18.03 Hour Exam II: March 17, 1999: Solutions

[10] **1. (a)** Find a particular solution (call it y_1) to $y'' + 2y' + y = x^2$.

Undetermined coefficients:

$$\begin{aligned} y &= ax^2 + bx + c \\ 2y' &= 4ax + 2b \\ y'' &= 2a \end{aligned}$$

Equating coefficients with $x^2 + 0x + 0$, $a = 1, b = -4, c = 6$. So $y_1 = x^2 - 4x + 6$.

[10] **(b)** Find a particular solution (call it y_2) to $y'' + 2y' + y = e^{3x} \sin x$.

The characteristic polynomial is $f(r) = (r+1)^2$, and $f(3+i) = (4+i)^2 = 15 + 8i \neq 0$, so $z'' + 2z + z = e^{(3+i)x}$ has solution

$$z_p = \frac{e^{(3+i)x}}{15+8i} = \frac{(15-8i)e^{3x}(\cos x + i \sin x)}{15^2 + 8^2}$$

whose imaginary part is

$$y_2 = e^{3x} \frac{15 \sin x - 8 \cos x}{15^2 + 8^2}.$$

[10] **(c)** Find a particular solution to $y'' + 2y' + y = 2x^2 + 3e^{3x} \sin x$ in terms of y_1 and y_2 from parts **(a)** and **(b)**. (You may leave your answer in terms of the symbols y_1, y_2!)

$2y_1 + 3y_2$.

[10] **2.** Find a pair of solutions y_1, y_2, to $y'' + 2y' + y = 0$ such that $y_1(0) = 1, y_1'(0) = 0, y_2(0) = 0, y_2'(0) = 1$.

The repeated root leads to general solution $ae^{-x} + bxe^{-x}$, with derivative $(-a+b)e^{-x} - bxe^{-x}$. Evaluating at 0 gives $a = 1, -a + b = 0$ for y_1 and $a = 0, -a + b = 1$ for y_2, so

$$y_1 = (1+x)e^{-x}, \quad y_2 = xe^{-x}.$$

[20] **3.** $y'' + py' + y = 0$ represents a free spring-mass-dashpot system. We can set the dashpot constant $p \geq 0$ as we like. Make a table showing what values of p lead to each of the following behaviors, and for each make a little sketch of a typical solution: Undamped oscillation, underdamping, overdamping.

The characteristic polynomial has $p^2 - 4$ in the square root, so:

$p = 0$ leads to pure oscillation

$0 < p < 2$ leads to underdamping or damped oscillation

$p > 2$ leads to overdamping (no oscillation)

4. Now we'll set $p = 1$ in the previous equation and drive this system with a periodic force: $y'' + y' + y = e^{i\omega_0 x}$.

[10] **(a)** What is the periodic or "steady state" solution? (You may leave your answer in terms of a complex exponential.)

The characteristic polynomial is $f(r) = r^2 + r + 1$, and $f(i\omega_0) = (1 - \omega_0^2) + i\omega_0$ is not zero, so the periodic solution is

$$e^{i\omega_0 x}/((1 - \omega_0^2) + i\omega_0).$$

[5] **(b)** In terms of ω_0, what is the gain, or amplitude multiplier? (The gain is the the ratio of the amplitude of the periodic solution to the amplitude of the driving term (which is 1 in this case).)

The gain is $1/|(1 - \omega_0^2) + i\omega_0| = 1/\sqrt{(1 - \omega_0^2)^2 + \omega_0^2} = 1/\sqrt{\omega_0^4 - \omega_0^2 + 1}$.

[5] **(c)** For what value of ω_0 is this gain the largest? (i.e., for what value of ω_0 do we observe "practical resonance"?)

Maximum occurs when the quadratic in the denominator is a minimum, namely at the line of symmetry of the parabola, $\omega_0^2 =$"$-b/2$"$= 1/2$. So

$$\omega_0 = 1/\sqrt{2}.$$

[10] **5. (a)** Assume that $y = e^{2x}u$ and $y'' + 2y' + y = e^{2x}q(x)$ for some function $q(x)$. Write down an ODE satisfied by u which is free of exponential terms.

ESL: The characteristic polynomial is $f(r) = (r + 1)^2$ again, so

$$e^{2x}q(x) = f(D)(e^{2x}u) = e^{2x}f(D + 2I)u.$$

Since $f(r + 2) = (r + 3)^2 = r^2 + 6r + 9$, the ODE for u is

$$u'' + 6u' + 9u = q(x).$$

[10] **(b)** Find the general solution to the inhomogeneous non-constant coefficient second order linear ordinary differential equation $xy'' + y' = 1$. (Hint: it's reducible.)

Write $v = y'$, so $xv' + v = 1$. This is separable: $dv/(1 - v) = dx/x$, or $-\ln|1 - v| = \ln|x| + c$, so $1 - v = a/x$ for a constant a: $v = 1 - a/x$. Integrating,

$$y = x - a\ln|x| + b.$$

Friday's lecture: music, jump ropes, and pancake stacks (EP 2.10)