Phase portraits in two dimensions

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It is convenient to represent the solutions to an autonomous system $\vec{x}' = \vec{f}(\vec{x})$ (where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$) by means of a *phase portrait.* The x, y plane is called the *phase plane* (because a point in it represents the state or phase of a system). The phase portrait is a representative sampling of trajectories of the system. A *trajectory* is the path traced out by a solution. It does *not* include information about the time at which solutions pass through various points (this will depend upon when the clock was set), nor does it display the speed at which the solution passes through the point. Still, it conveys essential information about the qualititative behavior of solutions of the system of equations.

The building blocks for the phase portrait of a general system will be the phase portraits of *homogeneous linear constant coefficient* systems: $\vec{x}' = A\vec{x}$, where A is a constant square matrix. Notice that this equation *is* autonomous!

The phase portraits of these linear systems display a startling variety of shapes. We'll want names for them, and the names I'll use differ slightly from the names used in the book and also from the names used in the notes. Hence this handout.

One thing that can be read off from the phase portrait is the stability properties of the system. A linear autonomous system is *unstable* if most of its solutions tend to infinity with time. (The meaning of "most" will become clearer below.) It is *asymptotically stable* if all of its solutions tend to 0 as t goes to ∞. This is the condition we were calling "stable" earlier in the course. Finally, it is *neutrally stable* if none of its solutions tend to infinity with time but most of them do not tend to zero either. It is an interesting fact that any linear autonomous system exhibits one of these three behaviors.

Recall that the general solution to a system $\vec{x}' = A\vec{x}$ is usually of the form $c_1 e^{\lambda_1 t}\vec{\alpha}_1 + c_2 e^{\lambda_2 t}\vec{\alpha}_2$, where λ_1, λ_2 are the eigenvalues of the matrix A and $\vec{\alpha}_1, \vec{\alpha}_2$ are corresponding nonzero eigenvectors. There are two caveats. First, this is not necessarily the case if the eigenvalues coincide. In two dimensions, when the eigenvalues coincide one of two things happens. (1) The *complete* case. Then $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, *every* vector is an eigenvector (for the eigenvalue $\lambda_1 = \lambda_2$), and the general solution is $e^{\lambda_1 t}\vec{\alpha}$ where $\vec{\alpha}$ is any vector. (2) The *incomplete* case. Then there is (up to multiple) only one eigenvector, $\vec{\alpha}_1$, and the general solution is $\vec{x} = e^{\lambda_1 t}(c_1\vec{\alpha}_1 + c_2(t\vec{\alpha}_1 + \vec{\alpha}_2))$, where $\vec{\alpha}_2$ is a vector such that $(A - \lambda_1 I)\vec{\alpha}_2 = \vec{\alpha}_1$. (Such a vector $\vec{\alpha}_2$ always exists in this situation, and is unique up to addition of a multiple of $\vec{\alpha}_1$.)

The second caveat is that the eigenvalues may be non-real. They will then form a complex conjugate pair. The eigenvectors will also be non-real, and if $\vec{\alpha}_1$ is an eigenvector for λ_1 then $\vec{\alpha}_2 = \overline{\vec{\alpha}_1}$ is an eigenvector for $\lambda_2 = \overline{\lambda_1}$. Independent real solutions may be obtained by taking the real and imaginary parts of either $e^{\lambda_1 t}\vec{\alpha}_1$ or $e^{\lambda_2 t}\vec{\alpha}_2$. (These two have the same real parts and their imaginary parts differ only in sign.) This will give solutions of the general form e^{at} times a vector whose coordinates are linear combinations of $\cos(\omega t)$ and $\sin(\omega t)$, where $\lambda_1 = a + i\omega$.

Structural stability

The two major classes of phase portraits here are: (1) Eigenvalues real and not equal (that is, proper nodes or saddle points), and (2) Eigenvalues neither real nor purely imaginary. This is because these are the "stucturally stable" examples.

This means the following. If we have a system which is modeled by the differential equation $\vec{x}' = A\vec{x}$, we probably don't know the coefficients of A with perfect accuracy. If we happen to have a matrix whose eigenvalues coincide, a very slight perturbation of the coefficients will result in a matrix whose eigenvalues are distinct. (By moving the coefficients in different directions, one can arrange that they become distinct and real or non-real complex conjugates with small imaginary parts.)

This means that in real life we probably will never get a system which exhibits a star node, an improper node, or either of the last two degenerate subcases—these have equal eigenvalues. By changing the matrix slightly a star node will deform into a proper node (with λ_1/λ_2 very near to 1) or (deforming in a different way) into a spiral (with very small frequency—arches looking very straight in medium scale, very tightly wound around the origin and showing huge arches on larger scale). Similarly, an improper node can deform into either a spiral or a proper node.

The degenerate cases with $\lambda_1 = \lambda_2 = 0$ are similar. These can deform into proper nodes, saddle points, or spirals (because the eigenvalues can deform into a pair of real eigenvalues with the same sign, a pair of real eigenvalues with opposite sign, or a pair of non-real complex conjugate eigenvalues).

Nor is it likely that the eigenvalues will have zero real part. If the eigenvalues are purely imaginary and not zero—so we get a center—they deform into a pair of complex conjugate nonreal eigenvalues—a spiral.

In the same way, if one of the eigenvalues happens to be zero, a very slight perturbation of the coefficients results in a matrix with nonzero distinct real eigenvalues—so the comb deforms into a proper node or a saddle point. None of the degenerate cases is structurally stable.

The portrait gallery

Now for the dictionary of phase portraits. In the pictures which accompany these descriptions some elements are necessarily chosen at random. These elements will of course be determined by the specific problem at hand. For one thing, most of the time there will be two independent eigendirections (i.e., lines through the origin made up of eigenvectors). Below, if these are real they will be the lines through $\vec{\alpha}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{\alpha}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If there is only one eigendirection (this only happens if $\lambda_1 = \lambda_2$ and is then called the "incomplete case") it will be the line through $\vec{\alpha}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If they are not real there are definitely two but the problem of how they influence the phase portrait is more complex and will not be addressed.

Case 1: λ_1, λ_2 **real and of the same sign**
Node
Unstable if $\lambda_1, \lambda_2 > 0$**, asymptotically stable if** $\lambda_1, \lambda_2 < 0$

I'll draw the case of $\lambda_1, \lambda_2 > 0$.

Subcase 1a: $\lambda_1 \neq \lambda_2$
Proper node

Say $\lambda_1 > \lambda_2$. The solutions lie along curves of the form $u\vec{\alpha}_1 + v\vec{\alpha}_2$, where $u = cv^{\lambda_1/\lambda_2}$ (where c is constant). If $\lambda_1/\lambda_2 = 2$ for example these will be parabolas which have been skewed so the axes become the eigendirections. If λ_1/λ_2 is larger, the curves will have more of a "shoulder" and seem to merge with the $\vec{\alpha}_2$ eigenline. If $\lambda_1/\lambda_2 < 2$ the curves will have sharper bases. In any case, the curves become tangent at 0 to the eigenline whose eigenvalue has smaller absolute value.

Example: $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

Subcase 1b: $\lambda_1 = \lambda_2$**, complete**
Star node

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Subcase 1c: $\lambda_1 = \lambda_2$**, incomplete**
Improper node

Example: $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

$\boxed{\begin{array}{l}\textbf{Case 2: } \lambda_1, \lambda_2 \textbf{ real and of opposite sign}\\ \textbf{Saddle point}\\ \textbf{Unstable}\end{array}}$

Example: $A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$

$\boxed{\begin{array}{l}\textbf{Case 3: } \lambda_1, \lambda_2 \textbf{ purely imaginary}\\ \textbf{Center}\\ \textbf{Neutrally stable}\end{array}}$

The direction of travel may be obtained by computing $\vec{x}' = A\vec{x}$ at a single point, such as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This is easy since $A\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the first column of A.

Example: $A = \begin{bmatrix} 2 & -3 \\ 2 & -2 \end{bmatrix}$

$\boxed{\begin{array}{l}\textbf{Case 4: } \lambda_1, \lambda_2 \textbf{ neither real nor purely imaginary}\\ \textbf{Spiral}\\ \textbf{Asymptotically stable if Re } \lambda_1 < 0\textbf{, Unstable if Re } \lambda_1 > 0\end{array}}$

The direction of travel may be obtained as above. The spirals move out from the origin if Re $\lambda_1 > 0$, into the origin if Re $\lambda_1 < 0$. I'll draw the case of Re $\lambda_1 > 0$.

Example: $A = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix}$

Case 5: $\lambda_1 = 0$
Degenerate

Subcase 5a: $\lambda_2 \neq 0$
Comb
Neutrally stable if $\lambda_2 < 0$**, Unstable if** $\lambda_2 > 0$

Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Subcase 5b: $\lambda_2 = 0$**, complete**
All solutions constant
Neutrally stable

The matrix is necessarily $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; the equation is $\dot{x} = 0, \dot{y} = 0$; every vector is an eigenvector and a constant solution.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Subcase 5c: $\lambda_2 = 0$**, incomplete**
Parallel lines
Unstable

In this case most solutions tend to infinity but linearly and not exponentially as they do when at least one eigenvalue has positive real part.

Example: $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$