Solutions to 18.03 Problem Set 4.5

Part II.

17. (F 12 Mar) (a) Remember, we know y_1 and wish to find a second solution, which we'll call y for the moment. So Abel's equation is

$$y' - \frac{y_1'}{y_1} y = \frac{e^{-\int p(x)dx}}{y_1}. \tag{1}$$

Multiply by an integrating factor (badly named w, since it's not the Wronskian) and demand of it that

$$wy' + w'y = (wy)' = wy' - \frac{wy_1'}{y_1} y$$

or $w' = -\frac{y_1'}{y_1} w$. This is separable, and leads to

$$\frac{dw}{w} = -\frac{y_1'}{y_1} dx,$$

or

$$\ln |w| = -\ln |y_1| + c$$

so, since we can choose c as we like, we can take w to be $1/y_1$. Inspecting (1) again we see that we could have seen this without doing the work. Anyway, this leads to

$$\left(\frac{y}{y_1}\right)' = \frac{1}{y_1^2} e^{-\int p(x)dx}$$

or

$$y = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx,$$

which is EP 2.6 (8).

(b) You have to know how to interpret $(xD)^2$: it's the operator you get by doing xD twice in succession: $(xD)^2 u = xD(xDu) = xD(xu') = x(xu'' + u') = x^2 u'' + xu' = (x^2 D^2 + xD)u$.

(c) Let $x = e^t$, and write $\dot{y} = dy/dt$, etc. Then $\dot{x} = x$,

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{x},$$

$$y'' = \frac{dy'}{dx} = \frac{1}{x} \frac{d}{dt} \frac{\dot{y}}{x} = \frac{1}{x} \frac{\ddot{y} - \dot{y}}{x^2} \frac{dx}{dt} = \frac{\ddot{y} - \dot{y}}{x^2}.$$

Thus $x^2 y'' + pxy' + qy = r(x)$ is, in terms of t, $(\ddot{y} - \dot{y}) + p\dot{y} + qy = r(e^t)$, or

$$\ddot{y} + (p-1)\dot{y} + qy = r(e^t).$$

This substitution converts an Euler-Cauchy equation to a constant coefficient equation. For example, $x(\ln x)^2 \sin(\ln x) = e^t t^2 \sin t$ and the given equation reads

$$\ddot{y} - 2\dot{y} + y = t^2 e^t \sin t.$$

Solve this by replacing it with another ODE having a (complex) exponential on the RHS:

$$Lz = \ddot{z} - 2\dot{z} + z = t^2 e^{st}, \qquad s = 1 + i; \qquad y = \text{Im } z.$$

This calls for ESL: $f(r) = (r-1)^2$, so replacing r by $(D + sI)$ we get $(D + (s-1)I)^2 = D^2 + 2iD - I$ (since $s - 1 = i$). Now write $z = e^{st}u$ and apply ESL:

$$t^2 e^{st} = f(D)(e^{st}u) = e^{st} f(D + sI)u = e^{st}(D^2 + 2iD - I)u$$

so we wish to solve $(D^2 + 2iD - I)u = t^2$. This we can handle by undetermined coefficients: $u = at^2 + bt + c$ implies

$$\begin{array}{rcrrr} -z & = & -at^2 & -bt & -c \\ 2iz' & = & & 4iat & +2bi \\ z'' & = & & & 2a \end{array}$$

Equating coefficients, $-a = 1$ so $a = -1$; $-b + 4ia = 0$ so $b = -4i$, $-c + 2bi + 2a = 0$ so $c = 6$: $u = -t^2 - 4it + 6$; so $z_p = (-t^2 - 4it + 6)e^{(1+i)t} = (-t^2 - 4it + 6)e^t(\cos t + i \sin t)$; so $y_p = \text{Im } z_p = e^t((-t^2 + 6)\sin t - 4t\cos t) = x(-(\ln x)^2 + 6)\sin(\ln x) - 4(\ln x)\cos(\ln x))$.

This is a particular solution. The homogeneous equation $x^2 y'' - xy' + y = 0$ is equivalent, under the same substition, to $\ddot{y} - 2\dot{y} + y = 0$. The characteristic polynomial $r^2 - 2r + 1$ has repeated root $r = 1$, so we have independent solutions e^t, te^t. In terms of x these are $x, x \ln x$; so the general solution is

$$= x(-(\ln x)^2 + 6)\sin(\ln x) - 4(\ln x)\cos(\ln x)) + ax + bx\ln x.$$

(I hope I didn't make an arithmetic mistake!)