Partial Solutions to 18.03 Practice Hour Exam II

1. **(a)** $(1+\sqrt{3}i)^6 = 2^6(\cos\pi/3 + i\sin\pi/3)^6 = 2^6(e^{i\pi/3})^6 = 2^6e^{2\pi i} = 2^6 = 64$. **(b)** $z = \ln 2 + 2\pi k i$ for all integers k.

2. **(a)** $-xe^x + Ae^x + Be^{2x}$. **(b)** $e^x(x^4/12 + Ax + B)$. **(c)** $\frac{1}{5}e^x(-2\cos x + \sin x) + A\cos x + B\sin x$. **(d)** The solutions are $x^{1+\sqrt{2}}$ and $x^{1-\sqrt{2}}$. The Wronskian is $y_1y_2' - y_1'y_2 = -2\sqrt{2}x$. We could have known that the Wronskian was a multiple of x, from Problem set 3 problem II.12.

3. **(a)** Physically, the spring constant q will be non-negative. If it is 0 (e.g. if someone stole the spring), there will be exponential decay to some equilibrium value (general solution $C_1e^{-4x} + C_2$). If $0 < q < 4$ (if the spring is weak), there will be exponential decay; the system is overdamped. If $q = 4$, the general solution is of the form $(Ax+B)e^{-2x}$ — critically damped. If $q > 4$ (so the discriminant $16 - q^2$ of the characteristic quadratic $r^2 + 4r + q = 0$ is negative), then the roots of the equation will have imaginary parts, and the system will be underdamped. The overall behavior will look fundamentally different: y will still tend to 0 as time passes, but it will have oscillatory behavior. (Aside: what would happen if you had a "reverse spring", i.e. q were negative?)

(b) The general solution of the homogeneous version is $Ae^{-x} + Be^{-2x}$. To find a specific solution, we instead solve $y'' + 3y' + 2y = e^{\omega_0 ix}$, and take the real part. We expect a solution of the form $Ae^{\omega_0 ix}$. Substituting into the differential equation, we find $A((2-\omega_0^2) + 3\omega_0 i) = 1$, i.e.

$$A = \frac{(2-\omega_0^2) - 3\omega_0 i}{(2-\omega_0^2)^2 + (3\omega_0)^2}.$$

Hence to find a solution to the original differential equation, we multiply A by $\cos(\omega_0 x) + i\sin(\omega_0 x)$, and take the real part:

$$\begin{aligned} y &= \frac{(2-\omega_0^2)\cos(\omega_0 x) + 3\omega_0\sin(\omega_0 x)}{(2-\omega_0^2)^2 + (3\omega_0)^2} \\ &= \frac{\cos(\omega_0 x - \phi)}{\sqrt{(2-\omega_0^2)^2 + (3\omega_0)^2}} \end{aligned}$$

where $\tan\phi = 3\omega_0/(2-\omega_0^2)$. This is the steady state solution. ϕ is the phase lag. The gain is $1/\sqrt{(2-\omega_0^2)^2 + (3\omega_0)^2}$. This is maximized ("practical resonance") when $(2-\omega_0^2)^2 + (3\omega_0)^2 = \omega_0^4 + 5\omega_0^2 + 4$ is minimized, which is when $\omega_0 = 0$. (Note: this is odd behavior, a result of incomplete proofreading of the practice exam.)

4. **(a)** Linear: x^2D, $Sy = y(\sin(x))$, $Ly = x^2Dy + \sin(x)y$, $M_f y = f(x)y$. Non-linear: $Ty = y + 1$.

(b) (Hint: Expand out $x\frac{d}{dx}(x\frac{d}{dx}(x\frac{d}{dx}f(x)))$, using the chain rule twice.) $(xD)^3 = xD + 3x^2D^2 + x^3D^3$.

(c) Notice that $(xD)x^r = rx^r$, so $(xD)^k x^r = r^k x^r$. Hence if $f(r) = a_l r^l + \ldots + a_1 r + a_0$, then $f(xD)x^r = a_l(xD)^l x^r + \ldots + a_1(xD)x^r + a_0x^r = (a_l r^l + \ldots + a_1 r + a_0)x^r = f(r)x^r$.

5. If $r^2 + pr + q = 0$ has a root r_1 with negative real part, then there is a solution of $y'' + py' + qy = 0$ which is not zero but which converges to zero as $x \to \infty$. If both roots r_1 and r_2 have negative real parts, then all nonzero solutions have this property. These conditions can be interpreted in terms of p and q; see the lecture summary for Class 12 on the web.

6. All solutions are in the back of the book except EP 46, in which case (using the method of p. 166) $y = \sqrt{A(x - B)^2 + 1/a}$.