18.03 Class 3, Monday, Feb 8: First order linear equations

The most important first order equations are *linear*: they can be put, by algebraic means, into the form

$$y' + p(x)y = q(x). \tag{1}$$

For example if $p = 0$ we have the solution $y = \int q(x)dx$. If $q = 0$ (the *homogenous* case) we can separate variables and get $y = ce^{-\int p(t)dt}$; especially, if $p(x)$ is constant, this gives the equation we studied on Friday. But in general this is not a separable equation.

I described the mortgage process: in the continuous approximation, one is charged interest $I(t)$ per year on the outstanding loan amount $l(t)$, and pays off the mortgage at $a(t)$ dollars per year. One is led to

$$l'(t) - I(t)l(t) = -a(t): \qquad p = -I, \quad q = a.$$

I urged the students to think of this example when they face a general linear first order ODE, and interpret $p(t)$ as an *exponential decay factor* and $q(t)$ as a *driving* or *forcing* term. The left hand member of (1) describes the system we are studying, and the right hand term $q(x)$ describes an external pressure being applied to it; it drives y up if it is positive, or down (as in the mortgage case) if it is negative.

I took the case in which I and a are both constant. This is actually separable, and you find

$$l(t) = \frac{a}{I} + ce^{It}.$$

Determining the constant of integration c is not done by the usual initial condition method in this case. In fact, there are two undetermined constants here: c and a. We are given an initial loan amount, $L = l(0)$, and a number of years T, at which time the loan amount should be reduced to 0; so we have the pair of equations

$$0 = l(T) = \frac{a}{I} + ce^{IT}, \qquad L = \frac{a}{I} + c.$$

The first shows that $c = -\frac{a}{I}e^{-IT}$, and the second then shows that

$$a = \frac{I}{1 - e^{-IT}}L.$$

The total payout is aT, which is the initial loan amount L times the ratio

$$\rho = \frac{IT}{1 - e^{-IT}}.$$

(I didn't point out to them that this is the generating function for Bernoulli numbers.) Thus for example for a 30 year mortgage at 7%, the total payout comes to about $\rho \simeq 2.393$, and consequently the monthly payments on a \$100,000 mortgage are \$664.73. This is very close to actual bank charges.

The I returned to the general case (1) and showed them the TRICK: multiply both sides by another (as yet unknown) function w, with the idea of writing the sum on the left as the two terms in the derivative of a product. Working this out lead to a differential equation for w: $w' = -p(x)w$. This is separable, and has

$$w = e^{\int p(t)dt}.$$

This is an *integrating factor*. I suggested remembering the trick rather than this expression for the integrating factor (and certainly not the expression (6) on EP p 47, which I didn't even write down). I carried out the trick on the equation

$$y' = e^x - y.$$

This equation has an exponential damping term, but also an exponentially increasing driving term. Which will win? The general solution is

$$y = \frac{1}{2}e^x + ce^{-x},$$

so the damping can't overcome the impressed force but does halve its effect.

If I have time I'll make the point that the solutions to (1) exist everywhere as long as the coefficients p and q are reasonable. This is in distinction to the kinds of equations we were looking at on the first two days, which were often nonlinear and whose solutions often blew up.