Partial Solutions to 18.03 Problem Set 6.5 Part II

27. If $\dot{x} = y$, then by the chain rule $\frac{dy}{dt} = \dot{x}\frac{dy}{dx} = yy'$. (We've seen this substitution before: see EP p. 167.) Hence $\ddot{x} + e^x = 0$ can be rewritten as $\dot{y} = -e^x$, $\dot{x} = y$. Substituting $\dot{y} = yy'$, we get $yy' = -e^x$. Separating variables, we get $ydy = -e^x dx$, so $\frac{y^2}{2} = -e^x + C_1$. Hence $\dot{x} = y = \sqrt{2(C_1 - e^x)}$. (Solving for x in terms of t is trickier. Separating variables:

$$\frac{dx}{\sqrt{2(C_1 - e^x)}} = dt.$$

This doesn't have an especially nice solution.)

28. (a) If $(a+bi)(x+yi) = (p+qi)$, then $p = ax - by$, $q = bx + ay$. We check that

$$A(a+bi)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax - by \\ bx + ay \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

as desired. Also, if $z = a + bi$ and $w = c + di$, so $zw = (ac - bd) + (ad + bc)i$, then

$$A(z)A(w) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{bmatrix} = A(zw),$$

and

$$A(z) + A(w) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a + c & -(b+d) \\ b + d & a + c \end{bmatrix} = A(z + w).$$

Convince yourself that $A(\cos\theta + i\sin\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has the effect of rotating counterclockwise by an angle θ: for example, $A(\cos\theta + i\sin\theta)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $A(\cos\theta + i\sin\theta)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ as one would expect. If $z = a + bi$, then the eigenvalues of $A(z)$ are $z = a + bi$ and $\overline{z} = a - bi$, and the corresponding eigenvectors are $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ respectively.

Let $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ be the eigenvalue matrix, so

$$P^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}.$$

Let $D = \begin{bmatrix} a + bi & 0 \\ 0 & a - bi \end{bmatrix}$ be the eigenvector matrix. Then $A(z) = PDP^{-1}$ (see EP p. 454), and

$$e^{A(z)} = Pe^{D}P^{-1}$$

$$\begin{aligned}
&= \frac{1}{2}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{a+bi} & 0 \\ 0 & e^{a-bi} \end{bmatrix}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{a+bi}+e^{a-bi}}{2} & -\frac{e^{a+bi}-e^{a-bi}}{2i} \\ \frac{e^{a+bi}-e^{a-bi}}{2i} & \frac{e^{a+bi}+e^{a-bi}}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{z}+\overline{e^{z}}}{2} & -\frac{e^{z}-\overline{e^{z}}}{2i} \\ \frac{e^{z}-\overline{e^{z}}}{2i} & \frac{e^{z}+\overline{e^{z}}}{2} \end{bmatrix} \\
&= A(e^{z}).
\end{aligned}$$

(b) We find the general solution to the system $u' = u + v$, $v' = v$. $v = C_1e^t$, so $u' = u + C_1e^t$, so $u = C_1te^t + C_2e^t$. The solution with initial condition $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has $C_1 = 0$, $C_2 = 1$, so $x(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$. The solution with initial condition $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has $C_1 = 1$, $C_2 = 0$, so $x(t) = \begin{bmatrix} te^t \\ e^t \end{bmatrix}$. Thus

$$e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} = e^t\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

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