18.03 Practice Hour Exam I

1. This problem concerns the differential equation $\frac{dx}{dt} = x^2 - x$.

(a) This is an autonomous ODE. Draw the graph of $\dot{x}$ against x. Indicate the critical points for the system and classify them—stable, unstable, or neutrally stable.

(b) Sketch some solution curves in the (t, x) plane, including all constant solutions. Indicate all points of inflection.

(c) What is the general solution?

(d) Find and solve a differential equation for the family of curves orthogonal to the solutions of this differential equation. Return to **1(b)** and sketch in some of these curves (using dotted lines).

2. (a) A certain species of sea-slug undergoes periodic fluctations in its fertility, with the result that the birth-rate in the population in the Sargasso Sea has the form $\alpha(\beta - \cos(\omega t))$ births per individual per year for certain constants α, β, ω. If the death-rate is constant, say δ deaths per individual per year, write down the ODE modeling the population.

(b) Before you attempt to solve this, identify which of the following adjectives describe this ODE correctly. (a) Linear. (b) Homogeneous Linear. (c) Homogeneous nonlinear. (d) Separable. (e) Autonomous.

(c) Now find the general solution.

(d) It is observed that while the population does fluctuate over the year, it doesn't explode or crash. What relation does this observation impose on the constants α, β, δ? Is this reasonable? In this "steady state," what is the ratio of the maximal population to the minimal one (in terms of the constants α, ω).

(e) Take $\omega = \alpha = \beta = \delta = 1$ and initial condition $x(0) = 1$, and use MATLAB's `ezplot` to plot the population function that you compute. [This is a *practice* exam, after all, right?] Use `hold on` and plot also the fertility rate function. (Remember to put single quotes around the functions you submit to `ezplot`.) The population maxima don't occur at the same time as the birth-rate maxima, do they? When do they occur?

3. Find the general solution to the homogeneous ODE $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$. Don't forget the "obvious" solution!

4. This problem pertains to the differential equation $\frac{dy}{dx} = x + y$.

(a) Sketch some isoclines (using dotted lines), the direction field, and some solution curves of the equation.

(b) Solve the initial value problem $\frac{dy}{dx} = x + y, y(0) = 2$. Announce at the outset the method you intend to use.