18.03 Class 2, Friday, Feb 5: Exponential growth, direction fields, MATLAB

I began by describing compound interest, with its difference equation $p(t + \Delta t) = p(t) + Ip(t)\Delta t$. Then I let the compounding period Δt go to zero, to get the *continuous approximation* and the ODE $p'(t) = Ip(t)$. Then I solved $dx/dt = ax$ with careful attention to how the c in $x = ce^{at}$ comes about. I reminded the students that this is a very common equation in nature. I returned to the original problem and inserted an initial value and a value $I = .1$, and found that after 20 years I have e^2 times as much as I deposited.

I wrote up the general form of a first-order ODE, $dy/dx = g(x, y)$, and gave as examples $dx/dt = ax$, $dy/dx = g(x)$, $dy/dx = y^2/2$ (from Wednesday's lecture), $dx/dt = x^2 - t$, and $dy/dx = y^2 - x^2 - 1$.

I made the point that in this course we'll study ODEs from three different but mutually reinforcing perspectives:

- Qualitative behavior—especially using graphics
- Analytic solutions
- Numerical methods.

I proposed to pursue the first approach now. I tried to give meaning to the equation $dy/dx = g(x, y)$: at each point in the plane we are specifying a slope. I sketched the "direction field" for $dx/dt = x$, and then threaded some solutions to see the exponentials appearing. I marked $c > 0$ above the t-axis, $c < 0$ below, and $c = 0$ on it.

Then I proposed to study the Riccati equation $dx/dt = x^2 = t$ in the same way. I pointed out that in the earlier case I didn't just plot the slope at random points; rather, I found all the points at which $m = 0, 1, 2, -1, -2$: the *isoclines*. So I did the same here, but it was harder and the solutions weren't so clear. I used this to motivate turning to MATLAB.

I fired up MATLAB, computed e^2 (using `format long` to get more impressive precision), the pulled up `dfield5` and accepted the default ODE $x' = x^2 - t$. I plotted the direction field and then some solutions. I pointed out what this was saying about the behavior of solutions: those above the parabola headed off to infinity at some point, while those inside it split from the upper branch at some point and joined the lower branch. Solutions below the parabola are asymptotic to it. I pointed out the *separatrix*.

In the 1:00 class I then returned to the `setup` box and replaced the ODE with $y' = y^2 - x^2 - 1$, plotted some solutions, observed their behavior, found the separatrices, and discovered the solution $y = -x$. This part of the 2:00 class was interrupted by laughter.