27.6 Generic freeness

Let A be a Noetherian integral domain. For the purposes of this discussion only, we say an A-algebra B satisfies $(\dagger)$ if for each finitely generated B-module M, there exists a nonzero $f \in A$ such that M_f is a free A_f-module.

27.6.1. Theorem (Generic freeness). — *Every finitely generated A-algebra satisfies $(\dagger)$.*

Proof. We prove generic freeness (Theorem 27.6.1) via a series of exercises.

27.6.A. EXERCISE. Show that A itself satisfies $(\dagger)$.

27.6.B. EXERCISE. Reduce the proof of Theorem 27.6.1 to the following statement: if B is a Noetherian A-algebra satisfying $(\dagger)$, then $B[T]$ does too. (Hint: induct on the number of generators of B as an A-algebra.)

We now prove this statement. Suppose B satisfies $(\dagger)$, and let M be a finitely generated $B[T]$-module, generated by the finite set S. Let M_1 be the sub-B-module of M generated by S. Inductively define

$$M_{n+1} = M_n + TM_n,$$

a sub B-module of M. Note that M is the increasing union of the B-modules M_n.

27.6.C. EXERCISE. Show that multiplication by T induces a surjection

$$\psi_n : M_n/M_{n-1} \to M_{n+1}/M_n.$$

27.6.D. EXERCISE. Show that for $n \gg 0$, ψ_n is an isomorphism. Hint: use the ascending chain condition on M_1.

27.6.E. EXERCISE. Show that there is a nonzero $f \in A$ such that $(M_{n+1}/M_n)_f$ is free as an A_f-module, for all n. Hint: as n varies, M_{n+1}/M_n passes through only finitely many isomorphism classes.

The following result concludes the proof of Theorem 27.6.1.

27.6.F. EXERCISE (NOT REQUIRING NOETHERIAN HYPOTHESES). Suppose M is an A-module that is an increasing union of submodules M_n, with $M_0 = 0$, and that M_n/M_{n-1} is free. Show that M is free. Hint: first construct compatible isomorphisms $\phi_n : \oplus_{i=1}^n M_i/M_{i-1} \to M_i$ by induction on n. Then show that the limit $\phi := \varinjlim \phi_n : \oplus_{i=1}^\infty M_i/M_{i-1} \to M$ is an isomorphism. More generally, your argument will show that if the M_i/M_{i-1} are all projective, then M is (non-naturally) isomorphic to their direct sum.

$\square$