18.03 Class 5, Friday, Feb 12: Autonomous equations, the phase line

I began by pointing out that if y_p is any solution to the linear ODE

$$y' + p(x)y = q(x) \tag{1}$$

and y_h is any nonzero solution to the "associated homogeneous linear equation"

$$y' + p(x)y = 0$$

then the *general* solution of (1) is

$$y = y_p + cy_h.$$

For example, in the tide example $dx/dt + kx = a\cos(\omega t)$ from Wednesday, we found a particular solution y_p; the associated homogenous equation $dx/dt + kx = 0$ has e^{-kt} as a nonzero solution; so the general solution to the original equation is $y_h + ce^{kt}$.

An ODE is *autonomous* if it's of the form $dx/dt = g(x)$: the independent variable, t, occurs only through the derivative. That is to say, the whole picture is time-independent. These are easy to solve, in principle: they are separable. But it is often just as useful to understand them quantitatively. Note that the isoclines are horizontal lines (or sets of them), and that any horizontal translate of a solution is another solution. (The pictures in EP, 1.38, 7.1, 7.3, do not show this clearly at all.) The simplest example is $dx/dt = kx$, with solutions given by $x = ce^{kt}$.

Another good example is the *logistic* equation. This is what happens when the growth rate, k, is no longer constant but rather depends upon the population size x. One reasonable dependence is of the form $k = b(a - x)$: so as the population approaches some limiting value a the rate of growth slows down. $dx/dt = b(a - x)x$. I drew the graph of $b(a - x)x$ and marked it's roots, drew the (t, x)-plane, some isoclines and some solutions. Solutions between 0 and a represent a population growing from some small value, exponentially at first but then slowing down to exponential convergence to $x = a$. Solutions above $x = a$ correspond to a relaxation from an artificially high population.

A *critical point* for the autonomous ODE $dx/dt = g(x)$ is a number c such that $g(c) = 0$. I pointed out that critical points exactly correspond to constant solutions. I drew the x-axis, vertically, marked the critical points, then drew in arrows indicating the direction of motion of the solution in those regions. This is the *phase line.*

A critical point is *stable* if $g(x) > 0$ for x just less than c and $g(x) < 0$ for x just greater than c. It is *unstable* if instead $g(x) < 0$ for x just less than c and $g(x) < 0$ for x just greater than c. I illustrated this in the case of the logistic equation: a is stable, 0 is unstable.

Solutions around a stable critical point converge asymtotically to that steady state. Their long term behavior is indepenent of initial conditions. Solutions around an unstable critical point diverge exponentially. Their long term behavior is extremely sensitive to initial condition.