18.03 Practice Final Examination
Spring, 1999

Here is a copy of the information which will appear on the final exam itself.

Laplace Transform—Definition:

$\mathcal{L}(f(t);s) = \int_0^\infty f(t)e^{-st}dt$ for Re s large.

Laplace Transform—Rules:

0. $\mathcal{L}(f(t);s)$ is unchanged if we alter any single value of $f(t)$.

1. $\mathcal{L}$ is linear: $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$ and if c is constant then $\mathcal{L}(cf) = c\mathcal{L}(f)$.

2. $\mathcal{L}$ essentially determines f: if f and g are piecewise continuous and $\mathcal{L}(f) = \mathcal{L}(g)$, then $f(a) = g(a)$ wherever both are continuous.

3. $\mathcal{L}$ preserves real and imaginary parts.

4. Scaling: For $a > 0$, $\mathcal{L}(f(t/a);s) = a\mathcal{L}(f(t);as)$.

5. t-shift rule: $\mathcal{L}(f_a(t);s) = e^{-at}\mathcal{L}(f(t);s)$, where $f_a(t) = \begin{cases} f(t-a) & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$.

6. s-shift rule: $\mathcal{L}(f(t);s-a) = \mathcal{L}(e^{at}f(t);s)$.

7. t-derivative rule: $\mathcal{L}(f'(t);s) = s\mathcal{L}(f(t);s) - f(0)$.

8. s-derivative rule: $-\frac{d}{ds}\mathcal{L}(f(t);s) = \mathcal{L}(tf(t);s)$.

9. $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$.

10. $\mathcal{L}(\delta) = 1$.

Laplace Transform—Values:

$$\mathcal{L}(1;s) = 1/s; \qquad \mathcal{L}(e^{at};s) = 1/(s-a)$$

$$\mathcal{L}(t^{a-1};s) = \Gamma(a)/s^a, \quad a > 0; \qquad \mathcal{L}(t^n;s) = n!/s^{n+1}, \quad n = 0, 1, 2, \ldots$$

Caution! The following formula has been corrected, and is different from an earlier version of the practice final, when sin and cos were swapped.

$$\mathcal{L}(\sin(\omega t);s) = \frac{\omega}{s^2+\omega^2}; \qquad \mathcal{L}(\cos(\omega t);s) = \frac{s}{s^2+\omega^2}$$

Γ function:

$\Gamma(a) = \int_0^\infty e^{-t}t^{a-1}dt$ for $a > 0$.

$a\Gamma(a) = \Gamma(a+1), a > 0$; $\Gamma(n) = (n-1)!$ for n a positive integer; $\Gamma(1/2) = \sqrt{\pi}$.

Heat equation:

$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(0) = u(\pi) = 0$ has solutions $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2t}\sin(nx)$.

The first four problems all concern the initial value problem

$$\frac{dy}{dx} + xy = x, \qquad y(0) = 0,$$

but they are independent of one another.

1. This equation is separable. Find the general solution by separating variables, and then find the particular solution with $y(0) = 0$.

2. It is also a first order linear equation. Find an appropriate integrating factor and solve the same initial value problem again.

3. Use Euler's method to estimate $y(0.2)$ using a stepsize of 0.1.

4. Now draw an axis system (with range $-3 \leq x, y \leq 3$), sketch the isoclines for $m = -1, 0, 1$, and the direction field along them, and sketch a few solution curves (including the one through $(0, 0)$).

5. **(a)** Use the Laplace transform to solve the initial value problem $\ddot{x} - x = 0$, $x(0) = 0$, $\dot{x}(0) = 1$.

(b) Express the solution to $\ddot{x} - x = r(t)$, $x(0) = 0$, $\dot{x}(0) = 0$, as a convolution using Duhamel's principle. Write this convolution as an integral (from the definition).

6. Find the general solution of $Ly = 0$, where L is the differential operator $L = (D+2I)^5$.

7. Let $A = \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix}$, where a is some real number, and consider the linear system $\dot{\vec{x}} = A\vec{x}$. Specify the values of a (and the answer may be "none") for which the phase portrait exhibits each of the following. (Every number a should appear.)

(i) A nonconstant periodic orbit.

(ii) A saddle.

(iii) A stable spiral (not a center).

(iv) An unstable spiral (not a center).

(v) A stable proper node.

(vi) An unstable proper node.

(vii) A stable improper node.

(viii) An unstable improper node.

8. Consider the homogeneous linear equation $y''' + y'' + y' + y = 0$.

(a) Determine the normal modes (i.e., the exponential solutions) of this equation. (Hint: is -1 a root of the characteristic polynomial?)

(b) Write down three independent *real* solutions.

9. Use the Laplace transform to find the solution to the same homogeneous linear equation as in **8.** (but written now using t for the independent variable and x for the independent variable), $x^{(3)} + \ddot{x} + \dot{x} + x = 0$, for which $x(0) = \dot{x}(0) = 0$, $\ddot{x}(0) = 1$.

10. Consider the linear autonomous system $\begin{cases} \dot{x} &= y \\ \dot{y} &= 2x - y. \end{cases}$

(a) Write down the second-order homogeneous linear equation which leads to this system by "anti-elimination."

(b) Write the system as a vector-valued differential equation of the form $\dot{\vec{x}} = A\vec{x}$. Find the eigenvalues of the matrix A, and for each eigenvalue find a nonzero eigenvector. Write down the normal modes (i.e., the exponential solutions).

(c) Write down a fundamental matrix for this system, and then normalize it to find e^{At}.

Not unrelated is:

10. Consider the nonlinear autonomous system $\begin{cases} \dot{x} &= y - 1 \\ \dot{y} &= x^2 - y. \end{cases}$

(a) Find the critical points of this system. Compute the Jacobian, evaluate it at each critical point, and decide the type of each critical point: saddle, unstable or stable spiral, unstable or stable node.

(b) Sketch the phase portrait of this system in a region including all the critical points. Mark the direction of travel on the trajectories.

11. Consider the Airy equation $y'' + xy = 0$. Let $\mathrm{ca}(x)$ be the solution satisfying $\mathrm{ca}(0) = 1, \mathrm{ca}'(0) = 0$, and let $\mathrm{sa}(x)$ be the solution satisfying $\mathrm{sa}(0) = 0, \mathrm{sa}'(0) = 1$. The Wronskian of these two solutions is $w(x) = \mathrm{ca}(x)\mathrm{sa}'(x) - \mathrm{ca}'(x)\mathrm{sa}(x)$.

Compute $w'(x)$ and $w(0)$, and then solve this initial value problem to find $w(x)$. Explain why $\mathrm{sa}(a) \neq 0$ whenever $\mathrm{ca}(a) = 0$. [Hint: suppose both *were* zero.]

(b) Explain why $\mathrm{ca}'(a) \neq 0$ whenever $\mathrm{ca}(a) = 0$. [Hint: suppose both *were* zero.]

(c) Then use this information to show that if a, b are numbers such that $a < b$ and $\mathrm{ca}(a) = \mathrm{ca}(b) = 0$ but $\mathrm{ca}(x) \neq 0$ for all x between a and b, then there is a number c such that $a < c < b$ and $\mathrm{sa}(c) = 0$. [This is analogous to the behavior of $\cos(x), \sin(x)$: their zeros alternate.]

12. Let $f(t)$ be the function which is periodic of period 2π and given by $f(t) = t$ for $-\pi/2 < t < \pi/2$, $f(t) = \pi - t$ for $\pi/2 < t < 3\pi/2$.

(a) Find the Fourier coefficients of $f(t)$.

(b) Suppose a system controlled by the differential operator $L = D^2 + kI$ is driven by $f(t)$. Find all solutions of $Lx = f(t)$ which are periodic of period 2π. Your answer will depend upon k.

(c) If we have a bar of metal extending from $x = 0$ to $x = \pi$, whose heat distribution is governed by the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, and we keep the ends fixed at 0 degrees, what is $u(x,t)$?