Solutions to 18.03 Problem Set 2 (Part II)

8. (a)

Let $h = 1/N$ be the step size.

Euler. $y_n = y_{n-1} + hy_{n-1} = (1 + \frac{1}{N})y_{n-1}$.

Heun. $y_n = y_{n-1} + \frac{h}{2}(k_1 + k_2)$, $k_1 = y_{n-1}$, $k_2 = y_{n-1} + hk_1 = (1+h)y_{n-1}$, so

$$y_n = y_{n-1} + \frac{h}{2}(2+h)y_{n-1} = \left(1 + h + \frac{h^2}{2}\right) y_{n-1} = \left(1 + \frac{1}{N} + \frac{1}{2N^2}\right) y_{n-1}.$$

Runge-Kutta (see EP p. 480). $k_1 = y_{n-1}$. $k_2 = y_{n-1} + \frac{1}{2}hy_{n-1} = (1 + \frac{h}{2})y_{n-1}$.

$$k_3 = y_{n-1} + \frac{1}{2}h(y_{n-1} + \frac{h}{2}y_{n-1}) = \left(1 + \frac{h}{2} + \frac{h^2}{4}\right) y_{n-1}.$$

$$k_4 = y_{n-1} + hk_3 = \left(1 + h + \frac{h^2}{2} + \frac{h^3}{4}\right) y_{n-1}.$$

Finally,

$$y_n = y_{n-1} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = y_{n-1}\left(1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}\right).$$

(Some of you may recognize $1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}$ as being the first five terms of a longer series.) Using $h = 1/N$, we have:

$$y_n = y_{n-1}\left(1 + \frac{1}{N} + \frac{1}{2N^2} + \frac{1}{6N^3} + \frac{1}{24N^4}\right).$$

(b) *Euler*: $(1+\frac{1}{N})^N$. *Heun*: $(1+\frac{1}{N}+\frac{1}{2N^2})^N$. *Runge-Kutta*: $\left(1 + \frac{1}{N} + \frac{1}{2N^2} + \frac{1}{6N^3} + \frac{1}{24N^4}\right)^N$.

(c) The three processes give the following values (for various N):

	$N=2$	$N=4$	$N=8$	$N=16$
Euler	2.25	2.44140625	2.5657845	2.6379285
Heun	2.640625	2.6948557	2.7118412	2.7165935
Runge-Kutta	2.7173462	2.7182099	2.7182768	2.7182815

(d) The difference between e and each entry in the above table (rounded) is:

	$N=2$	$N=4$	$N=8$	$N=16$
Euler	.4682818	.2768756	.1524973	.0803533
Heun	.0776568	.0234261	.0064406	.0016883
Runge-Kutta	.0009356	.0000719	.0000050	.0000003

Visibly, the entries in the Euler row decrease by roughly a factor of 2; for Heun by a factor of 4; for Runge-Kutta by a factor of 16.

(e) Heun(2)/Euler(4) is about 4. Heun(4)/Euler(8) is about 7. Heun(8)/Euler(16) is about 13. RK(2)/Heun(4) is about 25. RK(4)/Heun(8) is about 90. RK(8)/Heun(16) is about 338. Runge-Kutta is the clear winner.

8.5 —Corrected version. (b) 3.4885

(c) As $\frac{dx}{dt} = -1$,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = -(y^3 - 3y - x) = -y^3 + 3y - t.$$

(e) The value of y' changes drastically over distances much smaller than the step size.

9. *Warning:* you can't substitute *any* values of of x; they must lie between c and d.

(a) Pick two values x_0 and x_1 such that $\cos(x_i) \neq 0$ and $\tan(x_0) \neq \tan(x_1)$. Suppose $af + bg = 0$. Then substituting $x = x_0$, we get $a\sin(x_0) + b\cos(x_0) = 0$, so $a\tan(x_0) + b = 0$. Similarly $a\tan(x_1) + b = 0$. Hence $a\tan(x_0) = a\tan(x_1)$. As $\tan(x_0) \neq \tan(x_1)$, a must be 0. Hence $b = -a\tan(x_1)$ is also 0.

(b) Pick two distinct values x_0 and x_1 between c and d. Proceed as before: suppose $af + bg = 0$. Then $ae^{x_0} + bx_0e^{x_0} = 0$, so (as $e^{x_0} \neq 0$) $a = -bx_0$. Similarly, $a = -bx_1$, so $bx_0 = bx_1$. As $x_0 \neq x_1$, b must be 0, so $a = -bx_1$ must also be 0.

(c) This is similar to (b). Pick two distinct values x_0 and x_1 between c and d. Suppose $af + bg = 0$. Then $ax_0{}^m + bx_0x_0{}^n = 0$, so (as $x_0^m \neq 0$) $a = -bx_0^{n-m}$. Similarly, $a = -bx_1^{n-m}$, so $bx_0^{n-m} = bx_1^{n-m}$. As $x_0 \neq x_1$, b must be 0, so $a = -bx_1^{n-m}$ must also be 0.

10. (a)

$$\begin{aligned}
\int e^{kt}\cos(\omega t)dt &= \int \mathrm{Re}\ \left(e^{(k+i\omega)t}\right) dt + c \\
&= \mathrm{Re}\ \left(\int e^{(k+i\omega)t}dt\right) + c \\
&= \mathrm{Re}\ \left(\frac{1}{k+i\omega}e^{(k+i\omega)t}\right) + c \\
&= \mathrm{Re}\ \left(\frac{k-i\omega}{1}k^2 + \omega^2 e^{kt}(\cos(\omega t) + i\sin(\omega t))\right) + c \\
&= e^{kt}\mathrm{Re}\ \left(\frac{k\cos(\omega t) + \omega\sin(\omega t)}{k^2+\omega^2} + i\frac{k\sin(\omega t) - \omega\cos(\omega t)}{k^2+\omega^2}\right) + c \\
&= e^{kt}\mathrm{Re}\ \left(\frac{k\cos(\omega t) + \omega\sin(\omega t)}{k^2+\omega^2} + i\frac{k\sin(\omega t) - \omega\cos(\omega t)}{k^2+\omega^2}\right) + c \\
&= e^{kt}\frac{k\cos(\omega t) + \omega\sin(\omega t)}{k^2+\omega^2} + c
\end{aligned}$$

To rephrase the answers in terms of φ and r, we use $k^2 + \varphi^2 = r^2$, $k = r\cos\varphi$, and $\omega = r\sin\varphi$ (polar coordinates). Hence

$$\int e^{kt}\cos(\omega t)dt = e^{kt}\frac{r\cos\varphi\cos(\omega t) + r\sin\varphi\sin(\omega t)}{r^2} + c = \frac{1}{r}e^{kt}\cos(\omega t - \varphi) + c.$$

(b) The integrating factor is e^{kt}, so we want to solve

$$\frac{d}{dt}\left(e^{kt}x\right) = ke^{kt}\cos(\omega t).$$

Integrating both sides with respect to t, we get

$$e^{kt}x = ake^{kt}\frac{k\cos(\omega t)+\omega\sin(\omega t)}{k^2+\omega^2} + c$$

so

$$x = ak\frac{k\cos(\omega t)+\omega\sin(\omega t)}{k^2+\omega^2} + ce^{-kt}.$$

11. **(a)** As $c_1e^{\omega x} + c_2e^{-\omega x} = (a+b)e^{\omega x}(a-b)e^{-\omega x}$, $a+b = c_1$ and $a-b = c_2$. Thus $a = (c_1+c_2)/2$, $b = (-c_1+c_2)/2$.

(b) In (1), $y(0) = 1$ gives $a = 1$. Then

$$y' = -\frac{p}{2}e^{-px/2}\left(a\cos(\omega x) + b\sin(\omega x)\right) + e^{-px/2}\left(-a\omega\sin(\omega x) + b\omega\cos(\omega x)\right).$$

Then $y'(0) = 0$ gives $-\frac{p}{2}a + b\omega = 0$, so $b = \frac{pa}{2\omega} = \frac{p}{2\omega}$.

This gives the solution

$$e^{-px/2}\left(\cos(\omega x) + \frac{p}{2\omega}\sin(\omega x)\right).$$

In (2), $y(0) = 1$ gives $a = 1$ again. $y' = -\frac{p}{2}e^{-px/2}(a+bx) + be^{-px/2}$. Then $y'(0) = 0$ gives $-\frac{p}{2}a + b$, so $b = pa/2 = p/2$. This gives the solution $e^{-px/2}\left(1 + \frac{p}{2}x\right)$.

The case of (3) is similar to (1). Notice first that $\frac{d}{dz}\cosh(z) = \sinh(z)$ and $\frac{d}{dz}\sinh(z) = \cosh(z)$.

As before $y(0) = 0$ gives $a = 1$.

$$y' = -\frac{p}{2}e^{-px/2}\left(a\cosh(\omega x) + b\sinh(\omega x)\right) + e^{-px/2}\left(a\omega\sinh(\omega x) + b\omega\cosh(\omega x)\right)$$

Then $y'(0) = 0$ gives $-\frac{p}{2}a + b\omega = 0$, so $b = \frac{pa}{2\omega} = \frac{p}{2\omega}$ (as in (1)).

This gives the solution

$$e^{-px/2}\left(\cosh(\omega x) + \frac{p}{2\omega}\sinh(\omega x)\right).$$

(c) Some preliminaries: $q = 4$, so $\omega(p) = \sqrt{|(p/2)^2 - 4|}$. Notice that $\omega(4) = 0$; as p approaches 4 (from either above or below), $\omega(p)$ approaches 0 (from above).

If $p = 4$, we are in case (2), and $y_4(x) = e^{-2x}(1+2x)$ from (b).

If $p < 4$, we wish to show that

$$\lim_{p\to 4^-} e^{-px/2}\left(\cos(\omega x) + \frac{p}{2\omega}\sin(\omega x)\right) = e^{-2x}(1+2x).$$

As $\lim_{p\to 4^-} e^{-px/2}\cos(\omega(p)x) = e^{-2x}$, we wish to show that

$$\lim_{p\to 4^-} e^{-px/2}\left(\frac{p}{2\omega}\sin(\omega x)\right) = e^{-2x}(2x),$$

i.e.

$$\lim_{p\to 4^-} \frac{\sin(\omega x)}{\omega} = x.$$

This can be checked to be true using l'Hopital's rule. For example, a slick approach would be

$$\lim_{p\to 4^-} \frac{\sin(\omega x)}{\omega} = \lim_{\omega\to 0} \frac{\sin(\omega x)}{\omega} = \lim_{\omega\to 0} \frac{x\cos(\omega x)}{1} = x.$$

If $p > 4$, we wish to show that

$$\lim_{p\to 4^+} e^{-px/2} \left(\cosh(\omega x) + \frac{p}{2\omega}\sinh(\omega x)\right) = e^{-2x}(1 + 2x).$$

As $\lim_{p\to 4^+} e^{-px/2}\cosh(\omega(p)x) = e^{-2x}$, we wish to show that

$$\lim_{p\to 4^+} e^{-px/2} \left(\frac{p}{2\omega}\sinh(\omega x)\right) = e^{-2x}(2x),$$

i.e.

$$\lim_{p\to 4^+} \frac{\sinh(\omega x)}{\omega} = x.$$

This can be checked to be true using l'Hopital's rule.