Partial Solutions to 18.03 Problem Set 3

Part I 13.

33. $y(x) = 0$. 35. $y(x) = c_1 + c_2 e^{-5x}$. 37. $y(x) = c_1 e^{-x/2}$. 39. $y(x) = (c_1 + c_2 x)e^{-x/2}$. (These are all bounded: use l'Hopital's rule to check that $\lim_{x\to\infty} y(x) = 0$.) 41. $y(x) = c_1 e^{-4x/3}$.

Part II 12. This problem was revised so the Wronskian agreed with the definition given in the book ($w = y_1 y_2' - y_1' y_2$). This is a solution to the original problem; for the revised problem, some signs must be changed. Full credit will be given to solutions of either version.

(a) $w' = (y_1' y_2 - y_2' y_1)' = (y_1' y_2' + y_2 y_1'') - (y_1 y_2'' - y_1' y_2') = y_1'' y_2 - y_1 y_2''$.

(b)

$$\begin{aligned} w' &= y_1'' y_2 - y_1 y_2'' \\ &= (-p(x) y_1' - q(x) y_1) y_2 - (-p(x) y_2' - q(x) y_2) y_1 \\ &= -p(x)(y_1' y_2 - y_1 y_2') \\ &= -p(x) w. \end{aligned}$$

Hence $w' + p(x) w = 0$.

(c) Exponentials are never zero. So if $w(x) = ce^{-\int p(x)dx}$ is 0 for some x, then c is 0, so $w(x)$ is zero for *all* x.

If $p(x)$ is the 0 function, then $w' = -pw = 0$, so w is constant.

(d) If $a = 0$, then two linearly independent solutions are $y_1 = 1$ and $y_2 = x$. The Wronskian is -1.

If a is positive, then two linearly independent solutions are $y_1 = \sin(\sqrt{a}x)$ and $y_2 = \cos(\sqrt{a}x)$. The Wronskian is

$$y_1' y_2 - y_1 y_2' = \sqrt{a}\cos^2(\sqrt{a}x) + \sqrt{a}\sin^2(\sqrt{a}x) = \sqrt{a}.$$

If a is negative, then two linearly independent solutions are $y_1 = e^{\sqrt{a}x}$ and $y_2 = e^{-\sqrt{a}x}$. The Wronskian is

$$y_1' y_2 - y_1 y_2' = \left(\sqrt{a} e^{\sqrt{a}x}\right) e^{-\sqrt{a}x} - e^{\sqrt{a}x}\left(-\sqrt{a} e^{-\sqrt{a}x}\right) = 2\sqrt{a}.$$

(e) If $y_1 = x$, then $x^2 y_1'' + x y_1' - y_1 = 0 + x \cdot 1 - x = 0$ as desired.

In standard form, the differential equation is $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$, so $p(x) = \frac{1}{x}$. From Abel's formula, the Wronskian w (using $c = 1$) is:

$$w = e^{-\int p(x)dx} = e^{-\ln x} = \frac{1}{x}.$$

But also $w = y_1' y_2 - y_1 y_2'$, so $\frac{1}{x} = y_2 - x y_2'$. We now know how to find y_2: rewrite this as $y_2' - \frac{1}{x} y_2 = \frac{1}{x^2}$. Multiplying by an integrating factor of $1/x$, we find $(\frac{d}{dx}\frac{y_2}{x}) = \frac{1}{x} y_2' - \frac{1}{x^2} y_2 = \frac{1}{x^3}$. Hence $\frac{y_2}{x} = c_2 - \frac{1}{2x^2}$, so $y_2 = c_2 x - \frac{1}{2x}$. For any c, this is linearly independent of $y_1 = x$. With $c = 0$, we have $-2/x$. By linearity (and homogeneity), $y_2 = \frac{1}{x}$ is also a solution. Check: $y_2' = -\frac{1}{x^2}$, $y_2'' = \frac{2}{x^3}$, so

$$x^2 y_2'' + x y_2' - y_2 = x^2 \cdot \frac{2}{x^3} + x \cdot \left(-\frac{1}{x^2}\right) - \frac{1}{x} = 0.$$