The truth about jump ropes

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Edwards and Penney are very deceptive in their development (pp. 205–207) of the ODE satisfied by a jump rope, and they fooled me. I said some things that were simply wrong in lecture on Friday, March 19, and this note is designed to fix the lie and in fact to derive a nonlinear ODE satisfied by a swinging rope with a large amplitude and point out some features of ODEs of this type.

1. We have a rope, spinning around an axis—say the x-axis—with angular velocity ω. The portion of it between x and $x + \Delta x$ obeys Newton's law. We ignore gravity, air resistance, and friction, so the only force acting on it is the tension exerted by the rest of the rope. We make the critical assumption that the rope is in equilibrium: its only motion is the rotation around the axis. This means that the x-components of the tension are equal and opposite: this portion of the rope is not accelerating in the x direction. Thus the x-component of the tension is constant. But a consequence of this is that the tension itself is *not* constant; only its x component is. Write T for the x component of the tension. Then the y-component is given by $-y'(x)T$ at x and by $y'(x + \Delta x)T$ at $x + \Delta x$. The net force is thus $T\Delta y'$.

The acceleration is given by $-\omega^2 y$ as in EP. The mass on the other hand is given by the length of the segment times the linear density ρ. The length is $\sqrt{1 + y'(x)^2}\Delta x$ (ignoring terms which go to zero faster than Δx). The essence of the linear approximation is simply that $\sqrt{1 + y'^2}$ is near to 1 if y' is small.

Thus $F = ma$ reads

$$T\Delta y' = (\rho\sqrt{1 + y'^2}\Delta x)(-\omega^2 y).$$

Dividing through by Δx and taking the limit, we arrive at the nonlinear second order ODE

$$y'' + a^2 y\sqrt{1 + y'^2} = 0, \qquad a = \sqrt{\frac{\rho}{T}}\,\omega. \tag{1}$$

The linear approximation is

$$y'' + a^2 y = 0 \tag{2}$$

as claimed by EP and by me.

The tension at x is not given by the constant T in general, but rather by the magnititude of the force vector with horiziontal component T and slope $y'(x)$: namely, $T\sqrt{1 + y'(x)^2}$. If you are holding an end of this rope you will exert a force to counteract the tension. It will have x component T, but also a component perpendicular to this as well. The tension is maximal at points at which the derivative is maximal. Since the maxima of y' occur where $y'' = 0$, and $y'' = 0$ forces $y = 0$ according to the differential equation, these points all occur along the x-axis. These are the points at which the rope is most likely to break.

Just as in the linear case, we may express solutions to (1) for a general parameter a in terms of those for $a = 1$. In fact, if $y(x)$ is a solution to $y'' + y\sqrt{1 + y'^2} = 0$, then $w(x) = y(ax)/a$ is a solution to $w'' + a^2 w\sqrt{1 + w'^2} = 0$.

The solutions of

$$y'' + y\sqrt{1 + y'^2} = 0, \qquad y'(0) = 0, \tag{3}$$

cannot be expressed in terms of the standard functions of calculus. They can be expressed in terms of "elliptic functions," however. These functions participate in one of the most magnificent stories in mathematics, linking geometry, analysis, and algebra in a unique and beautiful way. They are in fact connected with the proof by Andrew Wiles of the Fermat Conjecture. In addition to their use in describing the jump rope, they arise in the study of the time evolution of the spinning body controlled by the Euler equations and also of the undamped nonlinear pendulum, both of which will be described soon in lecture.

Rather than entering into a general discussion of elliptic functions, I will simply say a few more things about the specific equation (3). Let's write $z = y'$ and consider it as a function of y. Then using the chain rule and (3),

$$\frac{dz}{dy} = \frac{dz/dx}{dy/dx} = \frac{z'}{z} = \frac{\sqrt{1 + z^2}y}{z}.$$

This is an ODE relating y and z, and it's separable. Integrating,

$$\frac{y^2}{2} + \int \frac{z\,dz}{\sqrt{1 + z^2}} = c. \tag{4}$$

The substitution $u = 1 + z^2$ leads to

$$\int \frac{z\,dz}{\sqrt{1 + z^2}} = \frac{1}{2}\int u^{-1/2}\,du = u^{1/2} + \text{const} = \sqrt{1 + z^2} + \text{const}$$

Thus the pair (y, z) satisfies the equation

$$\frac{y^2}{2} + \sqrt{1 + z^2} = c \tag{5}$$

or

$$z^2 = y^4/4 - cy^2 + (c^2 - 1). \tag{6}$$

Thus the function $y(x)$ together with its derivative $y'(x) = z(x)$ parametrize the curve defined by (6), just as $\cos(x)$ and its derivative $-\sin(x)$ parametrize the curve defined by $y^2 + z^2 = 1$.

For solutions with $y(0) = 0$, substituting $x = 0$ in (5) shows that

$$c = \sqrt{1 + y'(0)^2}. \tag{7}$$

We can determine the maximal value of y, since y is maximal when $z = dy/dx = 0$, so, by (5), $(y^2/2) + 1 = \sqrt{1 + y'(0)^2}$ or

$$y_{max} = \sqrt{2\left(\sqrt{1 + y'(0)^2} - 1\right)}. \tag{8}$$

It also shows that when $y'(0)$ is large, y_{max} is about $\sqrt{2y'(0)}$: it grows as the square root of $y'(0)$.

At the other extreme, when $y'(0)$ is small, the tangent line approximation gives $\sqrt{1+y'^2} \simeq 1 + y'^2/2$ and $\sqrt{1+y'(0)^2} \simeq 1 + y'(0)^2/2$, so after some cancellation (5) reads

$$\frac{y^2}{2} + \frac{y'^2}{2} = \frac{y'(0)^2}{2}. \tag{9}$$

Since for y small the solutions to (1) are close to those of (2), $y(x) \simeq y'(0)\sin(x)$, so (9) is equivalent to $\sin^2(x) + \cos^2(x) = 1$: (5) is a nonlinear generalization of this. Also, (8) is approximated by $y_{max} \simeq y'(0)$, as is the case for solutions of the linearized equation.

Proceeding further along these lines one can show that all solutions to (3) are periodic, and exhibit the same sort of symmetry properties enjoyed by trigonometric functions. By this I mean that if $y(a) = 0$, then y is skew-symmetric about a, $y(a-x) = -y(a+x)$; and if $y'(a) = 0$, then y is symmetric about a, $y(a-x) = y(a+x)$.

These features show up clearly in the plot below, constructed by MATLAB, for the case $y'(0) = 1000$. The value of y_{max} derived above is about 44.6990044631808. Notice how unlike a trigonometric function this looks—it's very "square-shouldered."