

18.03 Practice Hour Exam I

1. This problem concerns the differential equation $\frac{dx}{dt} = x^2 - x$.

(a) This is an autonomous ODE. Draw the graph of \dot{x} against x . Indicate the critical points for the system and classify them—stable, unstable, or neutrally stable.

There are two critical points, $x = 0$ and $x = 1$ (the roots of $x^2 - x = 0$). The phase portrait looks roughly as follows:

$$>>>>>> 0 <<<<<<< 1 >>>>>>$$

so we see that 0 is stable and 1 is unstable.

(b) Sketch some solution curves in the (t, x) plane, including all constant solutions. Indicate all points of inflection.

The point of inflection is where $\frac{d}{dx}(x^2 - x) = 2x - 1 = 0$, i.e. $x = 1/2$.

(c) What is the general solution?

We use separation of variables and partial fractions to get

$$\int \left(-\frac{1}{x} + \frac{1}{x-1} \right) dx = t + C$$

from which we get $\ln |(x-1)/x| = t + C$. As $(x-1)/x = 1 - 1/x$, we can rearrange this to get

$$x = \frac{1}{1 - C_2 e^t}.$$

Along with solution $x = 0$ (where did we lose it?), these are all the solutions. (Note that we temporarily lost the solution $x = 1$, but it turned up again in the final family; just take $C_2 = 0$.)

(d) Find and solve a differential equation for the family of curves orthogonal to the solutions of this differential equation. Return to **1(b)** and sketch in some of these curves (using dotted lines).

The family of orthogonal curves satisfies the differential equation

$$\frac{dx}{dt} = -\frac{1}{x^2 - x}.$$

Using separation of variables again, we get $(-x^2 + x)dx = dt$, from whence $-x^3/3 + x^2/2 = t + C$.

2. (a) A certain species of sea-slug undergoes periodic fluctuations in its fertility, with the result that the birth-rate in the population in the Sargasso Sea has the form $\alpha(\beta - \cos(\omega t))$ births per individual per year for certain constants α, β, ω . If the death-rate is constant, say δ deaths per individual per year, write down the ODE modeling the population.

$$x' = (\alpha(\beta - \cos(\omega t)) - \delta)x$$

(b) Before you attempt to solve this, identify which of the following adjectives describe this ODE correctly. (a) Linear. (b) Homogeneous Linear. (c) Homogeneous nonlinear. (d) Separable. (e) Autonomous.

Answer: (d) Separable.

(c) Now find the general solution.

We use separation of variables to get $x = 0$ or

$$\begin{aligned} \frac{dx}{x} &= (\alpha(\beta - \cos(\omega t)) - \delta) dt \\ \text{so } \ln|x| &= C + (\alpha\beta - \delta)t - \frac{\alpha}{\omega} \sin(\omega t) \\ \text{so } x &= C_2 e^{(\alpha\beta - \delta)t - \frac{\alpha}{\omega} \sin(\omega t)} \end{aligned}$$

(d) It is observed that while the population does fluctuate over the year, it doesn't explode or crash. What relation does this observation impose on the constants α, β, δ ? Is this reasonable? In this "steady state," what is the ratio of the maximal population to the minimal one (in terms of the constants α, ω).

The constraint is $\alpha\beta = \delta$. Then the ratio of the maximal population to the minimal one is $e^{2\alpha/\omega}$.

(e) Take $\omega = \alpha = \beta = \delta = 1$ and initial condition $x(0) = 1$, and use MATLAB's `ezplot` to plot the population function that you compute. [This is a *practice* exam, after all, right?] Use `hold on` and plot also the fertility rate function. (Remember to put single quotes around the functions you submit to `ezplot`.) The population maxima don't occur at the same time as the birth-rate maxima, do they? When do they occur?

3. Find the general solution to the homogeneous ODE $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$. Don't forget the "obvious" solution!

The "obvious" solution is $x = 0$. If $x \neq 0$, we rewrite the equation as

$$\frac{dy}{dx} = \frac{1}{2} \left(1 + \left(\frac{y}{x}\right)^2 \right)$$

and substitute $v = y/x$, so $y' = xv'$. Then

$$xv' = \frac{1 + v^2}{2} - v = \frac{(1 - v)^2}{2}.$$

We separate variables to get $2dv/(v - 1)^2 = \frac{dx}{x}$, and integrate to get

$$-\frac{2}{v - 1} = \ln|x| + C$$

Hence

$$y = x \left(-\frac{2}{\ln|x| + C} + 1 \right).$$

4. This problem pertains to the differential equation $\frac{dy}{dx} = x + y$.

(a) Sketch some isoclines (using dotted lines), the direction field, and some solution curves of the equation.

(The isoclines are straight lines, of the form $y = -x + C$.)

(b) Solve the initial value problem $\frac{dy}{dx} = x + y, y(0) = 2$. Announce at the outset the method you intend to use.

We use the substitution method, substituting $v = x + y$, so $y' = v' - 1$ (and our initial condition is $v(0) = 2$). Then the equation can be rewritten $v' - 1 = v$, so we can separate variables to get

$$\frac{dv}{v+1} = dx$$

as $v + 1 \neq 0$ (from our initial condition). Hence $\ln|v + 1| = x + C$, and by our initial condition $C = \ln 3$. Thus $|v + 1| = 3e^x$, and from the initial condition we know $v + 1$ is positive, so $v = 3e^x - 1$. Thus $y = 3e^x - x - 1$. We check that y' indeed $x + y$:

$$y' = 3e^x - 1 = (3e^x - x - 1) + x.$$