18.03 Practice Final Examination Solutions
Spring, 1999

The first four problems all concern the initial value problem

$$\frac{dy}{dx} + xy = x, \qquad y(0) = 0,$$

but they are independent of one another.

1. This equation is separable. Find the general solution by separating variables, and then find the particular solution with $y(0) = 0$.

Solution. As differentials, $\dfrac{dy}{1-y} = x\,dx$, so $-\ln|y-1| = x^2/2 + c$ or (replacing the excluded solution $y = 1$), $y = 1 + ce^{-x^2/2}$. The initial condition forces $c = -1$: $y = 1 - e^{-x^2/2}$.

2. It is also a first order linear equation. Find an appropriate integrating factor and solve the same initial value problem again.

Solution. We want w such that $\dfrac{d(wy)}{dx} = w\dfrac{dy}{dx} + xwy$. This forces $\dfrac{dw}{dx} = xw$, which is separable. $w = e^{x^2/2}$ is a solution, so the original ODE is equivalent to $\dfrac{d(e^{x^2/2}y)}{dx} = e^{x^2/2}x$. Thus $e^{x^2/2}y = \displaystyle\int e^{x^2/2}x\,dx = e^{x^2/2} + c$, and hence $y = 1 + ce^{-x^2/2}$ as before. $c = -1$ as before.

3. Use Euler's method to estimate $y(0.2)$ using a stepsize of 0.1.

Solution. The ODE is $dy/dx = f(x,y)$ with $f(x,y) = x(1-y)$. $h = .1$. Euler: $y_{n+1} = y_n + f(x_n, y_n)h$. $x_0 = 0, x_1 = .1, x_2 = .2$; $y_0 = 0, y_1 = y_0 + f(0,0)(.1) = 0$; $y(.02) \simeq y_2 = y_1 + f(.1,.1)(.1) = 0 + (.1)(1)(.1) = .01$.

4. Now draw an axis system (with range $-3 \le x, y \le 3$), sketch the isoclines for $m = -1, 0, 1$, and the direction field along them, and sketch a few solution curves (including the one through $(0,0)$).

Solution. The isocline for slope m is the hyperbola $y = 1 - m/x$. When $m = 0$ this degenerates to the lines $x = 0$ and $y = 1$. The solution through $(0,0)$ is an upside-down bell curve asymtotic to the line $y = 1$.

5. (a) Use the Laplace transform to solve the initial value problem $\ddot{x} - x = 0$, $x(0) = 0$, $\dot{x}(0) = 1$.

Solution. Using the t-derivative rule twice, $\mathcal{L}(f''(t);s) = s^2\mathcal{L}(f(t);s) - sf(0) - f'(0)$, so applying $\mathcal{L}$ gives $s^2X(s) - 1 - X(s) = 0$, or $X(s) = \dfrac{1}{s^2-1} = \dfrac{1}{2}\left(\dfrac{1}{s-1} - \dfrac{1}{s+1}\right)$. Thus $x(t) = (1/2)(e^t - e^{-t}) = \sinh(t)$.

(b) Express the solution to $\ddot{x} - x = r(t)$, $x(0) = 0$, $\dot{x}(0) = 0$, as a convolution using Duhamel's principle. Write this convolution as an integral (from the definition).

Solution. $x(t) = w(t) * r(t)$, where $w(t)$ is the weight function, the inverse Laplace transform of $1/p(s)$. We have just calculated that $w(t) = \sinh(t)$. As an integral, $x(t) = \int_0^t \sinh(\tau) r(t-\tau)\, d\tau$.

6. Find the general solution of $Ly = 0$, where L is the differential operator $L = (D+2I)^5$.

Solution. $y = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)e^{-2x}$.

7. Let $A = \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix}$, where a is some real number, and consider the linear system $\dot{\vec{x}} = A\vec{x}$. Specify the values of a (and the answer may be "none") for which the phase portrait exhibits each of the following. (Every number a should appear.)

(i) A nonconstant periodic orbit. *Ans:* $a = 0$

(ii) A saddle. *Ans:* None.

(iii) A stable spiral (not a center). *Ans:* $-2 < a < 0$.

(iv) An unstable spiral (not a center). *Ans:* $0 < a < 2$.

(v) A stable proper node. *Ans:* $a < -2$.

(vi) An unstable proper node. *Ans:* $2 < a$.

(vii) A stable improper node. *Ans:* $a = -2$.

(viii) An unstable improper node. *Ans:* $a = 2$.

Solution. The characteristic polynomial $\lambda^2 - a\lambda + 1$ has roots $a \pm \sqrt{(a/2)^2 - 1}$. These are real and distinct if $|a| > 2$, and then they have the same sign since their product is the constant term 1. This gives **(v)** and **(vi)**. If $|a| < 2$ the roots are not real; if $a \neq 0$ they are also not imaginary, and this gives **(iii)** anad **(iv)**. If $a = 0$ the roots are purely imaginary and this gives **(i)**. If $|a| = 2$, the matrix has repeated roots but is not diagonal, and this gives **(vii)** and **(viii)**.

8. Consider the homogeneous linear equation $y''' + y'' + y' + y = 0$.

(a) Determine the normal modes (i.e., the exponential solutions) of this equation. (Hint: is -1 a root of the characteristic polynomial?)

Solution. $p(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1 = (\lambda + 1)(\lambda^2 + 1)$, so the roots are $-1, i, -i$. The normal modes are thus e^{-x}, e^{ix}, and e^{-ix}.

(b) Write down three independent *real* solutions.

Solution. $e^{-x}, \sin(x), \cos(x)$.

9. Use the Laplace transform to find the solution to the same homogeneous linear equation as in **8.** (but written now using t for the independent variable and x for the independent variable), $x^{(3)} + \ddot{x} + \dot{x} + x = 0$, for which $x(0) = \dot{x}(0) = 0$, $\ddot{x}(0) = 1$.

Solution. Applying the formula given in **5.** above to $f'(t)$, and using the t-derivative rule yet again, $\mathcal{L}(f'''(t); s) = s^3\mathcal{L}(f(t); s) - s^2 f(0) - sf'(0) - f''(0)$: so $s^3X(s) - 1 + s^2X(s) + sX(s) + X(s) = 0$, or $X(s) = 1/p(s) = 1/(s+1)(s^2+1)$. Separate this using partial fractions: $\dfrac{1}{(s+1)(s^2+1)} = \dfrac{a}{s+1} + \dfrac{bs+c}{s^2+1}$. I'll illustrate the "cover-up method": Multiply through by $s+1$ and then set $s = -1$: $a = 1/2$. Then multiply through by

s^2+1 and then set $s=i$: $bi+c = \frac{1}{1+i} = \frac{1-i}{2}$, so $b=-1/2, c=1/2$: $\frac{1}{(s+1)(s^2+1)} = \frac{1}{2}\left(\frac{1}{s+1}+\frac{-s+1}{s^2+1}\right)$. Thus by the table of values, $x(t) = (1/2)(e^{-t}-\cos(t)+\sin(t))$.

10. Consider the linear autonomous system $\begin{cases} \dot{x} &= y \\ \dot{y} &= 2x - y. \end{cases}$

(a) Write down the second-order homogeneous linear equation which leads to this system by "anti-elimination."

Solution. $\ddot{x}+\dot{x}-2x = 0$.

(b) Write the system as a vector-valued differential equation of the form $\dot{\vec{x}} = A\vec{x}$. Find the eigenvalues of the matrix A, and for each eigenvalue find a nonzero eigenvector. Write down the normal modes (i.e., the exponential solutions).

Solution. $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$. The characteristic polynomial is $\lambda^2+\lambda-2 = (\lambda-1)(\lambda+2)$. The eigenvalues are $\lambda_1 = 1, \lambda_2 = -2$. An eigenvector for λ_1 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for λ_2 is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Normal modes are $e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(c) Write down a fundamental matrix for this system, and then normalize it to find e^{At}.

Solution. $\Phi(t) = \begin{bmatrix} e^t & e^{-2t} \\ e^t & -2e^{-2t} \end{bmatrix}$. $\Phi(0) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$; $\Phi(0)^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$; $e^{At} = \Phi(t)\Phi(0)^{-1} = \frac{1}{3}\begin{bmatrix} 2e^t+e^{-2t} & e^t-e^{-2t} \\ 2e^t-2e^{-2t} & e^t+2e^{-2t} \end{bmatrix}$.

Not unrelated is:

11. Consider the nonlinear autonomous system $\begin{cases} \dot{x} &= y-1 \\ \dot{y} &= x^2 - y. \end{cases}$

(a) Find the critical points of this system. Compute the Jacobian, evaluate it at each critical point, and decide the type of each critical point: saddle, unstable or stable spiral, unstable or stable node.

Solution. $\dot{x} = 0 \Rightarrow y = 1$, and then $\dot{y} = 0 \Rightarrow x = \pm 1$. $J(x,y) = \begin{bmatrix} 0 & 1 \\ 2x & -1 \end{bmatrix}$, so $J(-1,1) = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$. Its characteristic polynomial $\lambda^2+\lambda+2$ leads to a stable spiral, which winds clockwise. $J(1,1) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ is the matrix we analyzed above; it gives a saddle.

(b) Sketch the phase portrait of this system in a region including all the critical points. Mark the direction of travel on the trajectories.

12. Consider the Airy equation $y''+xy = 0$. Let $\mathrm{ca}(x)$ be the solution satisfying $\mathrm{ca}(0) = 1, \mathrm{ca}'(0) = 0$, and let $\mathrm{sa}(x)$ be the solution satisfying $\mathrm{sa}(0) = 0, \mathrm{sa}'(0) = 1$. The Wronskian of these two solutions is $w(x) = \mathrm{ca}(x)\mathrm{sa}'(x) - \mathrm{ca}'(x)\mathrm{sa}(x)$.

Compute $w'(x)$ and $w(0)$, and then solve this initial value problem to find $w(x)$. Explain why $\mathrm{sa}(a) \neq 0$ whenever $\mathrm{ca}(a) = 0$. [Hint: suppose both *were* zero.]

Solution. $w' = \mathrm{ca}'\mathrm{sa}' + \mathrm{ca}\mathrm{sa}'' - \mathrm{ca}''\mathrm{sa} - \mathrm{ca}'\mathrm{sa}' = \mathrm{ca}(-x\mathrm{sa}) - (-x\mathrm{ca})\mathrm{sa} = 0$. (This calculation works whenever there is no damping term.) Thus w is constant. Evaluating at $x = 0$ shows that $w = 1$. If $\mathrm{sa}(a) = \mathrm{ca}(a) = 0$, then both terms in the definition of $w(a)$ would vanish.

(b) Explain why $\mathrm{ca}'(a) \neq 0$ whenever $\mathrm{ca}(a) = 0$. [Hint: suppose both *were* zero.]

Solution. If $\mathrm{ca}'(a) = 0 = \mathrm{ca}(0)$ then again both terms in the definition of $w(a)$ would vanish.

(c) Then use this information to show that if a, b are numbers such that $a < b$ and $\mathrm{ca}(a) = \mathrm{ca}(b) = 0$ but $\mathrm{ca}(x) \neq 0$ for all x between a and b, then there is a number c such that $a < c < b$ and $\mathrm{sa}(c) = 0$. [This is analogous to the behavior of $\cos(x), \sin(x)$: their zeros alternate.]

Solution. $\mathrm{ca}'(a) \neq 0$ by **(b)**. Suppose $\mathrm{ca}'(a) > 0$. Then $\mathrm{ca}(x) > 0$ in (a, b), and $\mathrm{ca}'(b) < 0$. Now $1 = w(a) = -\mathrm{ca}'(a)\mathrm{sa}(a)$ implies that $\mathrm{sa}(a) < 0$, and $1 = w(b) = -\mathrm{ca}'(b)\mathrm{sa}(b)$ implies that $\mathrm{sa}(b) > 0$. Thus somewhere between a and b, sa must vanish (by the "intermediate value theorem"). The argument in case $\mathrm{ca}'(a) < 0$ is exactly the same.

13. Let $f(t)$ be the function which is periodic of period 2π and given by $f(t) = t$ for $-\pi/2 < t < \pi/2$, $f(t) = \pi - t$ for $\pi/2 < t < 3\pi/2$.

(a) Find the Fourier coefficients of $f(t)$.

Solution. $f(t)$ is odd so $a_n = 0$ for all n. $b_n = \frac{2}{\pi}\int_0^\pi f(t)\sin(nt)\,dt$. In the interval $[0, \pi]$, $f(t)$ is even around $\pi/2$, while $\sin(nt)$ is odd around $\pi/2$ if n is even, and even if n is odd. Thus $f(t)\sin(nt)$ is odd around $\pi/2$ if n is even, and even if n is odd, so $b_n = 0$ for n is even and if n is odd $b_n = \frac{4}{\pi}\int_0^{\pi/2} t\sin(nt)\,dt$. When we integrate this by parts the first term vanishes, leaving $b_n = \frac{4}{\pi}\left[\frac{1}{n^2}\sin(nt)\right]_0^{\pi/2}$. The lower limit is zero, and the upper limit depends upon the parity of $(n-1)/2$: for n odd, $b_n = \frac{4}{\pi}\frac{(-1)^{(n-1)/2}}{n^2}$. With $j = (n-1)/2$,

$$f(t) = \frac{4}{\pi}\sum_{n=1,\text{ odd}}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2}\sin(nt).$$

(b) Suppose a system controlled by the differential operator $L = D^2 + kI$ is driven by $f(t)$. Find all solutions of $Lx = f(t)$ which are periodic of period 2π. Your answer will depend upon k.

Solution By periodicity, $x = \frac{a_0}{2} + \sum_{n=1}^\infty (a_n\cos(nt) + b_n\sin(nt))$. Differentiating twice and adding kx, $\ddot{x} + kx = \frac{ka_0}{2} + \sum_{n=1}^\infty (k - n^2)(a_n\cos(nt) + b_n\sin(nt))$. Equating coefficients, if

k is not a square integer, $a_n = 0$, $b_n = 0$ for n even, and $b_n = \dfrac{4}{\pi}\dfrac{(-1)^{(n-1)/2}}{n^2(k-n^2)}$ if n is odd:

$$x = \frac{4}{\pi}\left(\frac{\sin(t)}{k-1} - \frac{\sin(3t)}{9(k-9)} + \cdots\right).$$

If k is an odd square, there is no possible value of b_n, and this shows that there are no periodic solutions: an overtone of the forcing term is in resonance with the system. If k is an even square, say $k = n^2$, the corresponding Fourier coefficients a_n, b_n, are zero in the forcing term, so there is no resonance. Instead, the displayed solution is not the only periodic one, but may be altered by adding $a\cos(nt) + b\sin(nt)$: the general solution is periodic.

(c) If we have a bar of metal extending from $x = 0$ to $x = \pi$, whose heat distribution is governed by the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, and we keep the ends fixed at 0 degrees, what is $u(x,t)$ if $u(x,0) = f(x)$?

Solution. The coefficients b_n in the solution $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx)$ are the Fourier coefficients of the odd extension of $u(x,0)$ to $[-\pi,\pi]$; so

$$u(x,t) = \frac{4}{\pi}\sum_{n=1,\text{ odd}}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2} e^{-n^2 t}\sin(nx).$$