18.03 Problem Set 5

Due by 1:00 P.M., Friday, April 2, 1999, in the boxes at 2-106, next to the Undergraduate Mathematics Office.

Syllabus

20. (F 19 Mar) Endpoint problems: Jump ropes and pancake stacks: EP 2.10; Airy functions handout.

Part III: Systems of First Order ODEs

21. (M 29 Mar) Elimination and the opposite: EP 5.1 (355–359).

22. (W 31 Mar) Eigenvalues and eigenvectors: Notes LS 1.1, 1.2.

23. (F 2 Apr) Complex or repeated eigenvalues: Notes LS 1.3, 1.4.

Part I.

20. (F 19 Mar) (a) Find the solution of $y'' + 2y' + y = 0$ for which $y(1) = 2, y'(1) = 3$.

(b) Suppose $x_0 \neq 0$. Find the solution of $y'' + 2y' + y = 0$ for which $y(0) = 0$ and $y(x_0) = 1$.

(c) Now suppose the equation is $y'' + 2y' + 2y = 0$. It's no longer true that for any $x_0 \neq 0$ there is a solution such that $y(0) = 0$ and $y(x_0) = 1$. What condition do you have to require of x_0?

21. (M 29 Mar) EP 5.1: 1, 3; EP 5.2: 1, 3, 5.

22. (W 31 Mar) EP 5.3: 1, 3; Notes LS 14, 22, 20, 28.

Part II.

20. (F 19 Mar) (a) The handout on the Airy functions discusses solutions $\mathrm{ca}(x)$ and $\mathrm{sa}(x)$ to the linear ODE $\ddot{x} + tx = 0$. In EP3.6 an application is discussed which requires a solution to the ODE $\ddot{y} + \lambda ty = 0$. Remember that (for $\lambda > 0$) $\cos(\sqrt{\lambda}x)$ and $\sin(\sqrt{\lambda}x)$ are solutions to $\ddot{y} + \lambda y = 0$. Analogously, find an expression for solutions to $\ddot{y} + \lambda ty = 0$, normalized at 0, in terms of ca and sa.

(b) Now use the asymptotic approximation for $\mathrm{ca}(x)$ on the bottom of p. 2 of the same handout to estimate the locations of the zeros (for $t > 0$) of the solution y_1 to $\ddot{y} + \lambda ty$ such that $y_1(0) = 1, y_1'(0) = 0$ that you found in **(a)**. (You'll use the fact that the zeros of the cosine lie at $(n + (1/2))\pi$, where n is an integer.)

21. (M 29 Mar) This problem continues the romance of Romeo and Juliet as described in class. We'll consider the situation of "selfless Juliet":

$$\begin{aligned} R' &= aR + bJ \\ J' &= cR \end{aligned}$$

where a, b, c are constants. Why is Juliet selfless in this model? (This and all subsequent psychological questions are optional!) Interpret a, b, c psychologically. (For example, a

is a measure of Romeo's self-doubt: when it is positive, his feelings tend to re-enforce themselves, while when it is negative there is a negative feedback.)

(a) Eliminate J from this system to obtain a second-order differential equation for R. Explain why this new equation is stable exactly when $a < 0$ and $bc < 0$. (Stable means that ultimately solutions are very nearly independent of initial condition. What we saw was that a constant coefficent linear ODE is stable exactly when the real parts of the roots of its characteristic polynomial are negative.) Give a psychological interpretation of this result. Assume henceforth that $a < 0, bc < 0$. If $|a|$ is small relative to $|bc|$, the second order equation will be underdamped and its general solution will be of the form

$$R(t) = re^{st}\cos(\omega t - \theta),$$

according to our work in the last section. Exactly what are the conditions on $|a|$ and $|bc|$ which lead to a solution of this form? r and θ are constants of integration, but s and ω are determined by a, b, c. What are they? (Careful of the sign under the square root in the formula for ω!) Interpret these results psychologically: What happens when $|a|$ is large? How is the frequency of Romeo's oscillation of feelings towards Juliet dependent upon the parameters a, b, c? (Specifically: if we increase $|a|$ what happens to ω?) Does this make sense psychologically? If we increase $|b|$ (or $|c|$), keeping the other parameters constant, what happens to ω? Does this make sense psychologically?

(b) Continue to assume that the system is underdamped. By choosing when to start the clock we can take $\theta = 0$, so $R(t) = R(0)e^{st}\cos(\omega t)$. Solve for $J(t)$. It has the form

$$J(t) = je^{st}\cos(\omega t - \phi).$$

Compute ϕ and j in terms of a, b, c. ϕ is a phase lag; it measures how far Juliet lags behind Romeo in their romantic circlings. $\rho = j/R(0)$ is a measure of the ratio of her involvement to his. Can you explain the values of ϕ and of ρ on emotional grounds?

(c) Still assuming the relationship is stable: what happens when $|a|$ is large relative to $|bc|$? Explain why the feelings of Romeo and Juliet for each other just cool off without much drama. Romeo's self-doubting just doesn't leave much room for the relationship to evolve.

(d) Of course, the Capulets and Montagues have an effect on this relationship as well. If this effect is constant over time we have the linear inhomogeneous autonomous system

$$\begin{aligned} R' &= aR + bJ + r_1 \\ J' &= cR + r_2 \end{aligned}$$

Assume that $bc \neq 0$ and explain how this adult supervision leads to a unique stable relationship between Romeo and Juliet, a constant solution. In fact, explain how by adjusting their attitude (as expressed by r_1 and r_2), the adults can produce any combination of feelings (any constant solutions R, J) they like.

22. (W 31 Mar) (a) Write out the system

$$\begin{aligned} R' &= aR + bJ \\ J' &= cR \end{aligned}$$

as a matrix equation using $x = \begin{bmatrix} R \\ J \end{bmatrix}$. Find the eigenvalues of the matrix involved.

(b) Take the case $a = -3, b = -1, c = 2$ in the above: so Romeo has moderately severe self-doubt and a negative response to Juliet's affection (is this masochism?), while Juliet actively reciprocates Romeo's advances. Find the eigenvalues and eigenvectors in this case, and write down the general solution in vector terms. What are the two normal modes?

(c) Finally, determine the particular solution determined by $x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ (so Romeo starts out disliking Juliet and Juliet is neutral towards Romeo). Write down $R(t)$ and $J(t)$. Does Romeo eventually come to like Juliet? At what moment is he fondest of her? How do Juliet's feelings towards Romeo evolve? Does she ever have better than neutral feelings towards him? Sketch a graph of $R(t)$ and of $J(t)$ for $t \geq 0$.