Partial Solutions to 18.03 Problem Set 4, Part II

14. $\{r_1, r_2\} = \{1, 2\}$. If we take $r_1 = 1$, $r_2 = 2$, we get $(D - I)u = x^3$, i.e. $u' - u = x^3$. The integrating factor is e^{-x}, so we get $\frac{d}{dx}(ue^{-x}) = x^3 e^{-x}$. Integrating, we get $ue^{-x} = e^{-x}(-x^3 - 3x^2 - 6x - 6) + C_1$, so

$$u = -x^3 - 3x^2 - 6x - 6 + C_1 e^x.$$

Then $y' - 2y = (D - 2I)y = -x^3 - 3x^2 - 6x - 6 + C_1 e^x$. The integrating factor is e^{-2x}, so we get

$$\frac{d}{dx} e^{-2x} y = (-x^3 - 3x^2 - 6x - 6)e^{-2x} + C_1 e^{-x}.$$

Integrating and multiplying by e^{2x}, we get the full set of solutions:

$$y = \frac{1}{2}x^3 + \frac{9}{4}x^2 + \frac{21}{4}x + \frac{45}{8} + C_2 e^{2x} - C_1 e^x.$$

If we instead take $r_1 = 2$, $r_2 = 1$, we would have $u' - 2u = (D - 2I)u = x^3$. The integrating factor is e^{-2x}, so we get $\frac{d}{dx}(e^{-2x}u) = e^{-2x}x^3$, from which

$$u = -\frac{1}{2}x^3 - \frac{3}{4}x^2 - \frac{3}{4}x - \frac{3}{8} + C_1 e^{2x}.$$

Then $y' - y = u$, so we use an integrating factor of e^{-x} to get:

$$\frac{d}{dx} e^{-x} y = e^{-x}\left(-\frac{1}{2}x^3 - \frac{3}{4}x^2 - \frac{3}{4}x - \frac{3}{8}\right) + C_1 e^x.$$

Integrating and multiplying by e^x, we get

$$y = \frac{1}{2}x^3 + \frac{9}{4}x^2 + \frac{21}{4}x + \frac{45}{8} + C_1 e^{2x} + C_2 e^x.$$

15. **(a)** The solutions for the homogeneous case are of the form $A\cos\omega_0 t + B\sin\omega_0 t$. A specific solution to the original equation $\alpha\cos\omega t$ for some constant α. Solving for α: $\alpha(\omega_0^2 - \omega^2)\cos\omega t = a\cos\omega t$, so $\alpha = a/(\omega_0^2 - \omega^2)$. Thus the general solution is

$$\frac{a}{\omega_0^2 - \omega^2}\cos\omega T + A\cos\omega_0 t + B\sin\omega_0 t.$$

Notice that the solution stays bounded.

(b), (c) The particular solution is

$$\frac{a}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t) = \frac{2a}{\omega_0^2 - \omega^2}\left(\sin\frac{(\omega_0 - \omega)t}{2}\right)\left(\sin\frac{(\omega_0 + \omega)t}{2}\right).$$

(d) We look for a solution of the form $x = \alpha t \sin\omega t$ for some α. Then $\ddot{x} + \omega^2 x = 2\alpha\omega\cos\omega x$, so $\alpha = \frac{a}{2\omega}$. Hence the general solution is

$$x = \frac{a}{2\omega}t\sin\omega t + A\cos\omega t + B\sin\omega t.$$

16. (a)

Rewrite $\frac{dy}{dt} = y^2 - a^2$ as

$$\frac{1}{2a}\left(\frac{1}{y-a} - \frac{1}{y+a}\right) dy = dt.$$

(We've lost the solutions $y = a$ and $y = -a$; we'll put them back in later.) Integrate to get

$$\frac{1}{2a} \ln |(y-a)/(y+a)| = t + C.$$

Exponentiate to get

$$\frac{y-a}{y+a} = \pm e^{2a(t+C)},$$

where C is any real number. By changing constants, we can rewrite this as

$$\frac{y-a}{y+a} = C_2 e^{2at}.$$

Note that we've returned the "lost" solution $y = a$ (the case $C_2 = 0$). Solving for y, we get:

$$y = a\frac{1 + C_2 e^{2at}}{1 - C_2 e^{2at}}$$

(as well as the "lost" solution $y = -a$). On the other hand, dsolve gives us

$$y = -a\frac{e^{2ta - 2C_1 a} - 1}{1 + e^{2ta - 2C_1 a}}.$$

These are the same (let $C_2 = -e^{-2C_1 a}$), except dsolve has lost some solutions (for example, $y = a$, $y = -a$, and the case where C_2 is positive)!

(b) The *correct* solution to problem 4 is

$$y = \frac{e^{-x^2}}{(\sqrt{\pi}/2)erf(x) + C} - x.$$

(The scaling factor was missing!) On the other hand, dsolve gives

$$\begin{aligned}
y &= -\frac{2xC_1 e^{x^2} - x\sqrt{\pi}erf(x)e^{x^2} - 2}{e^{x^2}(2C_1 - \sqrt{\pi}erf(x))} \\
&= \frac{-2}{e^{x^2}(2C_1 - \sqrt{\pi}erf(x))} - x \\
&= e^{-x^2}\frac{-2}{2C_1 - \sqrt{\pi}erf(x)} - x \\
&= e^{-x^2}\frac{1}{(\sqrt{\pi}/2)erf(x) + C} - x
\end{aligned}$$

(where $C = -C_1$) as desired.

(c) $y(2)$ is approximately -1.8447.

(d) $y_1 = e^{-x}(\cos(4x) + \frac{1}{4}\sin(4x))$, $y_2 = \frac{1}{4}e^{-x}\sin(4x)$.