18.03 Problem Set 3

Due by 1:00 P.M., Friday, March 5, 1999, in the boxes at 2-106, next to the Undergraduate Mathematics Office.

Syllabus: Part II: Second order linear equations

9. (M 22 Feb) Second order linear equations: EP 2.1, 2.2.

10. (W 24 Feb) Complex numbers: Notes C.

11. (F 26 Feb) Complex-valued functions; the case of real roots: EP 2.3.

12. (M 1 Mar) Non-real and repeated roots: 2.3, 2.4.

13. (W 3 Mar) Initial value problems, Wronskian, operators: Notes O 4–8.

14. (F 5 Mar) Operators; higher order equations: Notes O 4–8.

Part I.

12. (M 1 Mar) EP 2.1: 39, 41.

A spring weakens very slowly over time: the spring constant k decreases, but so slowly that the motion at any given moment is as if it were constant. Assume that we have a system with this spring, a dashpot with constant c, and a mass m, controlled by the ODE

$$m\ddot{x} + c\dot{x} + kx = 0.$$

At what value of k does the system stop exhibiting "underdamped" oscillatory behavior and transition to "overdamped" exponential decay?

13. (W 3 Mar) In each of EP 2.1: 33, 35, 37, 39, 41, determine all solutions which do not become arbitrarily large in absolute value as x goes to infinity.

Part II.

12. (M 1 Mar) This problem involves the "Wronskian," which is described in EP 2.1. Consider the homogeneous second order linear equation

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

defined on some interval, and let y_1 and y_2 be any two solutions to it. The "Wronskian" of these two solution is

$$w = y_1 y_2' - y_1' y_2.$$

Notice that if we multiply y_2 by a constant, then the Wronskian gets multiplied by the same constant.

(a) Show that $w' = y_1 y_2'' - y_1'' y_2$.

(b) Show that w satisfies the homogeneous *first order* linear equation

$$w' + p(x)w = 0. \tag{2}$$

This is very interesting, since (2) does not depend at all upon the specific solutions y_1 and y_2. By separation of variables,

$$w = ce^{-\int p(x)dx} \tag{3}$$

for some constant c. This constant of integration c is the only dependence w has on the pair of solutions. (3) is *Abel's formula*.

(c) Conclude that either $w(x)$ is zero for all x, or else $w(x)$ is zero for no x. Also: if $p = 0$, w is constant.

(d) For example, find a linearly independent pair of solutions of $y'' + ay = 0$ (whose form will depend upon the constant a) and verify that the Wronskian is constant.

(e) Abel's formula can sometimes be used to find a second solution to (1) if we have a first solution. For example consider

$$x^2y'' + xy' - y = 0 \tag{4}$$

on the half line where $x > 0$. Verify that $y_1 = x$ is a solution. Use Abel's formula (taking $c = 1$) to compute the Wronskian. Then use the *definition* of the Wronskian to write down an inhomogeneous first order linear differential equation satisfied by y_2. Finally, solve this equation and verify that its solution satisfies (4).

13. (W 3 Mar) Nothing new.