Solutions to 18.03 Problem Set 7 Part II

31. (a) $\mathcal{L}(f(at);s) = \int_0^\infty f(at)e^{-st}dt$. Let $u = at$, so $du = adt$. Then $\mathcal{L}(f(at);s) = \int_0^\infty f(u)e^{-su/a}\frac{du}{a} = \frac{1}{a}F(\frac{s}{a})$.

(b) $\mathcal{L}(e^{(k+i\omega)t};s) = \frac{1}{s-(k+i\omega)} = \frac{(s-k)+i\omega}{(s-k)^2+\omega^2}$, so $\mathcal{L}(e^{kt}\cos(\omega t);s) = \mathrm{Re}(\mathcal{L}(e^{(k+i\omega)t};s)) = \frac{s-k}{(s-k)^2+\omega^2}$ and $\mathcal{L}(e^{kt}\sin(\omega t);s) = \mathrm{Im}(\mathcal{L}(e^{(k+i\omega)t};s)) = \frac{\omega}{(s-k)^2+\omega^2}$.

32. (a) Using the s-derivative rule, $\mathcal{L}(t\sin(\omega t);s) = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2\omega s}{(s^2+\omega^2)^2}$ and $\mathcal{L}(t\cos(\omega t);s) = -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$. Hence

$$\mathcal{L}\left(\frac{t}{2\omega}\sin(\omega t);s\right) = \frac{s}{(s^2+\omega^2)^2}.$$

Also $\mathcal{L}(\frac{1}{\omega}\sin(\omega t);s) = \frac{s^2+\omega^2}{(s^2+\omega^2)^2}$, so

$$\mathcal{L}\left(\frac{1}{2\omega^2}\left(\frac{1}{\omega}\sin(\omega t) - t\cos(\omega t)\right);s\right) = \frac{1}{(s^2+\omega^2)^2}.$$

(b) Write $f(t) = \frac{e^{at}-1}{t}$, and $F(s) = \mathcal{L}(f(t);s)$. Then $\mathcal{L}(tf(t);s) = \mathcal{L}(e^{at}-1;s) = \frac{1}{s-a} - \frac{1}{s}$. Also $\mathcal{L}(tf(t);s) = -\frac{dF(s)}{ds}$ by the s-derivative rule, so $\frac{dF(s)}{ds} = \frac{1}{s} - \frac{1}{s-a}$. Hence $F(s) = \ln(s) - \ln(s-a) + c = -\ln(1-a/s) + c$ for some c. We know that $\lim_{s\to\infty} F(s) = 0$, which will allow us to find c. As $s \to \infty$, $-\ln(1-a/s) \to -\ln 1 = 0$, so $c = 0$.

For the second half, let $f(t) = \frac{\sin(\omega t)}{t}$, and let $F(s) = \mathcal{L}(f(t);s)$. Then $\mathcal{L}(tf(t);s) = \mathcal{L}(\sin(\omega t);s) = \frac{\omega}{s^2+\omega^2}$. Also $\mathcal{L}(tf(t);s) = -\frac{dF(s)}{ds}$, so $\frac{dF(s)}{ds} = \frac{-\omega}{s^2+\omega^2}$. Integrating: $F(s) = c - \arctan(s/\omega)$ for some c. As $s \to \infty$, $\arctan(s/\omega) \to \pi/2$, so $c = \pi/2$.

33. (a)

$$\begin{aligned}\mathcal{L}(t^2\sin(\omega t);s) &= \mathrm{Im}\left(\mathcal{L}(t^2e^{i\omega t};s)\right)\\ &= \mathrm{Im}\left(\frac{2}{(s-i\omega)^3}\right)\end{aligned}$$

Rationalize the denominator by multiplying top and bottom by $(s+i\omega)^3$:

$$\begin{aligned}\mathcal{L}(t^2\sin(\omega t);s) &= \mathrm{Im}\left(\frac{2(s+i\omega)^3}{(s^2+\omega^2)^3}\right)\\ &= \mathrm{Im}\left(\frac{2(s^3-3s\omega^2)+2i(3s^2\omega-\omega^3)}{(s^2+\omega^2)^3}\right)\\ &= \frac{2(3s^2\omega-\omega^3)}{(s^2+\omega^2)^3}.\end{aligned}$$

Similarly,

$$\mathcal{L}(te^t\cos(\omega t);s) = \mathrm{Re}\left(\mathcal{L}(te^{(1+i\omega)t};s)\right)$$

$$
\begin{aligned}
&= \operatorname{Re}\left(\frac{1}{(s-1-i\omega)^2}\right) \\
&= \operatorname{Re}\left(\frac{(s-1+i\omega)^2}{((s-1)^2+\omega^2)^2}\right) \\
&= \operatorname{Re}\left(\frac{(s-1)^2-\omega^2+2i(s-1)\omega}{((s-1)^2+\omega^2)^2}\right) \\
&= \frac{(s-1)^2-\omega^2}{((s-1)^2+\omega^2)^2}.
\end{aligned}
$$

Finally, $\mathcal{L}(t^{1/2}e^t; s) = \frac{\Gamma(3/2)}{(s-1)^{3/2}} = \frac{\sqrt{\pi}}{2(s-1)^{3/2}}$ (as $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \sqrt{\pi}/2$).

(b) The graph of the "bump" is a rectangle with base b and height $1/b$ (hence area 1). As $b \to 0$, the bump becomes tall and skinny. $\mathcal{L}(u_a; s) = e^{-as}/s$, so

$$
\mathcal{L}(d(t); s) = \frac{1}{bs}(e^{-as} - e^{-(a+b)s}) = \frac{e^{-as}}{s}\left(\frac{1-e^{-bs}}{b}\right).
$$

The limit $\lim_{b\to 0}(\frac{1-e^{-bs}}{b})$ is *by definition* the derivative of $-e^{-bs}$ with respect to b at $b = 0$, which we know is s. Alternatively, by L'Hopital's rule, $\lim_{b\to 0}\frac{1-e^{-bs}}{b} = s$. Thus $\lim_{b\to 0}\mathcal{L}(d(t); s) = e^{-as}$.