18.03 Study Outline for Second Hour Exam

1. Ideas of linearity

a. General form of a linear ODE:

$$Ly = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = r(x). \tag{1}$$

b. $p_1(x), \dots p_n(x)$ are the "coefficients." Homogeneous means $r(x) = 0$:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0. \tag{2}$$

c. Solutions y_p exist whenever the coefficients are reasonable. "Superposition": If y_1 solves $Ly = r_1(x)$, y_2 solves $Ly = r_2(x)$, and c is a constant, then $y_1 + y_2$ solves $Ly = r_1(x) + r_2(x)$ and cy_1 solves $Ly = cr_1(x)$.

d. There are n linearly independent solutions $y_1, \dots, y_n$ to (2). $\{y_1, \dots, y_n\}$ is linearly independent if the only way the linear combination $a_1y_1 + \cdots + a_ny_n$ can be 0 is for $a_1 = \cdots = a_n = 0$.

e. "Superposition": The general solution to (1) is $y_p + a_1y_1 + \cdots + a_ny_n$, for any choice of $y_p, y_1, \dots, y_n$ as above.

2. Complex numbers

a. $(1, 0)$ in the plane is called 1, $(0, 1)$ is called i, so (a, b) is called $a + bi$. This gives addition. Multiplication is determined by $i^2 = -1$ and the standard rules. Re $(a + bi) = a$, Im $(a + bi) = b$ (where a, b are understood to be real).

b. Complex conjugation $\overline{a + bi} = a - bi$ (a, b real). $\overline{z + w} = \bar{z} + \bar{w}, \overline{zw} = \bar{z}\bar{w}$. The magnitude or modulus or absolute value $|z|$ of z is the distance to 0, and $z\bar{z} = |z|^2$. Re $z = (z + \bar{z})/2$, Im $z = (z - \bar{z})/2i$. Divide by multiplying numerator and denominator by the conjugate of the denominator.

c. Polar rule for multiplication: Magnitudes multiply, angles add.

d. Compute derivatives of complex valued functions coordinatewise. For fixed complex r, e^{rt} is the solution to the IVP $\dot{z} = rz, z(0) = 1$. $e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$. $e^{z+w} = e^z e^2$, $e^{\bar{r}t} = \overline{e^{rt}}$.

3. Constant coefficient homogeneous equations

a. e^{rx} is a solution to (2) exactly when r is a root of the characteristic polynomial

$$r^n + p_1r^{n-1} + \cdots + p_{n-1}r + p_n. \tag{3}$$

These are linearly independent "normal modes."

b. If $r = a + bi$ is a root and $i \neq 0$, then $\bar{r} = a - bi$ is a root too and so the real and imaginary parts of e^{rx} are linearly independent solutions: $e^{ax}\cos(bx), e^{ax}\sin(bx)$.

c. If r has multiplicity k as a root of (3), then $e^{rx}, xe^{rx}, \dots, x^{k-1}e^{rx}$, are independent solutions to (2).

d. If Re $r < 0$ the normal mode is damped: it falls exponentially to zero as $x \to \infty$. If Re $r = 0$ it is periodic. If Re $r > 0$ it blows up. If Im $r \neq 0$ it is oscillatory. Damped oscillation is "underdamping"; if r is real and negative we have "overdamping."

4. Initial value problems: second order

a. The Wronskian of two functions y_1, y_2 is $w(x) = y_1'y_2 - y_1y_2'$. If y_1, y_2 are solutions to $y'' + p(x)y' + q(x) = 0$, then w is either never zero (and $\{y_1, y_2\}$ is linearly independent) or always zero (and $\{y_1, y_2\}$ is linearly dependent). Abel's equation: $w' + p(x)w = 0$: so if $p = 0$ then w is constant.

b. Given any x_0 in the range of definition of the equation and any a, b, there is a unique solution y to (1) such that $y(x_0) = a, y'(x_0) = b$. One finds it by solving for a and b in $y = y_p + ay_1 + by_2$.

c. y_1, y_2 is normalized at x_0 if $y_1(x_0) = 1, y_1'(x_0) = 0, y_2(x_0) = 0, y_2'(x_0) = 1$. Then $y = ay_1 + by_2$ has $y(x_0) = a, y'(x_0) = b$.

d. For $p > 0$ solutions to $y'' + py' + qy = e^{i\omega_0 x}$ have "steady state" and "transient" components. Steady state exhibits an amplitude multiplier ("gain") and a phase lag. Practical resonance means ω_0 is tuned to maximize gain. Beats, pure resonance may occur if $p = 0$.

5. Operators

a. Examples: $Dy = y'$; $Iy = y$ (the "identity operator"); $M_f y = f(x)y$, for a fixed function $f(x)$. Operators compose, so $D^2y = y''$, and add, so $(D^2+2D+3I)y = y''+2y'+3y$. An operator F is linear if $F(u+v) = Fu + Fv$ and $F(cu) = cFu$ (c constant). D, I, M_f are linear; $Sy = y^2, Ay = y + 1$ are not. The general CCLDO is $L = D^n + p_1D^{n-1} + \cdots + p_nI$, where $p_1, \dots, p_n$ are constants. Its characteristic polynomial is $f(r) = r^n + p_1r^{n-1} + \cdots + p_n$, and $L = f(D)$.

b. $De^{rx} = re^{rx}$. If L is a CCLDO with characteristic polynomial f, then $Le^{rx} = f(r)e^{rx}$. More generally, the Exponential Shift Law says $f(D)(e^{rx}u) = e^{rx}f(D + rI)u$.

6. Undetermined coefficients

a. If L is a CCLDO with $p_n \neq 0$ and $r(x)$ is a polynomial of degree m, then $Ly = r(x)$ has exactly one solution which is polynomial of degree m.

b. If 0 is a root of f of multiplicity k, then regard $Ly = r(x)$ as an equation of order $n - k$ for $y^{(k)}$, solve it, and integrate k times. ($Ly = r(x)$ is "reducible.")

c. If s is not a root of f, then $Ly = e^{sx}$ has solution $e^{sx}/f(s)$.

d. The ESL shows that $Ly = e^{sx}q(x)$ is equivalent to $f(D - sI)y = q(x)$. This is useful if s is a root of f (when it leads to a reducible equation) or if $s \neq 0$ and q is not constant (when it eliminates the exponential; if q is polynomial we are in case **b.** or **c.**).

7. Nonconstant coefficients, nonlinear equations

a. No normal modes, no characteristic polynomial; superposition still holds if linear.

b. Homogeneous Euler-Cauchy: $x^2y'' + pxy' + qy = 0$: $y = x^r$ is a solution exactly when $r(r-1) + pr + q = 0$. Use $x = e^{\ln x}$ (for $x > 0$) to give meaning to this, even if r is complex. Repeated roots can be handled by

c. Reduction of order: If y_1 is one solution of $Ly = 0$, try to find another by writing $y = y_1u$ and deriving an ODE for u.

d. A second order equation (even nonlinear) can be reduced to a pair of first order equations if it is reducible: missing y (when it is really a first order equation for $v = y'$) or y' (when it is a first order equation for y' as a function of y).