercise 25.5.G.

25.6 Flatness implies constant Euler characteristic

We come to an important consequence of flatness promised in §25.1. We will see that this result implies many answers and examples to questions that we would have asked before we even knew about flatness.

25.6.1. Important Theorem ($\chi(\mathcal{F})$ is constant in flat families). — *Suppose* $f : X \to Y$ *is a projective morphism of locally Noetherian schemes, and* $\mathcal{F}$ *is a coherent sheaf on* X*, flat over* Y*. Then* $\chi(X_y, \mathcal{F}_y) = \sum_{i \geq 0} (-1)^i h^i(X_y, \mathcal{F}|_y)$ *is a locally constant function of* $y \in Y$*.*

This is first sign that "cohomology behaves well in flat families." (We will soon see a second: the Semicontinuity Theorem 25.7.1. A second proof will be given after the semicontinuity discussion.)

The theorem also gives a necessary condition for flatness. Converses (yielding a sufficient condition) are given in Exercise 25.6.B(b)–(d).

Proof. The question is local on the target Y, so we may reduce to case Y is affine, say $Y = \operatorname{Spec} B$, so $X \hookrightarrow \mathbb{P}^n_B$ for some n. We may reduce to the case $X = \mathbb{P}^n_B$, by considering $\mathcal{F}$ as a sheaf on $\mathbb{P}^n_B$. We may reduce to showing that Hilbert polynomial $\mathcal{F}(m)$ is locally constant for all $m \gg 0$ (by Serre vanishing for $m \gg 0$, Theorem 20.1.3(b), the Hilbert polynomial agrees with the Euler characteristic). Twist by $\mathcal{O}(m)$ for $m \gg 0$, so that all the higher pushforwards vanish. Now consider the Čech complex $\mathcal{C}^\bullet(m)$ for $\mathcal{F}(m)$. Note that all the terms in the Čech complex are flat. As all higher cohomology groups (higher pushforwards) vanish, $\Gamma(\mathcal{C}^\bullet(m))$ is exact except at the first term, where the cohomology is $\Gamma(\pi_* \mathcal{F}(m))$. We add the module $\Gamma(\pi_* \mathcal{F}(m))$ to the front of the complex, so it is once again exact:

$$0 \longrightarrow \Gamma(\pi_* \mathcal{F}(m)) \longrightarrow \mathcal{C}^1(m) \longrightarrow \mathcal{C}^2(m) \longrightarrow \cdots$$

(We have done this trick of tacking on a module before, for example in (20.2.4.1).) Thus by Exercise 25.3.G, as we have an exact sequence in which all but the first terms are flat, the first term is flat as well. Thus $\pi_* \mathcal{F}(m)$ is a flat coherent sheaf on Y, and hence locally free (Corollary 25.4.3), and thus has locally constant rank.

Suppose $y \in Y$. We wish to show that the Hilbert function $h_{\mathcal{F}|_y}(m)$ is a locally constant function of y. To compute $h_{\mathcal{F}|_y}(m)$, we tensor the Čech resolution with

$\kappa(y)$ and take cohomology. Now the extended Čech resolution (with $\Gamma(\pi_*\mathcal{F}(m))$ tacked on the front) is an exact sequence of flat modules, and hence remains exact upon tensoring with $\kappa(y)$ (Exercise 25.3.D). (A vague interpretation that will be useful later: in this flat situation, cohomology commutes with arbitrary base change.) Thus $\Gamma(\pi_*\mathcal{F}(m)) \otimes \kappa(y) \cong \Gamma(\pi_*\mathcal{F}(m)|_y)$, so the Hilbert function $h_{\mathcal{F}|_y}(m)$ is the rank at y of a locally free sheaf, which is a locally constant function of y. $\square$

25.6.2. Corollary. — *Assume the same hypotheses and notation as in Theorem 25.6.1. Then the Hilbert polynomial of $\mathcal{F}$ is locally constant as a function of $y \in Y$.*

25.6.A. EXERCISE. Suppose $X \to Y$ is a projective flat morphism of locally Noetherian schemes, and Y is connected. Show that the following functions of $y \in Y$ are constant: (a) the degree of the fiber, (b) the dimension of the fiber, (c) the arithmetic genus of the fiber.

Another consequence of the corollary is something remarkably useful.

25.6.3. Corollary. — *An invertible sheaf on a flat projective family of connected curves has locally constant degree on the fibers.*

(Recall that the degree of a line bundle on a projective curve requires no hypotheses on the curve such as nonsingularity, see (20.4.8.1).)

Proof. An invertible sheaf $\mathcal{L}$ on a flat family of curves is always flat (as locally it is isomorphic to the structure sheaf). Hence $\chi(\mathcal{L}_y)$ is a constant function of y. By the definition of degree given in (20.4.8.1), $\deg(\mathcal{L}_y) = \chi(\mathcal{L}_y) - \chi(X_y)$. The result follows from the local constancy of $\chi(\mathcal{O}_{X_y})$ and $\chi(\mathcal{L}_y)$ (Theorem 25.6.1). $\square$

Before we get to the interesting consequences of Corollary 25.6.3, we briefly discuss a converse to Theorem 25.6.1.

25.6.B. EXERCISE (CONVERSES TO THEOREM 25.6.1). (We won't use this exercise for anything.)
(a) Suppose A is a ring, and $S_\bullet$ is a finitely generated A-algebra that is flat over A. Show that $\operatorname{Proj} S_\bullet$ is flat over A.
(b) Suppose $\pi : X \to Y$ is a projective morphism of locally Noetherian schemes (which as always includes the data of an invertible sheaf $\mathcal{O}_X(1)$ on X), such that $\pi_*\mathcal{O}_X(m)$ is locally free for all $m \geq m_0$ for some m_0. Show that π is flat. Hint: describe X as

$$\operatorname{Proj}\left(\mathcal{O}_Y \bigoplus (\oplus_{m \geq m_0} \pi_*\mathcal{O}_X(m))\right).$$

(c) More generally, suppose $\pi : X \to Y$ is a projective morphism of locally Noetherian schemes, and $\mathcal{F}$ is a coherent sheaf on X, such that $\pi_*\mathcal{F}(m)$ is locally free for all $m \geq m_0$ for some m_0. Show that $\mathcal{F}$ is flat over Y.
(d) Suppose $\pi : X \to Y$ is a projective morphism of locally Noetherian schemes, and $\mathcal{F}$ is a coherent sheaf on X, such that $\sum(-1)^i h^i(X_y, \mathcal{F}|_y)$ is a locally constant function of $y \in Y$. If Y is reduced, show that $\mathcal{F}$ must be flat over Y. (Hint: Exercise 14.7.J shows that constant rank implies local freeness in particularly nice circumstances.)

25.6.C. ⋆ EXERCISE: THE HILBERT SCHEME. As suggested in §25.1, the Hilbert functor of $\mathbb{P}^n$ parametrizes finitely presented closed subschemes of $\mathbb{P}^n$. (You may be more comfortable thinking about $\mathbb{P}^n_k$ for some field k, but this discussion applies equally well to the more general $\mathbb{P}^n_B$, so we deliberately leave the subscript out.) More precisely, it is a contravariant functor sending the scheme B to finitely presented closed subschemes of $X \times_{\mathbb{Z}} B$ flat over B (and sending morphisms $B_1 \to B_2$ to pullbacks of flat families).
(a) Suppose that the Hilbert functor is representable, by a scheme $\mathrm{Hilb}_{\mathbb{P}^n}$, the **Hilbert scheme**. (This is true; as mentioned in §25.1, [**M-CAS**] gives a readable construction.) Show that $\mathrm{Hilb}_{\mathbb{P}^n_k}$ is the disjoint union of schemes, each one corresponding to finitely presented closed subschemes with fixed Hilbert polynomial. Hint: Corollary 25.6.3.
(b) Suppose further that $\mathrm{Hilb}_{\mathbb{P}^n}$ is locally of finite type. Use the valuative criterion of properness (Theorem 13.5.6) to show that it is proper. Hint: Exercise 25.4.I. (In practice it is simpler to show that it is projective, and hence proper, by construction.)

25.6.4. ⋆ **Intersection numbers are constant in flat families.** (This discussion is only for those who have read starred Chapter 22 on intersection theory.)

25.6.5. *Exercise (intersection numbers are locally constant in flat families).* Suppose $\pi : X \to B$ is a proper morphism to a connected scheme; $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are line bundles on X; and $\mathcal{F}$ is a coherent sheaf on X flat over B such that the support of $\mathcal{F}$ when restricted to any fiber of π has dimension at most n. If b is any point of B, define (the temporary notation) $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F})_b$ to be the intersection on the fiber X_b of $\mathcal{L}_1, \ldots, \mathcal{L}_n$ with $\mathcal{F}|_{X_b}$ (Definition 22.1.1). Show that $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F})_b$ is independent of b.

25.6.6. *Line bundles have locally constant degree in proper flat families.* As an example, by Exercise 22.1.B(a), if X is a proper flat family of curves (i.e. π is flat of relative dimension 1, proper) over a connected base, and $\mathcal{L}$ is a line bundle on X, then the degree $\mathcal{L}$ on fibers is constant. This motivates the following definition.

25.6.7. *Definition.* Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles on a k-variety X. We say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are **algebraically equivalent** if there exists a connected k-variety B with two k-valued points p_1 and p_2, and a line bundle $\mathcal{L}$ on $X \times B$ such that the restriction of $\mathcal{L}$ to the fibers X_{p_1} and X_{p_2} are isomorphic to $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively.

25.6.D. EXERCISE. Show that "algebraic equivalence" is an equivalence relation. Show that the line bundles algebraically equivalent to $\mathcal{O}$ form a subgroup of $\operatorname{Pic} X$. This subgroup is denoted $\operatorname{Pic}^0 X$. Identify the group of line bundles $\operatorname{Pic} X$ modulo algebraic equivalence with $\operatorname{Pic} X / \operatorname{Pic}^0 X$.

This quotient is called the **Néron-Severi group**. This definition was promised in §20.4.11. Note that by Proposition 22.1.4, $\operatorname{Pic}^\tau X \subset \operatorname{Pic}^0 X$.

25.6.8. ⋆ **Hironaka's example of a nonsingular threefold over a field.**

In §17.4.8, we produced a proper nonprojective variety, but it was singular. We can use §25.6.4 to give a *nonsingular* example, due to Hironaka.

Inside $\mathbb{P}^3_k$, fix two conics C_1 and C_2, which meet in two (k-valued) points, p_1 and p_2. We construct a proper map $\pi : X \to \mathbb{P}^3_k$ as follows. Away from p_i, we blow up C_i and then the proper transform of C_{3-i}. This is well-defined, as away from p_1 and p_2, C_1 and C_2 are disjoint, blowing up one and then the other is the same as blowing up their union, and thus the order doesn't matter.

Note that π is proper, as it is proper away from p_1, and proper away from p_2, and the notion of properness is local on the base (Proposition 11.3.4(b)). As X is projective hence proper (over k), and compositions of proper morphisms are proper (Proposition 11.3.4(c)), X is proper.

25.6.E. EXERCISE. Show that X is nonsingular. (Hint: use Theorem 19.4.13 to show that it is smooth.) Let E_i be the preimage of $C_i \setminus \{p_1, p_2\}$. Show that $\pi|_{E_i} \to C_i \setminus \{p_1, p_2\}$ is a $\mathbb{P}^1$-bundle (and flat).

25.6.F. EXERCISE. Show that $\overline{E_i} \to C_i$ is flat. (Hint: Exercise 25.4.H.)

25.6.G. EXERCISE. Show that $\pi^*(p_i)$ is the union of two $\mathbb{P}^1$'s, say Y_i and Z_i, meeting at a point, such that $Y_i, Y_{3-i}, Z_{3-i} \in E_i$ but $Z_i \notin E_i$.

25.6.H. EXERCISE. Show that X is not proper as follows. Suppose otherwise $\mathcal{L}$ is a very ample line bundle on X, so $\mathcal{L}$ has positive degree on every curve (including the Y_i and Z_i). Using flatness of $E_i \to C_i$, and constancy of degree in flat families (Exercise 25.6.6), show that $\deg_{Y_i} \mathcal{L} = \deg_{Y_{3-i}} \mathcal{L} + \deg_{Z_{3-i}} \mathcal{L}$. Obtain a contradiction. (This argument may remind you of the argument of §17.4.8.)

25.6.9. *The notion of "projective morphism" is not local on the target.* Note that $\pi : X \to \mathbb{P}^3$ is not projective, as otherwise X would be projective (as the composition of projective morphisms is projective, if the final target is quasicompact, Exercise 18.3.E). But away from p_i, π *is* projective (as it is a composition of blowups, which are projective by construction, and the final target is quasicompact, so Exercise 18.3.E applies). Thus the notion of "projective morphism" is not local on the target.

25.6.10. *Unimportant remark.* You can construct more fun examples with this idea. For example, we know that projective surfaces can be covered by three affine open sets (see the proof of Theorem 20.2.6. This can be used to give an example of (for any N) a proper surface that requires at least N affine open subsets to cover it.

25.7 Cohomology and base change: Statements and applications

Higher pushforwards are easy to define, but it is hard to get a geometric sense of what they are, or how they behave. For example, given a reasonable morphism $\pi : X \to Y$, and a quasicoherent sheaf on $\mathcal{F}$, you might reasonably hope that the fibers of $R^i\pi_*\mathcal{F}$ are the cohomologies of $\mathcal{F}$ along the fibers. More precisely, given $f : y \to Y$ corresponding to the inclusion of a point (better: $f : \operatorname{Spec} \mathcal{O}_{Y,y} \to Y$), yielding the fibered diagram

(25.7.0.1)
$$\begin{tikzcd} X_y \arrow[r, "f'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ y \arrow[r, "f"] & Y \end{tikzcd},$$

one might hope that the morphism

$$\boxed{\phi^p_y : f^*(R^p\pi_*\mathcal{F}) \to H^p(X_y, (f')^*\mathcal{F})}$$

(given in Exercise 20.7.B(a)) is an isomorphism. We could then picture $R^i\pi_*\mathcal{F}$ as somehow fitting together the cohomology groups of fibers into a coherent sheaf.

It would also be nice if $H^p(X_y, (f')^*\mathcal{F})$ was constant, and ϕ^p_y put them together into a nice locally free sheaf (vector bundle) $f^*(R^p\pi_*\mathcal{F})$.

There is no reason to imagine that the particular choice of base change $f : y \mapsto Y$ should be special. As long as we are dreaming, we may as well hope that in good circumstances, given a fiber diagram (20.7.2.1)

(25.7.0.2)
$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Z \arrow[r, "f"] & Y, \end{tikzcd}$$

the natural map

$$\boxed{\phi^p_Z : f^*(R^p\pi_*\mathcal{F}) \to R^p\pi'_*(f')^*\mathcal{F}}$$

of sheaves on Z (Exercise 20.7.B(a)) is an isomorphism. (In some cases, we can already address this question. For example, cohomology commutes with flat base change, Theorem 25.2.8, so the result holds if g is flat. Also, if $\mathcal{F}$ is flat over Y, then the Euler characteristic of $\mathcal{F}$ on fibers is locally constant, Theorem 25.6.1.)

There is no point in dreaming if we are not going to try to make them come true. So let's formalize our dreams. Suppose $\mathcal{F}$ is a coherent sheaf on X, $\pi : X \to Y$ is projective, Y (hence X) is Noetherian, and $\mathcal{F}$ is flat over Y. We formalize our dreams into three nice properties that we might wish in this situation. We will see that they are closely related.

(a) Given a fibered square (25.7.0.1), is $\phi^p_y : R^p\pi_*\mathcal{F} \otimes \kappa(y) \to H^p(X_y, \mathcal{F}_y)$ an isomorphism?
(b) Given a fibered square (25.7.0.2), is $\phi^p_Z : f^*(R^p\pi_*\mathcal{F}) \to R^p\pi'_*(f')^*\mathcal{F}$ an isomorphism?
(c) Is $R^p\pi_*\mathcal{F}$ locally free?

We turn first to property (a). The dimension of the left side $R^p\pi_*\mathcal{F} \otimes \kappa(y)$ is an upper semicontinuous function of $y \in Y$ by upper semicontinuity of rank of finite type quasicoherent sheaves (Exercise 14.7.I). The Semicontinuity Theorem states that the dimension of the right is also upper semicontinuous. More formally:

25.7.1. Semicontinuity theorem. — *Suppose $X \to Y$ is a proper morphism of Noetherian schemes, and $\mathcal{F}$ is a coherent sheaf on X flat over Y. Then for each $p \geq 0$, the function $Y \to \mathbb{Z}$ given by $y \mapsto \dim_{\kappa(y)} H^p(X_y, \mathcal{F}_y)$ is upper semicontinuous on Y.*

Translation: cohomology groups are upper semicontinuous in proper flat families. (A proof will be given in the §25.8.4.)

You may already have seen an example of cohomology groups jumping, at §25.4.7. Here is a simpler example, albeit not of the structure sheaf. Let (E, p_0) be an elliptic curve over a field k, and consider the projection $\pi : E \times E \to E$. Let $\mathcal{L}$ be the invertible sheaf (line bundle) corresponding to the divisor that is the diagonal, minus the section $p_0 \in E$. Then $\mathcal{L}_{p_0}$ is trivial, but $\mathcal{L}_p$ is non-trivial for any $p \neq p_0$ (as we showed in our study of genus 1 curves, in §21.8). Thus $h^0(E, \mathcal{L}_p)$ is 0 in general, but jumps to 1 for $p = p_0$.

25.7.2. *Side remark.* Cohomology of $\mathcal{O}$ doesn't jump in flat families in characteristic 0 if the fibers are nonsingular varieties. (Such maps will be called *smooth morphisms* soon.) Over $\mathbb{C}$, this is because Betti numbers are constant in connected families, and (23.4.11.1) from Hodge theory expresses the Betti constants h^k_{Betti} as sums of upper semicontinuous (and hence constant) functions $\sum_{i+j=k} h^j(\Omega^i)$, so the Hodge numbers $h^j(\Omega^i)$ must be constant. The general characteristic 0 case can be reduced to $\mathbb{C}$. But cohomology groups of $\mathcal{O}$ (for flat families of varieties) *can* jump in positive characteristic. Also, the example of §25.4.7 shows that "smoothness" is necessary.

25.7.3. Grauert's theorem. If $R^p\pi_*\mathcal{F}$ is locally free (property (c)) and ϕ^p_y is an isomorphism (property (a)), then $h^p(X_y, \mathcal{F}_y)$ is locally constant. The following is a partial converse.

25.7.4. Grauert's Theorem. — *If π is proper, $\mathcal{F}$ is flat over Y, Y is reduced, and $h^p(X_y, \mathcal{F}_y)$ is a locally constant function of $y \in Y$, then $R^p\pi_*\mathcal{F}$ is locally free, and ϕ^p_y is an isomorphism for all $y \in Y$.*

In other words, if cohomology groups of fibers have locally constant dimension (over a reduced base), then they can be fit together to form a vector bundle. (No Noetherian hypotheses are needed.) The following statement is even more magical.

25.7.5. Cohomology and Base Change Theorem. — *Suppose π is proper, $\mathcal{F}$ is coherent and flat over Y, and ϕ^p_y is surjective. Then the following hold.*

- (i) *Then ϕ^p_y is an isomorphism, and the same is true nearby.*
- (ii) *Furthermore, ϕ^{p-1}_y is surjective (hence isomorphic by (a)) if and only if $R^p\pi_*\mathcal{F}$ is locally free. This in turn implies that h^p is locally constant.*

(Proofs of Theorems 25.7.4 and 25.7.5 will be given in the next section.)

This is amazing: the hypothesis involve what happens only over *reduced* points, and it has implications over the (possibly nonreduced) scheme as a whole! This might remind you of the local criterion for flatness (Theorem 25.9.3), and indeed that is the key ingredient of the proof.

Here are some consequences.

25.7.A. EXERCISE. Suppose $h^0(X_y, \mathcal{F}_y)$ is constant. Show that $\pi_*\mathcal{F}$ is locally free. (If Y were reduced, then this would just be Exercise 14.7.J.) Informal translation: if a flat sheaf has a constant number of global sections, the pushforward sheaf is a vector bundle fitting together the spaces of global sections on the fibers.

25.7.B. EXERCISE. Suppose $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$. Show that ϕ^{p-1} is an isomorphism for all $y \in Y$.

25.7.C. EXERCISE. Suppose $R^p\pi_*\mathcal{F} = 0$ for $p \geq p_0$. Show that $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$, $p \geq p_0$.

25.7.D. $\star$ EXERCISE (THE HODGE BUNDLE). Suppose $\pi : X \to B$ is a flat family of nonsingular curves of genus g. Serre duality for families involves a unique (up to isomorphism) invertible sheaf $\omega_{X/B}$ that restricts to the dualizing sheaf on each fiber. The case where B is $\operatorname{Spec} k$ yields the dualizing sheaf discussed in Theorem 20.4.6. This sheaf behaves well with respect to pullback: given a fibered square

(25.7.5.1)
$$\begin{tikzcd} X' \arrow[r, "\rho"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ B' \arrow[r, "\sigma"] & B, \end{tikzcd}$$

there is an isomorphism $\omega_{X'/B'} \cong \rho^*\omega_{X/B}$. Thus the fibers of $\omega_{X/B}$ are the dualizing sheaves of the fibers. Assuming all this, show that $\pi_*\omega_{X/B}$ is a locally free sheaf of rank g. This is called the **Hodge bundle**. Show that the construction of the Hodge bundle commutes with base change, i.e. given (25.7.5.1), describe an isomorphism $\sigma^*\pi_*\omega_{X/B} \cong \pi'_*\omega_{X'/B'}$.

25.8 $\star$ Universally computing cohomology of flat sheaves using a complex of vector bundles, and proofs of cohomology and base change theorems

The key to proving the Semicontinuity Theorem 25.7.1, Grauert's Theorem 25.7.4, and the Cohomology and Base Change Theorem 25.7.5 is the following wonderful result of Mumford's [**MAV**]. It turns questions of pushforwards (and how they behave under arbitrary base change) into something computable involving vector bundles (linear algebra). After stating it, we will interpret it.

25.8.1. Key Theorem. — *Suppose* $\pi : X \to \operatorname{Spec} B$ *is a proper morphism, and* $\mathcal{F}$ *is a coherent sheaf on* X*, flat over* $\operatorname{Spec} B$*. Then there is a complex*

(25.8.1.1)
$$\cdots \to K^{-1} \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely generated free B*-modules and an isomorphism of functors*

(25.8.1.2)
$$H^p(X \times_B A, \mathcal{F} \otimes_B A) \cong H^p(K^\bullet \otimes_B A)$$

for all p*, for all ring maps* $B \to A$*.*

Because (25.8.1.1) is an exact sequence of *free* B-modules, all of the information is contained in the maps, which are matrices with entries in B. This will turn questions about cohomology (and base change) into questions about linear algebra. For example, semicontinuity will turn into the fact that ranks of matrices (with functions as entries) drop on closed subsets.

Although the complex (25.8.1.1) is infinite, by (25.8.1.2) it has no cohomology in negative degree, even after any ring extension $B \to A$ (as the left side of (25.8.1.2) is 0 for $p < 0$).

The idea behind the proof is as follows: take the Čech complex, produce a complex mapping to it with the same cohomology (a quasiisomorphic complex, cf. §20.2.3) of free modules. We will first construct the complex so that (25.8.1.2) holds for $B = A$, and then show the same complex works for general A later. We begin with a lemma.

25.8.2. Lemma. — *Let $C^\bullet$ be a complex of B-modules such that $H^i(C^\bullet)$ are finitely generated B-modules, and such that that $C^p = 0$ for $p > n$. Then there exists a complex $K^\bullet$ of finitely generated free B-modules such that $K^p = 0$ for $p > n$, and a homomorphism of complexes $\phi : K^\bullet \to C^\bullet$ such that ϕ induces isomorphisms $H^i(K^\bullet) \to H^i(C^\bullet)$ for all i.*

Proof. We build this complex inductively

(25.8.2.1)
$$\begin{tikzcd} & & & K^{m+1} \arrow[r, "\delta^{m+1}"] \arrow[d, "\alpha^{m+1}"] & K^{m+2} \arrow[r] \arrow[d, "\alpha^{m+2}"] & \cdots \\ \cdots \arrow[r] & C^{m-1} \arrow[r] & C^m \arrow[r, "\delta^m"'] & C^{m+1} \arrow[r, "\delta^{m+1}"'] & C^{m+2} \arrow[r] & \cdots . \end{tikzcd}$$

Assume we have defined $(K^p, \alpha^p, \delta^p)$ for $p \geq m+1$ such that the squares ("α and δ") commute, and the top row is a complex, and ϕ^p defines an isomorphism of cohomology $H^q(K^\bullet) \to H^q(C^\bullet)$ for $q \geq m+2$ and a surjection $\ker \delta^{m+1} \to H^{m+1}(C^\bullet)$, and the K^p are finitely generated B-modules. (Our base case is $m = p$; we just take $K^n = 0$ for $n > p$.)

We construct $(K^m, \delta^m, \alpha^m)$. Choose generators of $H^m(C^\bullet)$, say $c_1, \dots, c_M$. Let $D^{m+1} = \ker(\delta^{m+1} : H^{m+1}(K^\bullet) \to H^{m+1}(C^\bullet))$. Choose generators of D^{m+1}, say $d_1, \dots, d_N$. Let $K^m = B^{\oplus(M+N)}$. Define α^m by sending the first M generators of $B^{\oplus(M+N)}$ to (lifts of) $c_1, \dots, c_M$. Send the last N generators to 0. Define δ^m by sending the last N generators to (lifts of) $d_1, \dots d_N$. Send the first M generators to 0. Then by construction, we have completed our inductive step:

$$\begin{tikzcd} & & K^m \arrow[r, "\delta^m"] \arrow[d, "\alpha^m"] & K^{m+1} \arrow[r, "\delta^{m+1}"] \arrow[d, "\alpha^{m+1}"] & K^{m+2} \arrow[r] \arrow[d, "\alpha^{m+2}"] & \cdots \\ \cdots \arrow[r] & C^{m-1} \arrow[r] & C^m \arrow[r, "\delta^m"'] & C^{m+1} \arrow[r, "\delta^{m+1}"'] & C^{m+2} \arrow[r] & \cdots . \end{tikzcd}$$

□

25.8.3. Lemma. — *Suppose $\alpha : K^\bullet \to C^\bullet$ is a morphism of complexes of* **flat** *B-modules inducing isomorphisms of cohomology (a "quasiisomorphism", cf. 20.2.3). Then for every B-algebra A, the maps $H^p(C^\bullet \otimes_B A) \to H^p(K^\bullet \otimes_B A)$ are isomorphisms.*

Proof. The mapping cone $M^\bullet$ of $\alpha : K^\bullet \to C^\bullet$ is exact by Exercise 2.7.E. Then $M^\bullet \otimes_B A$ is still exact, by Exercise 25.3.D. But $M^\bullet \otimes_B A$ is the mapping cone of

$\alpha \otimes_B A : K^\bullet \otimes_B A \to C^\bullet \otimes_B A$, so by Exercise 2.7.E, $\alpha \otimes_B A$ induces an isomorphism of cohomology (is a quasiisomorphism) too. $\square$

Proof of Key Theorem 25.8.1. Choose a finite affine covering of X. Take the Čech complex $C^\bullet$ for $\mathcal{F}$ with respect to this cover. Recall that Grothendieck's Coherence Theorem 20.8.1 showed that the cohomology of $\mathcal{F}$ is coherent. (That Theorem required serious work. If you need Theorem 25.8.1 only in the projective case, the analogous statement with projective hypotheses Theorem 20.7.1(d), was much easier.) Apply Lemma 25.8.2 to get the nicer version $K^\bullet$ of the same complex $C^\bullet$. Apply Lemma 25.8.3 to see that if you tensor with B and take cohomology, you get the same answer whether you use $K^\bullet$ or $C^\bullet$. $\square$

We now use Theorem 25.8.1 to prove some of the fundamental results stated earlier: the Semicontinuity theorem 25.7.1, Grauert's theorem 25.7.4, and the Cohomology and base change theorem 25.7.5. In the course of proving Semicontinuity, we will give a new proof of Theorem 25.6.1, that Euler characteristics are locally constant in flat families. This short proof will have the additional advantage that once we prove Theorem 25.8.1 with proper rather than projective hypotheses, we will also know Theorem 25.6.1 with these hypotheses.

25.8.4. *Proof of the Semicontinuity Theorem 25.7.1.* The result is local on Y, so we may assume Y is affine. Let $K^\bullet$ be a complex as in Key Theorem 25.8.1.

Then for $y \in Y$,

$$\begin{aligned}\dim_{\kappa(y)} H^p(X_y, \mathcal{F}_y) &= \dim_{\kappa(y)} \ker(d^p \otimes_A \kappa(y)) - \dim_{\kappa(y)} \operatorname{im}(d^{p-1} \otimes_A \kappa(y)) \\ &= \dim_{\kappa(y)}(K^p \otimes \kappa(y)) - \dim_{\kappa(y)} \operatorname{im}(d^p \otimes_A \kappa(y)) \\ &\qquad - \dim_{\kappa(y)} \operatorname{im}(d^{p-1} \otimes_A \kappa(y)) \end{aligned} \tag{25.8.4.1}$$

Now $\dim_{\kappa(y)} \operatorname{im}(d^p \otimes_A \kappa(y))$ is a lower semicontinuous function on Y. (Reason: the locus where the dimension is less than some number q is obtained by setting all $q \times q$ minors of the matrix $K^p \to K^{p+1}$ to 0.) The same is true for $\dim_{\kappa(y)} \operatorname{im}(d^{p-1} \otimes_A \kappa(y))$. The result follows. $\square$

25.8.5. A new proof of Theorem 25.6.1 that Euler characteristics of flat sheaves are locally constant.

If $K^\bullet$ were finite "on the left" as well — if $K^p = 0$ for $p \ll 0$ — then we would have a short proof of Theorem 25.6.1. By taking alternating sums (over p) of (25.8.4.1), we would have that

$$\chi(X_y, \mathcal{F}_y) = \sum (-1)^p h^p(X_y, \mathcal{F}_y) = \sum (-1)^p \operatorname{rank} K^p,$$

which is locally constant. The only problem is that the sums are infinite. We patch this problem as follows. Define a $J^\bullet \to K^\bullet$ by $J^p = K^p$ for $p \geq 0$, $J^p = 0$ for $p < -1$,

$$J^{-1} := \ker(K^0 \to K^1),$$

and the obvious map $J^\bullet \to K^\bullet$. Clearly this induces an isomorphism on cohomology (as $J^\bullet$ patently has the same cohomology as $K^\bullet$ at step $p \geq 0$, and both have 0 cohomology for $p < 0$). Thus $J^\bullet \to C^\bullet$ induces an isomorphism on cohomology.

Now J^{-1} is coherent (as it is the kernel of a map of coherent modules). Consider the mapping cone $M^\bullet$ of $J^\bullet \to C^\bullet$:

$$0 \to J^{-1} \to C^{-1} \oplus J^0 \to C^0 \oplus J^1 \to \cdots \to C^{n-1} \oplus J^n \to C^n \to 0.$$

From Exercise 2.7.E, as $J^\bullet \to C^\bullet$ induces an isomorphism on cohomology, the mapping cone has no cohomology (is exact). All terms in it are flat except possibly J^{-1} (the C^p are flat by assumption, and J^i is free for $i \neq -1$). Hence J^{-1} is flat too, by Exercise 25.3.G. But flat coherent sheaves over a Noetherian ring are locally free (Theorem 25.4.3). Then Theorem 25.6.1 follows from

$$\chi(X_y, \mathcal{F}_y) = \sum (-1)^p h^p(X_y, \mathcal{F}_y) = \sum (-1)^p \operatorname{rank} J^p.$$

□

25.8.6. Proof of Grauert's Theorem 25.7.4 and the Cohomology and Base Change Theorem 25.7.5.

Thanks to Theorem 25.8.1.2, Theorems 25.7.4 and 25.7.5 are now statements about complexes of free modules over a Noetherian ring. We begin with some general comments on dealing with the cohomology of a complex

$$\cdots \longrightarrow K^p \xrightarrow{\delta^p} K^{p+1} \longrightarrow \cdots.$$

We define some notation for functions on a complex.

- Let Z^p be the kernel of the pth differential of a complex, so for example $Z^p K^\bullet = \ker \delta^p$.
- Let B^{p+1} be the image of the pth differential, so for example $B^{p+1}K^\bullet = \operatorname{im} \delta^p$.
- Let W^{p+1} be the cokernel of the pth differential, so for example $W^{p+1}K^\bullet = \operatorname{coker} \delta^p$.
- As usual, let H^p be the homology at the pth step.

We have exact sequences

(25.8.6.1) $$0 \longrightarrow Z^p \longrightarrow K^p \longrightarrow K^{p+1} \longrightarrow W^{p+1} \longrightarrow 0$$

(25.8.6.2) $$0 \longrightarrow Z^p \longrightarrow K^p \longrightarrow B^{p+1} \longrightarrow 0$$

(25.8.6.3) $$0 \longrightarrow B^p \longrightarrow Z^p \longrightarrow H^p \longrightarrow 0$$

(25.8.6.4) $$0 \longrightarrow B^p \longrightarrow K^p \longrightarrow W^p \longrightarrow 0$$

(25.8.6.5) $$0 \longrightarrow H^p \longrightarrow W^p \longrightarrow B^{p+1} \longrightarrow 0$$

We proceed by a series of exercises, some of which were involved in the proof of the FHHF Theorem (Exercise 2.6.H). Suppose $C^\bullet$ is any complex in an abelian category $\mathcal{A}$ with enough projectives, and suppose F is any right-exact functor from $\mathcal{A}$.

25.8.A. EXERCISE (COKERNELS COMMUTE WITH RIGHT-EXACT FUNCTORS). Describe an *isomorphism* $\gamma^p : FW^pC^\bullet \xrightarrow{\sim} W^pFC^\bullet$. (Hint: consider $C^{p-1} \to C^p \to W^pC^\bullet \to 0$.)

25.8.B. EXERCISE. (a) Describe a map $\beta^p : FB^pC^\bullet \to B^pFC^\bullet$. Hint: (25.8.6.4) induces

$$\begin{tikzcd}
R^1FW^pC^\bullet \arrow[r] & FB^pC^\bullet \arrow[r] \arrow[d, dotted, "\beta^p"] & FC^p \arrow[r] \arrow[d, "="] & FW^pC^\bullet \arrow[r] \arrow[d, "\gamma^p"', "\sim"] & 0 \\
0 \arrow[r] & B^pFC^\bullet \arrow[r] & FC^p \arrow[r] & W^pFC^\bullet \arrow[r] & 0.
\end{tikzcd}$$

(b) Show that β^p is surjective. Possible hint: use Exercise 2.7.B, a weaker version of the snake lemma, to get an exact sequence

$$\boxed{\begin{tikzcd}
R^1FC^p \arrow[r] & R^1FW^pC^\bullet \arrow[r] & \ker \beta^p \arrow[r] & 0 \arrow[r] & \ker \gamma^p \\
 & {} \arrow[r] & \operatorname{coker} \beta^p \arrow[r] & 0 \arrow[r] & \operatorname{coker} \gamma^p \arrow[r] & 0.
\end{tikzcd}}$$

25.8.C. EXERCISE. (a) Describe a map $\alpha^p : FZ^pC^\bullet \to Z^pFC^\bullet$. Hint: use (25.8.6.2) to induce

$$\begin{tikzcd}
R^1FB^{p+1}C^\bullet \arrow[r] & FZ^pC^\bullet \arrow[r] \arrow[d, dotted, "\alpha^p"] & FC^p \arrow[r] \arrow[d, "="] & FB^{p+1}C^\bullet \arrow[r] \arrow[d, "\beta^{p+1}"] & 0 \\
0 \arrow[r] & Z^pFC^\bullet \arrow[r] & FC^p \arrow[r] & B^{p+1}FC^\bullet \arrow[r] & 0
\end{tikzcd}$$

(b) Use Exercise 2.7.B to get an exact sequence

$$\boxed{\begin{tikzcd}
R^1FC^\bullet \arrow[r] & R^1FB^{p+1}C^\bullet \arrow[r] & \ker \alpha^p \arrow[r] & 0 \arrow[r] & \ker \beta^{p+1} \\
 & {} \arrow[r] & \operatorname{coker} \alpha^p \arrow[r] & 0 \arrow[r] & \operatorname{coker} \beta^{p+1} \arrow[r] & 0.
\end{tikzcd}}$$

25.8.D. EXERCISE. (a) Describe a map $\phi^p : FHK^p \to HFK^p$. (This is the FHHF Theorem, Exercise 2.6.H(a).) Hint: (25.8.6.3) induces

$$\begin{tikzcd}
R^1FH^pC^\bullet \arrow[r] & FB^pC^\bullet \arrow[r] \arrow[d, "\beta^p"] & FZ^pC^\bullet \arrow[r] \arrow[d, "\alpha^p"] & FH^pC^\bullet \arrow[r] \arrow[d, dotted, "\phi^p"] & 0 \\
0 \arrow[r] & B^pFC^\bullet \arrow[r] & Z^pFC^\bullet \arrow[r] & H^pFC^\bullet \arrow[r] & 0
\end{tikzcd}$$

(b) Use Exercise 2.7.B to get an exact sequence:

$$\boxed{\begin{tikzcd}
R^1FZ^pC^\bullet \arrow[r] & R^1FH^pC^\bullet \arrow[r] & \ker \beta^p \arrow[r] & \ker \alpha^p \arrow[r] & \ker \phi^p \\
 & {} \arrow[r] & \operatorname{coker} \beta^p \arrow[r] & \operatorname{coker} \alpha^p \arrow[r] & \operatorname{coker} \phi^p \arrow[r] & 0.
\end{tikzcd}}$$

25.8.7. *Back to the theorems we want to prove.* Recall the properties we discussed at the start of §25.7.

(a) Given a fibered square (25.7.0.1), is $\phi^p_y : R^p\pi_*\mathcal{F} \otimes \kappa(y) \to H^p(X_y, \mathcal{F}_y)$ an isomorphism?

(b) Given a fibered square (25.7.0.2), is $\phi^p_Z : f^*(R^p\pi_*\mathcal{F}) \to R^p\pi'_*(f')^*\mathcal{F}$ an isomorphism?
(c) Is $R^p\pi_*\mathcal{F}$ locally free?

We reduce to the case Y and Z are both affine, say $Y = \operatorname{Spec} B$. We apply our general results of §25.8.6 to the complex (25.8.1.1) of Theorem 25.8.1.

25.8.E. EXERCISE. Suppose $W^pK^\bullet$ and $W^{p+1}K^\bullet$ are flat. Show that the answer to (b), and hence (a), is yes. Show that the answer to (c) is yes if Y is reduced or locally Noetherian. Hint: (You will take F to be the functor $\otimes_B A$, where A is some B-algebra.) Use (25.8.6.4) (shifted) to show that $B^{p+1}K^\bullet$ is flat, and then (25.8.6.5) to show that $H^pK^\bullet$ is flat. By Exercise 25.8.A, the construction of the cokernel $W^\bullet$ behaves well under base change. The flatness of B^{p+1} and H^p imply that their constructions behave well under base change as well — apply F to the (25.8.6.4) and (25.8.6.5) respectively. (If you care, you can check that $Z^pK^\bullet$ is also locally free, and behaves well under base change.)

25.8.F. EXERCISE. Prove Grauert's Theorem 25.7.4. (You won't need Noetherian hypotheses.) Hint: By (25.8.4.1), $W^pK^\bullet$ and $W^{p+1}K^\bullet$ have constant rank. But finite type quasicoherent sheaves having constant rank on a reduced scheme are locally free (Exercise 14.7.J), so we can invoke Exercise 25.8.E. Conclude that $H^pK^\bullet$ is flat of constant rank, and hence locally free.

25.8.8. *Proof of the Cohomology and Base Change Theorem 25.7.5.* Keep in mind that we now have locally Noetherian hypotheses. We reduce to the case Y and Z are both affine, say $Y = \operatorname{Spec} B$. Let $F = \cdot \otimes_B \kappa(y)$ The key input is the local criterion for flatness (Theorem 25.9.3): $R^1FW^qK^\bullet = 0$ if and only if $FW^qK^\bullet$ is flat at y, by the local criterion for flatness (and similarly with W replaced by other letters). In particular, $R^1FK^q = 0$ for all q.

25.8.G. EXERCISE. Look at the boxed snakes in §25.8.6 (with $C^\bullet = K^\bullet$), and show the following in order, starting from the assumption that $\operatorname{coker} \phi^p = 0$:

- $\operatorname{coker} \alpha^p = 0$, $\ker \beta^{p+1} = 0$, $R^1FW^{p+1}K^\bullet = 0$;
- $W^{p+1}K^\bullet$ is flat, $B^{p+1}K^\bullet$ is flat (use (25.8.6.4) with the indexing shifted by one), $Z^pK^\bullet$ is flat (use (25.8.6.3));
- $R^1FB^{p+1}K^\bullet = 0$;
- $\ker \alpha^p = 0$, $\ker \phi^p = 0$.

It might be useful for later to note that

$$R^1FW^pK^\bullet \cong \ker \beta^p \cong R^1FH^pK^\bullet$$

At this point, we have shown that ϕ^p_y is an isomorphism — half of part (i) of the theorem. Also, ϕ^{p-1}_y surjective implies $W^pK^\bullet$ is flat (in the same way that you showed ϕ^p_y surjective implies $W^{p+1}K^\bullet$ is flat), so we get H^p is free by Exercise 25.8.E, yielding half of (ii).

25.8.H. EXERCISE. For the other direction of (b), shift the grading of the last two boxed snakes down by one, to obtain further isomorphisms

$$\ker \beta^p \cong \operatorname{coker} \alpha^{p-1} \cong \operatorname{coker} \phi^{p-1}.$$

For the other direction of (a), note that if the stalks $W^p K^\bullet$ and $W^{p+1} K^\bullet$ at y are flat, then they are locally free (as they are coherent, by Theorem 25.4.2), and hence $W^p K^\bullet$ and $W^{p+1} K^\bullet$ are locally free in a *neighborhood* of y by Exercise 14.7.E. Thus the stalks of $W^p K^\bullet$ and $W^{p+1} K^\bullet$ are flat in a neighborhood of y, and the same argument applies for any point in this neighborhood to show that $W^{p+1} K^\bullet$, $B^{p+1} K^\bullet$, and $Z^p K^\bullet$ are all flat.

25.8.I. EXERCISE. Use this to show the following, possibly in order:

- $R^1 FC^{p+1} = R^1 FB^{p+1} = R^1 Z^p = 0$.
- $\ker \beta^{p+1} = 0$, $\operatorname{coker} \alpha^p = 0$, $\operatorname{coker} \phi^p = 0$.

25.8.J. EXERCISE. Put all the pieces together and verify that the Cohomology and Base Change Theorem 25.7.5 is now proved.

□

25.9 Harder facts: local and slicing criteria for flatness, and applications

25.9.1. A hard but useful fact: The local criterion for flatness.
In the case of a Noetherian local ring, there is a greatly improved version of the ideal-theoretic criterion given in Theorem 25.4.1: we need check only one ideal — the maximal ideal. The price we pay for the simplicity of this criterion, called the *local criterion for flatness*, is that it is much harder to prove.

25.9.2. Theorem (the local criterion for flatness). — *Suppose $(A, \mathfrak{m})$ is a Noetherian local ring, and M is a finitely generated A-module. Then M is flat if and only if $\operatorname{Tor}_1^A(M, A/\mathfrak{m}) = 0$.*

This is a miraculous statement: flatness over $\operatorname{Spec} A$ is determined by what happens over the closed point.

We now give a more general statement, as well as a consequence that sometimes also goes by the name of the local criterion for flatness. Assume that $(B, \mathfrak{n}) \to (A, \mathfrak{m})$ is a local morphism of local Noetherian rings (i.e. a ring homomorphism with $\mathfrak{n}A \subset \mathfrak{m}$), and that M is a finitely generated A-module. Of course we picture this in terms of geometry:

$$\begin{array}{c} \tilde{M} \\ \Big| \\ \operatorname{Spec}(A, \mathfrak{m}) \\ \Big\downarrow \\ \operatorname{Spec}(B, \mathfrak{n}). \end{array}$$

25.9.3. Theorem (local criterion for flatness, [E, p. 168]). — *The A-module M is B-flat if and only if $\operatorname{Tor}_1^B(B/\mathfrak{n}, M) = 0$.*

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