18.03 Problem Set 8.5

Final Exam: Wednesday, May 19, 9:00–12:00

This is another set of problems not to be handed in, just to give you practice on these last subjects. The Matlab material is optional.

Syllabus

V. Fourier Series and Partial Differential Equations

36. (W 5 May) Fourier series: EP 8.1.

37. (F 7 May) Applications: EP 8.4.

38. (M 10 May) Heat Equation: EP 8.5.

39. (W 12 May) Wave Equation: EP 8.6 (631–639).

Part I.

37. (F 7 May) EP 8.3: 19, 20.

38. (M 10 May) Nothing new.

39. (W 12 May) EP 8.6: 1, 3, 6.

Part II.

Here is a corrected version of the routine `cf.m` offered up in **36.** In effect it uses the trapezoidal rule.

```
function y=cf(vector,n)
N=length(vector);
t=linspace(-pi,pi,N);
y(1)=(sum(vector)-(vector(1)+vector(N))/2)/(N-1);
for k=2:n+1
y(k)=2*(sum(cos((k-1)*t).*vector)-(vector(1)+vector(N))/2)/(N-1);
y(n+1+k)=2*sum(sin((k-1)*t).*vector))/(N-1);
end
```

37. (F 7 May) (a) Using integration by parts, find the Fourier coefficients of the function defined by $f(t) = t$ for $-\pi < t < \pi$, $f(t + 2\pi) = f(t)$. [Answer: $a_n = 0$, $b_n = -2(-1)^n/n$; see EP 8.3 (16).]

(b) Now suppose we have the ODE $\ddot{x} + kx = f(t)$, $k > 0$, where $f(t)$ is as in **(a)**. Work out the Fourier coefficients of the solutions to this ODE which are periodic of period 2π, if such exist. [Answer: $a_n = 0$, $b_n = -2(-1)^n/n(k - n^2)$, provided that k is not a square integer; there is then a unique periodic solution. If k is a square integer then no periodic solutions exist.]

(c) Use MATLAB as suggested in **36.** to see how well the Fourier series approximates $f(t)$. Then create a routine like the one printed above to create the vector of the Fourier coefficients through a_n, b_n you found in **(b)**. Hint: here's the routine `csq.m` I used to produce the vector of the Fourier coefficients through a_n, b_n for the periodic solution to $\ddot{x} + kx = \mathrm{sq}(t)$, as described in class:

```
function v=csq(n,k)
for j=1:n
v(n+2+j)=-2*(1-(-1)^j)/(pi*j*(j*j-k));
end
```

Finally, take n pretty large, say $n = 100$, and experiment with what happens as you vary k.

(d) Recall the square wave $\mathrm{sq}(t) = -1$ for $-\pi < t < 0$, $\mathrm{sq}(t) = 1$ for $0 < t < \pi$, and its Fourier series

$$\mathrm{sq}(t) = \sum_{n=1,\mathrm{odd}}^{\infty} \frac{\sin(nt)}{n}. \tag{1}$$

Differentiate this Fourier expression term-by-term to get an expression for the derivative of $\mathrm{sq}(t)$. [Answer: $\mathrm{sq}'(t) = \frac{4}{\pi} \sum_{n=1,\mathrm{odd}}^{\infty} \cos(nt)$.]

(e) We can consider a "generalized periodic function" which we denote (by an abuse of notation) δ_a, with the characteristic property that

$$\int_{-\pi}^{\pi} \delta_a(t) g(t) dt = g(a) \tag{2}$$

for any decent periodic function g. Using this, compute the Fourier coefficients of δ_0 and of δ_π. [Answers: $\delta_0(t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos(nt)$, $\delta_\pi(t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^n\cos(nt)$.] It looks as though we could write $\mathrm{sq}'(t) = 2(\delta_0(t) - \delta_\pi(t))$. Check that the Fourier coefficients are right for this.

(f) Construct a routine which creates a vector consisting of the Fourier coefficients through a_n, b_n for δ_0, suitable for feeding to `fsum.m`. Then feed this to `fsum.m` to create a vector of values, create a corresponding vector of `t` values, and plot the result, for $n = 2, 4, 8, 100$. Do these seem to approximate the delta function?

38. (M 10 May) (a) Starting with (2) again, calculate:

$$\delta_{\pi/2}(t) = \frac{1}{2\pi} + \frac{1}{\pi}(\sin(t) - \cos(2t) - \sin(3t) + \cos(4t) + \sin(5t) - \cos(6t) - + + \cdots)$$

$$\delta_{-\pi/2}(t) = \frac{1}{2\pi} + \frac{1}{\pi}(-\sin(t) - \cos(2t) + \sin(3t) + \cos(4t) - \sin(5t) - \cos(6t) + + - \cdots)$$

$$\delta_{\pi/2}(t) - \delta_{-\pi/2}(t) = \frac{2}{\pi}\sum_{k=0}^{\infty}(-1)^k \sin((2k+1)t)$$

$$\delta_{\pi/2}(t) + \delta_{-\pi/2}(t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k \cos(2kt)$$

(b) Use **(a)** to write down the solution $u(x,t)$ to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{3}$$

with boundary conditions $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \delta_{\pi/2}(t)$: a very hot spot at the center of the interval, and the ends kept cold. The heat spreads from the hotspot. [Answer: $u(x,t) = \frac{2}{\pi}\sum_{k=0}^{\infty}(-1)^k e^{-(2k+1)^2 t}\sin((2k+1)x)$. Notice that almost immediately all the overtones are damped out to a much greater degree than the fundamental, and that consequently the heat distribution is well approximated by $(2/\pi)e^{-t}\sin(x)$.]

(c) Use **(a)** to write down a solution $u(x,t)$ to (3) subject to the boundary conditions $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(l,t)$, $u(x,0) = \delta_{\pi/2}(x)$: a very hot spot in the middle, and the ends insulated. [Answer: $u(x,t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k e^{-4k^2 t}\cos(2kx)$. Again, almost immediately this gets very close to $1/\pi - (2/\pi)e^{-4t}\cos(2x)$. This in turn decays rather quickly to the constant function whose value is the average value of the heat content of the bar, $1/\pi$.]