18.03 Problem Set 1.5

These problems are not to be turned in. They are just to keep you on your toes and off the street during the space of time before Wednesday's Hour Exam.

Syllabus

5. (F 12 Feb) Autonomous equations; the phase line: EP 1.8, 7.1.

6. (T 16 Feb) Review

7. (W 17 Feb) Hour Exam I This exam will occur during the lecture hours. You must attend the exam during the lecture hour for which you are registered. Each class will be split, according to the following scheme. Go to the room corresponding to your section.

1:00 class	Walker	Bressler, Consani, Derksen, Early, Freedman, Gerhardt, Greenblatt, Kunin, Liu, Lototsky
1:00 class	54-100	Postnikov, Soloviev, Vakil, Yang, Young, Zhang
2:00 class	Walker	Bressler, Consani, Derksen, Freedman, Gerhardt, Greenblatt, Kunin, Liu
2:00 class	10-250	Early, Lototsky, Postnikov, Soloviev, Vakil, Yang, Young, Zhang

Part I.

5. (F 12 Feb) EP 7.1: 5, 6, 12, 15: but rather than solving these analytically, sketch the graph of the right hand side $f(x)$, indicate which intervals give stable and unstable behavior (as in Fig 7.2, p. 508), and sketch the solutions (as in Fig. 7.3).

6. (T 17 Feb) Nothing new.

Part II.

5. (F 12 Feb) This problem will use the following terminology. We have an autonomous differential equation $dx/dt = g(x)$. A *critical point* for the DE is a number c such that $g(c) = 0$. It is *stable* if $g(x) > 0$ for x just less than c and $g(x) < 0$ for x just greater than c. It is *unstable* if instead $g(x) < 0$ for x just less than c and $g(x) < 0$ for x just greater than c. It is *neutrally stable* if $g(x)$ is of the same sign for x on both sides of c. (We do not consider the case of non-isolated critical points.)

Harvesting (of fish from a population, for example) can be crudely modeled by the constant coefficient Riccati equation

$$\frac{dx}{dt} - ax + bx^2 = -k$$

where a, b, and k are positive numbers. Here a reflects the natural rate of growth of the population; bx^2 reflects the logistic limiting factor of the natural environment; and k is the harvesting rate (which we assume constant). This equation is *autonomous*, i.e., time-independent. I write the equation in this way to stress the fact that (as in the case of the standard form of a linear equation) the left-hand side relates to the population "in isolation," while the right-hand side is a driving term which represents an effect coming from outside the system itself.

The "phase portrait" of this system differs depending on whether $a^2 - 4bk$ is positive, zero, or negative. These correspond to low, medium, and high harvesting rates. Sketch a graph of $g(x) = -bx^2 + ax - k$ in each case. (Be careful about which side of the y axis you put the zeros of $g(x)$ on!) Then sketch a bunch of solution curves in each of the three cases. Decide whether each critical point which occurs is stable, unstable, or neutrally stable.

Interpret your results in terms of public policy. Is the steady state present in the case $a^2 - 4bk = 0$ of any use? Can a policy of harvesting (i.e. a value for k) be given which guarantees the survival of the population independent of the initial condition?

6. (T 16 Feb) Nothing new.