Partial Solutions to 18.03 Problem Set 5

Part I. 20. (c) The general solution to $y''+2y'+2y=0$ is of the form $y=e^{-x}(A\cos x+B\sin x)$. If $y(0)=0$, then $A=0$, so $y(x)=e^{-x}B\sin x$. If furthermore x_0 is not a multiple of π (so $\sin x_0 \neq 0$), then if $B=e^{x_0}/\sin x_0$, then $y(x_0)=1$. But if x_0 is a multiple of 2π, no such B exists. Hence the condition is "x_0 is not a multiple of π".

Part II.

20. (a) (All derivatives in this solution will be with respect to t.) Let's look for a solution of the form $y=\mathrm{ca}(\alpha t)$ for some α. We know that $\mathrm{ca}(t)$ is a solution of $x''+tx=0$, so $\mathrm{ca}''(t)=-t\,\mathrm{ca}(t)$. Then $y'=\alpha\mathrm{ca}'(\alpha t)$, and $y''=\alpha^2\mathrm{ca}''(\alpha t)$. But $\mathrm{ca}''(z)=-z\,\mathrm{ca}(z)$, so taking $z=\alpha t$, $\mathrm{ca}''(\alpha t)=-\alpha t\,\mathrm{ca}(\alpha t)$. Hence $y''=-\alpha^3 t\,\mathrm{ca}(\alpha t)$. If $y''+t\lambda y=0$, then $-\alpha^3 t\,\mathrm{ca}(\alpha t)+t\lambda\mathrm{ca}(\alpha t)=0$. This is true if $\alpha^3=\lambda$, i.e. $\alpha=\sqrt[3]{\lambda}$, so $y=\mathrm{ca}(\sqrt[3]{\lambda}t)$. Note that $y(0)=1$ and $y'(0)=0$.

Similarly, $y=\mathrm{sa}(\alpha t)=\mathrm{sa}(\sqrt[3]{\lambda}t)$ is a solution; in this case, $y(0)=0$, $y'(1)=\sqrt[3]{\lambda}$.

Hence the normalized solutions are $y_1(t)=\mathrm{ca}(\sqrt[3]{\lambda}t)$, $y_2(t)=\frac{1}{\sqrt[3]{\lambda}}\mathrm{sa}(\sqrt[3]{\lambda}t)$.

(b) As before, $y_1(t)=\mathrm{ca}(\sqrt[3]{\lambda}t)$. From the Airy handout, $\mathrm{ca}(t)\sim at^{-1/4}\cos\left(\frac{2t^{3/2}}{3}-\frac{\pi}{12}\right)$ asymptotically, so

$$\mathrm{ca}(\sqrt[3]{\lambda}t)\sim a(\sqrt[3]{\lambda}t)^{-1/4}\cos\left(\frac{2(\sqrt[3]{\lambda}t)^{3/2}}{3}-\frac{\pi}{12}\right).$$

This is 0 when $\frac{2(\sqrt[3]{\lambda}t)^{3/2}}{3}-\frac{\pi}{12}=(n+1/2)\pi$ where n is some integer, i.e.

$$t=\frac{1}{\sqrt[3]{\lambda}}\left(\frac{3}{2}\left(n+\frac{7}{12}\right)\pi\right)^{2/3}.$$

We can do the same thing with $y_2(t)=\frac{1}{\sqrt[3]{\lambda}}\mathrm{sa}(\sqrt[3]{\lambda}t)$, which has zeros roughly when

$$\frac{2(\sqrt[3]{\lambda}t)^{3/2}}{3}+\frac{\pi}{12}=n\pi$$

(using the Airy handout again), which leads to zeros when $t=\frac{1}{\sqrt[3]{\lambda}}\left(\frac{3}{2}\left(n-\frac{1}{12}\right)\pi\right)^{2/3}$.

21. (a) $R''-aR'-bcR=0$. Since $x''+px'+qx=0$ is stable exactly when $p,\ q>0$, stability occurs precisely when $a<0$ and $bc<0$. The equation is underdamped if $p^2<4q$, i.e. $|a|^2<4|bc|$. The roots of $r^2-ar-bc=0$ are of the form $s\pm\omega i$ for $s=a/2$, $\omega=\sqrt{|bc|-(a/2)^2}$. As $|a|$ grows towards $2\sqrt{|bc|}$, the frequency ω shrinks to 0. As $|bc|$ increases, ω increases.

(b) $R=R(0)e^{at/2}\cos(\omega t)$, so $J=(R'-aR)/b=-\frac{1}{b}R(0)e^{at/2}(\frac{a}{2}\cos(\omega t)+\omega\sin(\omega t))$. Consider a right triangle with sides $a/2$ and ω and hypotenuse r, with angle ϕ opposite side ω, so $a/2=r\cos\phi$ and $\omega=r\sin\phi$. Then $r=\sqrt{(a/2)^2+\omega^2}=\sqrt{|bc|}$. Thus $J(t)=-\frac{1}{b}R(0)r\cos(\omega t-\phi)$.

$$\rho=\frac{-\frac{1}{b}R(0)r}{R(0)}=-r/b=\pm\sqrt{|c/b|}.$$

(c) If $|a|^2 > 4|bc|$ the roots are real; by stability the solutions fall to 0 without oscillation.

(d) If R and J are constant, then $R' = J' = 0$, so we are to solve $aR + bJ = -r_1$, $cR = -r_2$. Since

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -bc \neq 0$$

this has a unique solution.

22. (a)

$$\begin{bmatrix} R \\ J \end{bmatrix}' = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\begin{bmatrix} R \\ J \end{bmatrix}.$$

The characteristic polynomial of $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ is $\lambda^2 - a\lambda - bc$ with eigenvalues $\frac{a}{2} \pm \sqrt{(a/2)^2 + bc}$. (Look familiar?)

(b) $A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$. $f_A(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$.

With eigenvalue $\lambda_1 = -1$, $A - \lambda_1 I = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$, and $\vec{\alpha_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (or any nonzero multiple thereof).

With eigenvalue $\lambda_2 = -2$, $A - \lambda_2 I = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$, and $\vec{\alpha_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (or any nonzero multiple thereof).

The normal modes are $e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The general solution is $c_1 e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(c) $\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $c_1 = 1$, $c_2 = -2$, so $\vec{x} = e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 2e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence $R = e^{-t} - 2e^{-2t}$, $J = -2e^{-t} + 2e^{-2t}$.

Romeo comes to like Juliet when $R(t) = 0$ i.e. $e^{-t} = 2e^{-2t}$, i.e. $t = \ln(2)$. Thereafter Romeo likes Juliet. Romeo is fondest when $0 = R'(t) = -e^{-t} + 4e^{-2t}$, i.e. $t = 2\ln(2)$. For $t > 0$, $e^{-t} > e^{-2t}$, so $J < 0$ always. Graphs: