Partial Solutions to 18.03 Hour Exam III: April 21, 1999

A number of these problems will use the following list of qualities of linear phase portraits.

a. spiral
b. center
c. improper node
d. star node
e. proper node
f. saddle
g. unstable
h. neutrally stable
i. asymptotically stable
j. structurally stable
k. structurally unstable
l. clockwise
m. counterclockwise

1. I have in mind a certain two-by-two matrix A with real entries. One of its eigenvalues is $\lambda_1 = 1 + 2i$.

(a) What is the other eigenvalue? **A:** $\lambda_2 = 1 - 2i$.

(b) What is the characteristic polynomial of A? **A:** $\lambda^2 - 2\lambda + 5$.

(c) From the front page of this exam, list the letters of the qualities which the corresponding phase portrait is guaranteed to exhibit. **A:** (a) spiral, (g) unstable, (j) structurally stable

(d) Give an example of such a matrix A. **A:** One example is $\begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix}$.

2. Consider the homogeneous linear ODE $\dot{\vec{u}} = A\vec{u}$, where $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$.

(a) If $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, write down a nonzero second order ODE satisfied by x. **A:** $\ddot{x} + \dot{x} - 2x = 0$.

(b) What are the eigenvalues of A? **A:** The characteristic polynomial of A is $\lambda^2 + \lambda - 2$, which has roots 1 and -2.

(c) From the list on the front page, pick out the letters corresponding to the qualities the phase portrait of this system exhibits. **A:** (f) saddle, (g) unstable, (j) structurally stable

(d) For each eigenvalue find a nonzero eigenvector. **A:** A non-zero eigenvector for $\lambda_1 = 1$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. A non-zero eigenvector for $\lambda_2 = -2$ is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(e) Which picture best approximates the phase portrait of this system? **A:** The figure in the upper right.

(f) Find a fundamental matrix for this system and normalize it to obtain e^{At}.

A: One possibility is: $\Phi(t) = \begin{bmatrix} e^t & e^{-2t} \\ e^t & -2e^{-2t} \end{bmatrix}$. In this case, $\Phi(0) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, so $\Phi(0)^{-1} = -\frac{1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$. Hence $e^{At} = \Phi(t)\Phi(0)^{-1} = \frac{1}{3}\begin{bmatrix} 2e^t + e^{2t} & e^t - e^{-2t} \\ 2e^t - 2e^{-2t} & e^t + 2e^{-2t} \end{bmatrix}$

(g) Solve the initial value problem $\dot{\vec{u}} = A\vec{u}, \quad \vec{u}(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. **A:**

$$\vec{u} = e^{At}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{5}{3}\begin{bmatrix} 2e^t + e^{-2t} \\ 2e^t - 2e^{-2t} \end{bmatrix}.$$

You could also solve it directly, as you know the eigenvalues and eigenvectors of A.

3. We have a nonlinear autonomous system

$$\begin{cases} \dot{x} &= y + x^2 \\ \dot{y} &= 2x - y. \end{cases}$$

The origin $(0, 0)$ is a critical point of this system, and you have analyzed the linearization there in problem **2.**

(a) There is one more critical point. Find it and write down the matrix of the linear approximation at that point.

A: To find the critical point, solve $y + x^2 = 0$ and $2x - y = 0$. Substituting the latter into the former, we get $2x + x^2 = 0$, so $x = 0$ or $x = -2$. In the first case, $y = 0$ (the solution described in the question), and in the second case $y = -4$. Hence the other critical point is at $(-2, -4)$.

Using the substitution method: Substitute $x = -2 + u$, $y = -4 + v$. Then

$$\dot{\vec{u}} = \dot{\vec{x}} = (-4 + v) + (-2 + u)^2 = v - 4u + u^2$$

and

$$\dot{\vec{v}} = \dot{\vec{y}} = 2(-2 + u) - (-4 + v) = 2u - v.$$

Hence the linear part is: $\dot{\vec{u}} = -4u + v$, $\dot{\vec{v}} = 2u - v$, so the matrix is $\begin{bmatrix} -4 & 1 \\ 2 & 1 \end{bmatrix}$.

Using the Jacobian method: $J = \begin{bmatrix} 2x & 1 \\ 2 & -1 \end{bmatrix}$. At (-2,-4), $J = \begin{bmatrix} -4 & 1 \\ 2 & -1 \end{bmatrix}$.