18.03 Class 4, Wednesday, Feb 10: Linear equations: another example; tides.

I discussed the integrating factor algorithm again, using $xy' + y = \cos(x)$. I pointed out that there is a singularity in all but one solution to this, but it occurs at $x = 0$, where p and q have singularities themselves, so what can you expect? This is in contrast to the nonlinear example from day 1, however, $dy/dx - y^2/2$, in which every solution develops a singularity despite the fact that the ODE has nice coefficients. So regularity is a consequence of linearity.

Then I discussed the tides in Boston Harbor. In the harbor the water level is $x(t)$. It communicats with the ocean, whose water level is $y(t)$, via a narrow channel. A simple model of how the ocean affects the bay leads to

$$\frac{dx}{dt} = k(y - x)$$

where k is an appropriate "coupling constant." The same equation controls many other diffusion or mixing problems. It's a first order linear problem, $x' + kx = ky$, with decay term kx and forcing term ky. Tides are roughly sinusoidal, so suppose $y = a\cos(\omega t)$. I used the terms *amplitude* for a, *period* $2\pi/\omega$, and *frequency* $\omega/2\pi$. Many students were unfamiliar with these words.

Rather than go through the integrating factor rigamarole, I announced that I would look for a sinusoidal solution of the same frequency: an *ansatz*. The most general such is

$$x = b\cos(\omega t - \varphi)$$

and I proposed to find b and φ. The point is that we have a pretty good qualitative idea of what the solution is; we'll use 18.03 to find the detailed quantitative dependence of the behavior of the tide in the harbor on ω and k.

I differentiated and substituted into the equation. By replacing t by $t + \varphi/\omega$ you arrive at

$$ak\cos(\omega t + \varphi) = bk\cos(\omega t) - b\omega\sin(\omega t),$$

which holds if

$$\begin{aligned} ak\cos(\varphi) &= bk \\ ak\sin(\varphi) &= b\omega, \end{aligned}$$

i.e. $\tan(\varphi) = \omega/k$ and $b = a\sin(\varphi)$. I drew a right triangle with base 1, opposite side ω/k, and angle φ; so the hypotenuse is $\sqrt{1 + (\omega/k)^2}$ and $b = a/\sqrt{1 + (\omega/k)^2}$. We conclude that a solution is given by

$$x = \frac{a}{\sqrt{1 + (\omega/k)^2}}\cos(\omega t - \varphi), \quad \text{where } \tan(\varphi) = \omega/k. \tag{1}$$

We learn that the period of the tide in the bay is the same as in the ocean, but (1) the amplitude of the tide is less by a factor of $1/\sqrt{1 + (\omega/k)^2}$, and (2) the tide is delayed by a "phase shift" φ. The amplitude factor is always less than one, and much less if ω is large: so if the tide changes very rapidly in the ocean (relative to the strength of the coupling constant, which we have normalized to 1), then there is very little tide in the bay. The phase factor is roughly ω/k for small ω, but grows towards $\pi/2$ as the frequencey of the tide increases. In this limit the tide in the bay is slack when it is high or low in the ocean.

I pointed out that this can't be the only solution, since we know that any first order ODE has infinitely many; there's no room for initial conditions. I stated without justification that the general solution is (1) $+ce^{-kt}$. I'll return to this Friday.

I didn't get to talk about a second application: As in EP pp 63–64, with $a = 1$. (It is to be understood that this is one dumb navigational strategy.) I would have written s (for "slope"—I wrote $dy/dx = m$ for the equation of an isocline on Friday, so using m again for this different purpose would be confusing) where they write v. And I would have used similar triangles instead of introducing and then eliminating trig functions. I would have used this as a motivation for considering "homogeneous" equations (and cautioned the students that this is a *different* use of the word than when you say "homogeneous linear equation"—this is a *nonlinear* equation). Their isoclines are straight lines through the origin. The end result is

$$y = \frac{1}{2}(x^{1+k} - x^{1-k}).$$

The destination airport would like to know from which direction to expect the plane, and when. This solution tell us that as long as the plane arrives at all ($0 < k < 1$) it arrives from due north; but it doesn't tell us how long it takes.

In an addition to the amendment to PS1 I wrote out some MATLAB code which will plot some of these trajectories.