Solution to 18.03 Problem Set 1.5, Problem II.5

II.5. (F 12 Feb) This problem will use the following terminology. We have an autonomous differential equation $dx/dt = g(x)$. A *critical point* for the DE is a number c such that $g(c) = 0$. It is *stable* if $g(x) > 0$ for x just less than c and $g(x) < 0$ for x just greater than c. It is *unstable* if instead $g(x) < 0$ for x just less than c and $g(x) < 0$ for x just greater than c. It is *neutrally stable* if $g(x)$ is of the same sign for x on both sides of c. (We do not consider the case of non-isolated critical points.)

Harvesting (of fish from a population, for example) can be crudely modeled by the constant coefficient Riccati equation

$$\frac{dx}{dt} - ax + bx^2 = -k$$

where a, b, and k are positive numbers. Here a reflects the natural rate of growth of the population; bx^2 reflects the logistic limiting factor of the natural environment; and k is the harvesting rate (which we assume constant). This equation is *autonomous*, i.e., time-independent. I write the equation in this way to stress the fact that (as in the case of the standard form of a linear equation) the left-hand side relates to the population "in isolation," while the right-hand side is a driving term which represents an effect coming from outside the system itself.

The "phase portrait" of this system differs depending on whether $a^2 - 4bk$ is positive, zero, or negative. These correspond to low, medium, and high harvesting rates. Sketch a graph of $g(x) = -bx^2 + ax - k$ in each case. (Be careful about which side of the y axis you put the zeros of $g(x)$ on!) Then sketch a bunch of solution curves in each of the three cases. Decide whether each critical point which occurs is stable, unstable, or neutrally stable.

Interpret your results in terms of public policy. Is the steady state present in the case $a^2 - 4bk = 0$ of any use? Can a policy of harvesting (i.e. a value for k) be given which guarantees the survival of the population independent of the initial condition?

Solution.

The phase portrait will tell us everything, and we'll get the phase portrait from information about the quadratic $-bx^2 + ax - k$. Recall that the graph $y = -bx^2 + ax - k$ is a parabola, and it has a very simple shape. For example, if it doesn't meet the y-axis, then it is entirely positive (i.e. the parabola "opens upward"), in which case the phase portrait looks like this:

$$>>>>>>>>>>>>>>>$$

or it is entirely negative (i.e. the parabola "opens downward"), in which case the phase portrait looks like this:

$$<<<<<<<<<<<<<<< \,.$$

In this case, the leading co-efficient $-b$ is negative, so we know we're in the second case. In conclusion, if the parabola misses the x-axis, i.e. the discriminant $a^2 - 4bk$ is negative, then no matter what the population is initially, it will drop (to zero). This corresponds to a high harvesting rate: the population has no chance to replenish itself.

Next, what if the parabola meets the x-axis at two points? Because the leading co-efficient of the quadratic is negative, the phase portrait must look like this:

$$<<<<< r >>>>> s <<<<<$$

where r and s are the roots of the quadratic. (Important fact: both roots are positive! There are various ways to see this. One way is this: At $x = 0$, the quadratic $-bx^2+ax-k$ is negative, and the slope $-2bx + a$ is positive. If you think about the shape of the parabola, you'll see that the roots must be to the right of $x = 0$.) We can now interpret what the phase portrait tells us. If the population is bigger than s, it will drop to s. If it is between r and s, it will rise to s. If it is less than r, it will drop to 0. If it is *exactly* r, it will remain at r, but this equilibrium is unstable.

Finally, what if the parabola meets the x-axis at one point? Then, for similar reasons as above, the parabola opens downward, and is tangent to the x-axis at a point $x = r$, with $r > 0$. Then the phase portrait looks like this:

$$<<<<< r <<<<<$$

so if the population is initially greater than r, it will drop to r, and if it is initially less than r, it will drop to 0.

In terms of policy, the case $a^2 - 4bk$ is not good to rely on; if circumstances change slightly (i.e. a, b, and k change so that $a^2 - 4bk$ becomes slightly negative), you would lose the equilibrium point r, and the population would drop to 0. Also, from the previous case, there is no value for k which guarantees the survival of the population independent of the initial condition: if the initial population is small enough (less than r), it will not survive.