18.03 Problem Set 7

Due by 1:00 P.M., Friday, April 30, 1999, in the boxes at 2-106, next to the Undergraduate Mathematics Office.

Syllabus

IV. The Laplace Transform

31. (F 23 Apr) Basic properties: EP 4.1.

32. (M 26 Apr) Solution of IVPs: EP 4.2 (302–306), 4.3.

33. (W 28 Apr) Discontinuous functions: EP 4.5 (328–333).

34. (F 30 Apr) Convolution and the delta function: EP 4.4 (320–322), 4.6 (341–348).

35. (M 3 May) Transfer functions and Duhamel's principle: Notes LT, EP 4.6 (349–350).

There is a table of evaluations of the Laplace transform on p. 296 of Edwards and Penney. You will not be expected to memorize specific Laplace transforms, but you should learn (by memory) its *properties*—chiefly: linearity, the s-translation theorem (on p. 313), the relationship of the Laplace tranform to differentiation and to convolution, the formula for the derivative of a Laplace transform (Theorem 2 on p. 323), and **31(a)** below.

Part I.

31. (F 23 Apr) Notes L: 2, 3cd, 4, 6a, 8.

32. (M 26 Apr) Notes L: 9, 10, 11bce, 12, 15a, 16.

33. (W 28 Apr) EP 339: 11, 13, 15; 1, 3, 5 (but in these last three write your answer $f(t)$ as an alternative: $f(t) = \ldots$ if $0 \leq t <??$, $f(t) = \ldots$ if $t \geq ??$, etc.)

Part II.

31. (F 23 Apr) (a) If $F(s)$ is the Laplace transform of $f(t)$, what is the Laplace transform of $f(at)$ (where $a > 0$)? Explain your reasoning.

(b) The Laplace transform extends without surprises to *complex* valued functions defined on the half line $[0, \infty)$. It preserves real and imaginary parts (by virtue of its linearity). Use this principle and the fact that the Laplace transform of e^{at} is $1/(s-a)$ to compute the Laplace transforms of $e^{kt}\sin(\omega t)$ and $e^{kt}\cos(\omega t)$.

32. (M 26 Apr) (a) Using the values of the Laplace transform of $\cos(\omega t)$ and of $\sin(\omega t)$ and the basic properties of the Laplace transform, write down the transforms of $t\cos(\omega t)$ and $t\sin(\omega t)$. Then form linear combinations of these four functions to obtain the inverse Laplace transforms of

$$\frac{1}{(s^2+\omega^2)^2} \quad \text{and} \quad \frac{s}{(s^2+\omega^2)^2}.$$

(b) Show that

$$\mathcal{L}\left(\frac{e^{at}-1}{t}; s\right) = -\ln\left(1-\frac{a}{s}\right)$$

and

$$\mathcal{L}\left(\frac{\sin(\omega t)}{t}; s\right) = \frac{\pi}{2} - \arctan\left(\frac{s}{\omega}\right).$$

Use the following facts: if $f(t)$ has Laplace transform $F(s)$ (for $s > a$) then (1) $tf(t)$ has Laplace transform $-F'(s)$ (for $s > a$), and (2) $\lim_{s\to\infty} F(s) = 0$ (the Corollary on p. 299 of Edwards and Penney).

33. (W 28 Apr) (a) The s-translation theorem together with $\mathcal{L}(t^n; s) = n!/s^{n+1}$ gives us $\mathcal{L}(t^n e^{at}; s) = n!/(s-a)^{n+1}$. This too is valid for a complex. Use this to compute $\mathcal{L}(t^2 \sin(\omega t); s)$, $\mathcal{L}(te^t \cos(\omega t); s)$, and $\mathcal{L}(\sqrt{t}e^t; s)$.

(b) Now let a, b be positive numbers and sketch the graph of the "bump function" $d(t) = (u_a - u_{a+b})/b$. Compute $\mathcal{L}(d(t); s)$. What happens to $d(t)$ as $b \to 0$ (in words)? What happens to its Laplace transform? (This is a computation of a limit, and the definition of the derivative of a function should come in useful once again!)