18.03 Practice Hour Exam I

1. This problem concerns the differential equation $\dfrac{dx}{dt} = x^2 - x$.

(a) This is an autonomous ODE. Draw the graph of $\dot{x}$ against x. Indicate the critical points for the system and classify them—stable, unstable, or neutrally stable.

There are two critical points, $x = 0$ and $x = 1$ (the roots of $x^2 - x = 0$). The phase portrait looks roughly as follows:

$$>>>>>>\ 0 <<<<<<\ 1 >>>>>>$$

so we see that 0 is stable and 1 is unstable.

(b) Sketch some solution curves in the (t, x) plane, including all constant solutions. Indicate all points of inflection.

The point of inflection is where $\frac{d}{dx}(x^2 - x) = 2x - 1 = 0$, i.e. $x = 1/2$.

(c) What is the general solution?

We use separation of variables and partial fractions to get

$$\int \left(-\frac{1}{x} + \frac{1}{x-1} \right) dx = t + C$$

from which we get $\ln |(x-1)/x| = t + C$. As $(x-1)/x = 1 - 1/x$, we can rearrange this to get

$$x = \frac{1}{1 - C_2 e^t}.$$

Along with solution $x = 0$ (where did we lose it?), these are all the solutions. (Note that we temporarily lost the solution $x = 1$, but it turned up again in the final family; just take $C_2 = 0$.)

(d) Find and solve a differential equation for the family of curves orthogonal to the solutions of this differential equation. Return to **1(b)** and sketch in some of these curves (using dotted lines).

The family of orthogonal curves satisfies the differential equation

$$\frac{dx}{dt} = -\frac{1}{x^2 - x}.$$

Using separation of variables again, we get $(-x^2+x)dx = dt$, from whence $-x^3/3 + x^2/2 = t + C$.

2. (a) A certain species of sea-slug undergoes periodic fluctuations in its fertility, with the result that the birth-rate in the population in the Sargasso Sea has the form $\alpha(\beta - \cos(\omega t))$ births per individual per year for certain constants α, β, ω. If the death-rate is constant, say δ deaths per individual per year, write down the ODE modeling the population.

$$x' = (\alpha(\beta - \cos(\omega t)) - \delta)x$$

(b) Before you attempt to solve this, identify which of the following adjectives describe this ODE correctly. (a) Linear. (b) Homogeneous Linear. (c) Homogeneous nonlinear. (d) Separable. (e) Autonomous.

Answer: (d) Separable.

(c) Now find the general solution.

We use separation of variables to get $x = 0$ or

$$\begin{aligned} \frac{dx}{x} &= (\alpha(\beta - \cos(\omega t)) - \delta)\, dt \\ \text{so } \ln|x| &= C + (\alpha\beta - \delta)t - \frac{\alpha}{\omega}\sin(\omega t) \\ \text{so } x &= C_2 e^{(\alpha\beta-\delta)t - \frac{\alpha}{\omega}\sin(\omega t)} \end{aligned}$$

(d) It is observed that while the population does fluctuate over the year, it doesn't explode or crash. What relation does this observation impose on the constants α, β, δ? Is this reasonable? In this "steady state," what is the ratio of the maximal population to the minimal one (in terms of the constants α, ω).

The constraint is $\alpha\beta = \delta$. Then the ratio of the maximal population to the minimal one is $e^{2\alpha/\omega}$.

(e) Take $\omega = \alpha = \beta = \delta = 1$ and initial condition $x(0) = 1$, and use MATLAB's `ezplot` to plot the population function that you compute. [This is a *practice* exam, after all, right?] Use `hold on` and plot also the fertility rate function. (Remember to put single quotes around the functions you submit to `ezplot`.) The population maxima don't occur at the same time as the birth-rate maxima, do they? When do they occur?

3. Find the general solution to the homogeneous ODE $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$. Don't forget the "obvious" solution!

The "obvious" solution is $x = 0$. If $x \neq 0$, we rewrite the equation as

$$\frac{dy}{dx} = \frac{1}{2}\left(1 + (\frac{y}{x})^2\right)$$

and substitute $v = y/x$, so $y' = xv'$. Then

$$xv' = \frac{1 + v^2}{2} - v = \frac{(1 - v)^2}{2}.$$

We separate variables to get $2dv/(v - 1)^2 = \frac{dx}{x}$, and integrate to get

$$-\frac{2}{v - 1} = \ln|x| + C$$

Hence

$$y = x\left(-\frac{2}{\ln|x| + C} + 1\right).$$

4. This problem pertains to the differential equation $\dfrac{dy}{dx} = x + y$.

(a) Sketch some isoclines (using dotted lines), the direction field, and some solution curves of the equation.

(The isoclines are straight lines, of the form $y = -x + C$.)

(b) Solve the initial value problem $\dfrac{dy}{dx} = x + y, y(0) = 2$. Announce at the outset the method you intend to use.

We use the substitution method, substituting $v = x + y$, so $y' = v' - 1$ (and our initial condition is v(0) = 2). Then the equation can be rewritten $v' - 1 = v$, so we can separate variables to get

$$\frac{dv}{v+1} = dx$$

as $v + 1 \neq 0$ (from our initial condition). Hence $\ln|v + 1| = x + C$, and by our initial condition $C = \ln 3$. Thus $|v + 1| = 3e^x$, and from the initial condition we know $v + 1$ is positive, so $v = 3e^x - 1$. Thus $y = 3e^x - x - 1$. We check that y' indeed $x + y$:

$$y' = 3e^x - 1 = (3e^x - x - 1) + x.$$