Partial Solutions to 18.03 Practice Hour Exam III
Spring, 1999

1.

(a), (b) Let $A = \begin{bmatrix} 3 & -4 \\ 6 & -7 \end{bmatrix}$. The eigenvalues of A are the solutions to $\det \begin{bmatrix} 3-\lambda & -4 \\ 6 & -7-\lambda \end{bmatrix} = 0$, so $\lambda_1 = -1$, and $\lambda_2 = -3$. The eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) The adjectives which correctly describe the phase portrait of the autonomous system $\dot{\vec{x}} = A\vec{x}$ are: proper node, asymptotically stable, structurally stable.

2. (a) We can use the eigenvalues and eigenvectors to construct a fundamental matrix:

$$\Phi(t) = \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}.$$

Then $\Phi(0)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, so

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2e^t - e^{2t} & -e^t + e^{2t} \\ 2e^t - 2e^{2t} & -e^t + 2e^{2t} \end{bmatrix}.$$

One can also do this directly: the matrix of eigenvectors is $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, so $P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ (look familiar?), and the matrix of eigenvectors is $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2e^t - e^{2t} & -e^t + e^{2t} \\ 2e^t - 2e^{2t} & -e^t + 2e^{2t} \end{bmatrix}.$$

(b)

$$\vec{x}(t) = e^{At}\vec{x}(0) = \begin{bmatrix} 2e^t - e^{2t} & -e^t + e^{2t} \\ 2e^t - 2e^{2t} & -e^t + 2e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3e^t - 2e^{2t} \\ 3e^t - 4e^{2t} \end{bmatrix}.$$

(c)

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

3. (a) The critical points are where $\dot{x} = \dot{y} = 0$, i.e. where $y - x^2 = x - y^2 = 0$. Then $y = x^2 = y^4$, so $y^4 - y = 0$, so $y = 0$ or 1. Then $x = y^2$, so $(x, y) = (0, 0)$ or $(1, 1)$.

(b) Near $(0, 0)$, we can linearize (i.e. approximate) to get $\dot{x} = y$, $\dot{y} = x$. The corresponding matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which has eigenvalues 1 and -1. As the eigenvalues are real and of opposite sign, this is a saddle, which is unstable.

(c) Let $f(x, y) = y - x^2$ and $g(x, y) = x - y^2$, so $f_x(x, y) = -2x$, $f_y(x, y) = 1$, $g_x(x, y) = 1$, $g_y(x, y) = -2y$. If $x = 1 + u$ and $y = 1 + v$, then when u and v are very small (i.e. x and y are very close to 1), the following is approximately true:

$$\dot{u} = f_x(1, 1)u + f_y(1, 1)v,$$

$$\dot{v} = g_x(1,1)u + g_y(1,1)v.$$

Hence $\dot{u} = -2u + v$, $\dot{v} = u - 2v$. The corresponding matrix of coefficients is $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, which has eigenvalues -1 and -3 and corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(d) This is a proper node.

4. Let $y = x'$, so we have a system $x' = y$, $y' = -2y - 2x$. The corresponding matrix of coefficients is $\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$, which has eigenvalues $-1 + i$ and $-1 - i$. The phase portrait is a spiral. The spiral is inwards, as the real part of the eigenvalues is negative. To see if it is clockwise or counterclockwise, we plug in a random point $(x, y) = (1, 0)$ where $(x', y') = (0, -2)$, so we see that the spiral is clockwise.