# Partial solutions to 18.03 Problem Set 6 Part II

**23. (a)** The characteristic polynomial is $\lambda^2+2\lambda+a=0$. The eigenvalues are $\lambda_1=-1+\sqrt{1-a}$, $\lambda_2=-1-\sqrt{1-a}$. An eigenvector corresponding to $\lambda_1$ is $v_1=\begin{bmatrix}1\\ \sqrt{1-a}-1\end{bmatrix}$. An eigenvector corresponding to $\lambda_2$ is $v_2=\begin{bmatrix}1\\ -\sqrt{1-a}-1\end{bmatrix}$.

- If $a=1$, there is a repeated eigenvalue: $\lambda_1=\lambda_2=1$. Then $A=\begin{bmatrix}0&1\\-1&-1\end{bmatrix}$. $A$ doesn't have independent eigenvectors, as $A\neq\lambda_1 I$.
- If $a<1$, the eigenvalues are real and distinct.
- If $a>1$, the eigenvalues are nonreal complex: $-1+\sqrt{a-1}i$ and $-1-\sqrt{a-1}i$. They are indeed complex conjugates.

If $a=0$, 0 is an eigenvalue. The other eigenvalue is -2.

**(b)** The general real solution is given as follows.

(i) If $a<1$, $\begin{bmatrix}x\\y\end{bmatrix}=C_1e^{(-1+\sqrt{1-a})t}\begin{bmatrix}1\\ \sqrt{1-a}-1\end{bmatrix}+C_2e^{(-1-\sqrt{1-a})t}\begin{bmatrix}1\\ -\sqrt{1-a}-1\end{bmatrix}$.

(ii) If $a>1$, $\begin{bmatrix}x\\y\end{bmatrix}$ is a linear combination of

$$e^{-t}\left(\cos(\sqrt{a-1}t)+i\sin(\sqrt{a-1}t)\right)\begin{bmatrix}1\\ \sqrt{a-1}i-1\end{bmatrix}$$

and its conjugate. Hence it is a linear combination of its real part and its imaginary part:

$$\begin{aligned}\begin{bmatrix}x\\y\end{bmatrix} &= e^{-t}\Bigg(C_1\left(\cos(\sqrt{a-1}t)\begin{bmatrix}1\\-1\end{bmatrix}+\sin(\sqrt{a-1}t)\begin{bmatrix}0\\-\sqrt{a-1}\end{bmatrix}\right)\\ &\quad +C_2\left(\cos(\sqrt{a-1}t)\begin{bmatrix}0\\ \sqrt{a-1}\end{bmatrix}+\sin(\sqrt{a-1}t)\begin{bmatrix}1\\-1\end{bmatrix}\right)\Bigg).\end{aligned}$$

(iii) If $a=1$, one solution is $e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}$. The other solution is of the form $e^{-t}\left(t\begin{bmatrix}1\\-1\end{bmatrix}+\beta\right)$ for some vector $\beta$. $\beta$ can be found (see Notes LS.15) by solving $(A-\lambda_1 I)\beta=\begin{bmatrix}1\\-1\end{bmatrix}$; one solution is $\beta=\begin{bmatrix}1\\0\end{bmatrix}$. Hence the general solution is

$$\begin{bmatrix}x\\y\end{bmatrix}=C_1e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}+C_2e^{-t}\left(t\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}\right).$$

**24. (a)**

| $a$ | behavior |
|---|---|
| -8 | saddle point, unstable |
| -1 | saddle point, unstable |
| 0 | degenerate, comb, neutrally stable |
| .5 | asymptotically stable, proper node |
| 1 | asymptotically stable, improper node |
| 2 | spiral, asymptotically stable |
| 10 | spiral, asymptotically stable |

**(b)** If $a < 1$, we have real distinct roots. The two rays from the origin are the eigendirections: $y = (\sqrt{1-a} - 1)x$ and $y = (-\sqrt{1-a} - 1)x$.

**(c)** In this case, $\begin{bmatrix} x \\ y \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} C_1 + C_2 e^{-2t} \\ -2C_2 e^{-2t} \end{bmatrix}$. The long-term behavior: the second normal mode dies out, so a particle will tend to the line $y = 0$ (this is visible on `pplane5`). The slope of the path is -2. The constant solutions occur when $C_2 = 0$: $x = C_1$, $y = 0$. There are so many constant solutions because the matrix has 0 as an eigenvalue, i.e. is singular. Any first order homogeneous linear system has the constant solution $(x, y) = (0, 0)$.

**(d)** The "straight line" solution corresponds to $C_2$ in (iii) above. The line is $y = -x$. If $C_2 \neq 0$ (so the solution is neither the constant solution nor the rays), every solution crosses the $x$-axis exactly once (where $y = 0$, i.e. $e^{-t}(-C_1 - C_2 t) = 0$, i.e. $t = -C_1/C_2$), and the $y$-axis exactly once (where $x = 0$, i.e. $e^{-t}(C_1 + C_2 t + C_2) = 0$, i.e. $t = -C_1/C_2 - 1$). We also see that *every* solution (except the constant one and the rays) crosses the $x$-axis exaclty one time-unit after it crosses the $y$-axis.