The Euler Equations

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These notes describe a derivation of the Euler differential equations describing the motion of a rigid body. For simplicity I assume that this object is floating in space, so to speak, without any torques being applied.

1. Angular velocity. The first piece of information is that there is a point in the body about which all rotations occur. This is the *center of mass*.

We will always consider coordinate systems having this point as the origin. An *inertial* coordinate system is one in which objects without forces acting on them do not accelerate. This is the usual system within which physics is described. We assume that three mutually perpendicular vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, have been chosen, each of length one; and we fix things so that $e_1 \times \vec{e}_2 = \vec{e}_3$ (the "right hand rule").

The motion of the body can the be described by specifying a *rotation matrix* $A(t)$ for each time t. This describes the rotation the body has undergone relative to its position at time 0. The first column of $A(t)$ is the coordinate vector of the position to which $\vec{e}_1$ has moved at time t; the second is the position to which $\vec{e}_2$ has moved, and the third is the position to which e_3 has moved. The fact that the object is rigid says that this is still a mutually perpendicular set of vectors satisfying the right hand rule. A convenient way to express this is:

$$A(t)^T A(t) = I, \tag{1}$$

where of course I denotes the identity matrix. This implies that $\det A = \pm 1$, and the right hand rule specifies that $\det A = 1$. A matrix satisifying these properties is called a *rotation matrix*. The equation (1) says that $A(t)^T = A(t)^{-1}$, and since the inverse of a matrix commutes with the matrix (1) can be rewritten as $A(t)A(t)^T = I$.

If we differentiate this equation we get:

$$A'(t)A(t)^T + A(t)A'(t)^T = 0.$$

Since $(AB)^T = B^T A^T$ and $A(t)^T = A(t)^{-1}$, this equation is equivalent to the fact that the matrix $A'(t)A(t)^{-1}$ (which we write $B(t)$) is *skew-symmetric*: $B(t)^T = -B(t)$.

Let $\vec{r}(t)$ denote the position at time t of some particular point on the body, so that $\vec{r}(t) = A(t)\vec{r}(0)$. Then $\vec{r}'(t) = A'(t)\vec{r}(0) = A'(t)A(t)^{-1}\vec{r}(t)$, or

$$\vec{r}'(t) = B(t)\vec{r}(t).$$

From the skew-symmetry of $B(t)$ we can write it in the form

$$B(t) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

(where $\omega_1, \omega_2, \omega_3$ are functions of t). I choose to write it in such a fashion because then

$$B(t)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{2}$$

Thus, writing $\vec{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$,

$$\vec{r}'(t) = \vec{\omega}(t) \times \vec{r}(t). \tag{3}$$

The vector $\vec{\omega}$ is called the *angular velocity*. For steady rotations, it points in the direction of the axis of rotation, pointing in such a way that if one's right thumb points in the same direction then the fingers curl in the direction of motion. Moreover, its length is the angular speed.

Euler's idea is that one should use a coordinate system which rotates with the body. Then a vector $\vec{l}$ which is fixed with respect to an inertial system seems to rotate, and it obeys the same geometric law with $\vec{\omega}$ replaced by $-\vec{\omega}$:

$$\vec{l}' = -\vec{\omega} \times \vec{l}. \tag{4}$$

2. Angular momentum. So far we have just set up some geometry; physics has not entered yet, but now it does.

The angular velocity is not generally fixed for a freely rotating body. Instead, there is another vector, determined from the geometry and mass distribution of the object and the angular velocity, which is fixed. This is the *angular momentum* vector $\vec{l}$. It is obtained by integrating the vector $\vec{r} \times \rho\vec{r}'$ over the body, where ρ is the density. By (3) this is $-\rho\vec{r} \times (\vec{r} \times \vec{\omega})$. Form a skew-symmetric matrix C out of the entries of $\vec{r}$ as in (2), so that $\vec{r} \times \vec{v} = C\vec{v}$. Then $-\rho\vec{r} \times (\vec{r} \times \vec{\omega}) = -\rho C^2\vec{\omega}$. The matrix C^2 is *symmetric*: $C^T = C$. $-C^2$ is a matrix-valued function defined on the body, and if we integrate $-\rho C^2$ over the body we obtain a matrix L such that $\vec{l} = L\vec{\omega}$. L is the *inertia tensor*.

It is easy to see that the eigenvalues of a skew symmetric matrix such as C are purely imaginary, and that those of C^2 are the squares of those of C; so the eigenvalues of C^2 are real and nonpositive. Thus the matrix $-\rho C^2$ is symmetric and its eigenvalues are nonnegative. The same is therefore true of the integral L. Write $\lambda_1, \lambda_2, \lambda_3$ for the eigenvalues of L.

The fact that the angular momentum is constant is a law of motion, but it's hard to use because the matrix L depends upon time since the object whose shape controls it is rotating. Here is where Euler's idea comes in: change to a coordinate system which is attached to the body. In that system $\vec{l}$ is not constant; rather, it obeys (4). The shape of the body *is* constant, though, so the matrix L is constant. Thus:

$$L\vec{\omega}' = -\vec{\omega} \times L\vec{\omega}. \tag{5}$$

This is the "Euler equation." It arises by combining simple geometric considerations with the physical information that angular momentum is constant.

There is a general theorem, called the *spectral theorem*, which guarantees that any 3×3 symmetric matrix has 3 nonzero orthogonal eigenvectors. The eigendirections of L are called the *principal axes* of the body. If we use them for coordinate axes, then

$$\vec{l} = L\vec{\omega} = \begin{bmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3 \end{bmatrix}.$$

The Euler equation is easy to make explicit using these axes:

$$\begin{bmatrix} \lambda_1\omega_1' \\ \lambda_2\omega_2' \\ \lambda_3\omega_3' \end{bmatrix} = -\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \begin{bmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3 \end{bmatrix},$$

or

$$\begin{cases} \lambda_1\omega_1' &= (\lambda_2-\lambda_3)\omega_2\omega_3 \\ \lambda_2\omega_2' &= (\lambda_3-\lambda_1)\omega_3\omega_1 \\ \lambda_3\omega_3' &= (\lambda_1-\lambda_2)\omega_1\omega_2. \end{cases} \tag{6}$$

If L is nondegenerate (so that none of its eigenvalues are zero), we may choose to use the angular momentum vector instead of the angular velocity. Then

$$\begin{cases} l_1' &= \alpha_1 l_2 l_3 \\ l_2' &= \alpha_2 l_3 l_1 \\ l_3' &= \alpha_3 l_1 l_2 \end{cases} \tag{7}$$

where

$$\alpha_1 = \frac{\lambda_2-\lambda_3}{\lambda_2\lambda_3}, \quad \alpha_2 = \frac{\lambda_3-\lambda_1}{\lambda_3\lambda_1}, \quad \alpha_3 = \frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2}.$$

Since $||\vec{l}||$ is constant, the angular momentum vector is constrained to lie on a sphere.

3. Energy. Energy is another integral of motion. We have only kinetic energy here. It can be computed by integrating $\frac{1}{2}\rho||\vec{r}'||^2$ over the body. But

$$||\vec{r}'||^2 = \vec{r}' \cdot \vec{r}' = (\vec{\omega} \times \vec{r}) \cdot \vec{r}' = \vec{\omega} \cdot (\vec{r} \times \vec{r}'),$$

so the integral is just the quadratic form associated to the inertia operator:

$$E = \tfrac{1}{2}\vec{\omega} \cdot \vec{l} = \tfrac{1}{2}\vec{\omega} \cdot L\vec{\omega}.$$

It's easy to see that E is indeed constant: since L is symmetric, $\frac{1}{2}(\vec{\omega} \cdot L\vec{\omega})' = \vec{\omega} \cdot L\vec{\omega}' = -\vec{\omega} \cdot (\vec{\omega} \times L\vec{\omega})$, which is zero since the cross product is orthogonal to its factors.

If we use the principal axis system, this is

$$E = \tfrac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2).$$

or, in terms of the angular momentum vector expressed in the frame of the body,

$$E = \tfrac{1}{2}\left(\frac{l_1^2}{\lambda_1} + \frac{l_2^2}{\lambda_2} + \frac{l_3^2}{\lambda_3}\right).$$

The surfaces of constant energy are concentric ellipsoids, and the angular momentum is constrained to lie on the curve of intersection of the $||\vec{l}||$ = constant sphere and the E= constant ellipsoid. The precise motion through time is complicated, and its description involves the use of elliptic functions.