Asymptotics of the Airy Equation

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These notes will briefly discuss a single example of a second order linear ODE and indicate some of the beautiful and powerful methods which exist to get information about its solutions: the so-called homogeneous Airy equation

$$\ddot{x} + tx = 0. \tag{1}$$

This equation arose first in an analysis of the intensity of light in the neighborhood of a "caustic" (one of those intense lines that forms when light is defracted in a glass of water, for example). It occurs also in the analysis of the statics of a standing column.

The general theory of linear ODEs shows that there is a pair of solutions to (1), $\mathrm{ca}(t), \mathrm{sa}(t)$, which is "normalized at $t = 0$":

$$\begin{array}{ll} \mathrm{ca}(0) = 1, & \mathrm{sa}(0) = 0 \\ \mathrm{ca}'(0) = 0, & \mathrm{sa}'(0) = 1. \end{array}$$

By analogy with the cosine and sine functions we will refer to these as "Airy cosine" and "Airy sine" functions.

A great many second order linear ODEs may be transformed by one or another substitution to "Bessel's equation," which is

$$t^2\ddot{z} + t\dot{z} + (t^2 - \nu^2)z = 0, \tag{2}$$

where ν is a constant. The Airy equation is no exception, and you can verify for yourself that if $z = z(t)$ is a solution of Bessel's equation (2) with $\nu = \pm 1/3$, then

$$x = t^{1/2} z(2t^{3/2}/3)$$

is a solution of the Airy equation (1) (for $t > 0$).[1]

The "asymptotic expansion" of solutions of the Bessel equation were discovered by H. Hankel in 1868. The story of asymptotic expansions is one of the great achievements of 19th century mathematics. It allows you to calculate the values of solutions of differential equations for *large* values of t with great accuracy. This is of course impossible by direct numerical methods, which get swamped by iterative errors after a while. We well just give the very crudest part of the approximation for the Airy cosine and sine functions which result from this work.

Here is an argument to show roughly what we might expect. First of all, the Airy equation looks a lot like $\ddot{x} + \omega^2 x = 0$, whose solutions (normalized at 0) are $\cos(\omega t), \sin(\omega t)$.

[1] For Bessel function enthusiasts I remark that for $t > 0$

$$\mathrm{ca}(t) = 3^{-1/3}\Gamma(2/3)t^{1/2}J_{-1/3}\left(\frac{2t^{3/2}}{3}\right), \qquad \mathrm{sa}(t) = 3^{1/3}\Gamma(4/3)t^{1/2}J_{1/3}\left(\frac{2t^{3/2}}{3}\right).$$

It is as if the parameter ω is increasing with t. So we should expect *oscillatory* behavior, with period *decreasing* slowly with t.

In keeping with one of our general principles, try to write the solution to (1) as an exponential:

$$u = e^{is(t)}.$$

The novelty here is that the exponent $is(t)$ is not assumed to be a linear function of t, as it was when we found normal modes for homogeneous linear equations with *constant* coefficients. Rather, it is some more general function, which we will try to understand. Since we expect oscillatory behavior, we expect that the exponent will take on imaginary values and so we have chosen to write it as $is(t)$. Then $\dot{u} = i\dot{s}e^{is}, \ddot{u} = (i\ddot{s} - \dot{s}^2)e^{is}$, so s satisfies the ODE

$$i\ddot{s} = \dot{s}^2 - t.$$

This doesn't look good at first: we've traded a second order *linear* equation for a second order *nonlinear* one! But you'll notice that s doesn't occur in it: so writing $v = \dot{s}$, it is

$$i\dot{v} = v^2 - t, \tag{3}$$

a Riccati equation. At least we've reduced the order.

Still, it doesn't have solutions in elementary terms. The next observation is due to Carlini, Liouville, and Green, in the first part of the 19th Century: for *large* t, we should assume that our function s is such that $|\ddot{s}(t)|$ is much smaller than $|\dot{s}(t)^2|$. For example, if $s(t) = t^n$, then $\ddot{s}(t) = n(n-1)t^{n-2}$ while $\dot{s}(t)^2 = n^2t^{2n-2}$, so as long as $n > 0$ our expectation is satisfied. If this holds, then for large t the equation (3) is well approximated by

$$0 = v^2 - t.$$

That is, $v = \pm\sqrt{t}$. Recalling that $v = \dot{s}$, this implies

$$s(t) = \pm\frac{2t^{3/2}}{3} + c.$$

and so

$$x = ce^{\pm\frac{2it^{3/2}}{3}}.$$

These are complex conjugate solutions, having as linear combinations the real and imaginary parts

$$\cos\left(\frac{2t^{3/2}}{3}\right), \qquad \sin\left(\frac{2t^{3/2}}{3}\right).$$

The actual formula is the following:

$$\mathrm{ca}(t) \sim at^{-1/4}\cos\left(\frac{2t^{3/2}}{3} - \frac{\pi}{12}\right), \qquad \mathrm{sa}(t) \sim bt^{-1/4}\sin\left(\frac{2t^{3/2}}{3} + \frac{\pi}{12}\right)$$

as $t \to \infty$, where[2]

$$a = \frac{3^{1/6}\Gamma(2/3)}{\sqrt{\pi}} \simeq 0.91749089788326, \qquad b = \frac{3^{5/6}\Gamma(4/3)}{\sqrt{\pi}} \simeq 1.25854168258591.$$

[2]The Gamma function is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. Thus $\Gamma(0) = 1$, and one can see that $\Gamma(x+1) = x\Gamma(x)$, so for positive integers n we find $\Gamma(n) = (n-1)(n-2)\cdots 1 = (n-1)!$.

Our guesswork picked up the right oscillatory behavior but was not sharp enough to see the decreasing amplitude factor $t^{-1/4}$. This formula is spectacularly vindicated by the MATLAB plot attached. This is actually only the beginning of an even more accurate estimate of the solutions of this ODE.

Some further comments:

Afficionados of power series will realize that

$$\mathrm{ca}(t) = 1 - \frac{t^3}{3 \cdot 2} + \frac{t^6}{6 \cdot 5 \cdot 3 \cdot 2} - \cdots,$$

$$\mathrm{sa}(t) = t - \frac{t^4}{4 \cdot 3} + \frac{t^7}{7 \cdot 6 \cdot 4 \cdot 3} - \cdots.$$

A glance at the graphs, or the power series, hints that this pair of functions ought to have a third companion, with power series

$$\mathrm{ha}(t) = \frac{t^2}{2} - \frac{t^5}{5 \cdot 4 \cdot 2} + \frac{t^8}{8 \cdot 7 \cdot 5 \cdot 4 \cdot 2} - \cdots.$$

This function is not a solution of the Airy equation, but it *is* a solution of a third order equation obtained by "prolonging" or differentiating the Airy equation: apply d/dt to it and you get

$$x^{(3)} + t\dot{x} + x = 0.$$

The solutions to this homogeneous linear equation, normalized at 0, are

$$\begin{array}{rlll} \mathrm{ca}(t) & \text{with} \;\; \mathrm{ca}(0) = 1, & \mathrm{ca}'(0) = 0, & \mathrm{ca}''(0) = 0, \\ \mathrm{sa}(t) & \text{with} \;\; \mathrm{sa}(0) = 0, & \mathrm{sa}'(0) = 1, & \mathrm{sa}''(0) = 0, \\ \mathrm{ha}(t) & \text{with} \;\; \mathrm{ha}(0) = 0, & \mathrm{ha}'(0) = 0, & \mathrm{ha}''(0) = 1. \end{array}$$

Notice that by definition $\frac{d}{dt}(\ddot{x} + tx) = 0$, so $\ddot{x} + tx$ is a constant, and evaluating at $t = 0$ shows that this constant is 1: so ha is the solution to the *inhomogeneous second order* linear ODE

$$\ddot{x} + tx = 1$$

satifying the initial condition $\mathrm{ha}(0) = \mathrm{ha}'(0) = 0$.

This third function ha must be related to Bessel functions and have an analogous asymptotic expansion, but I don't know how to do this.

You might imagine that more and more interesting functions will arise as solutions of further derivatives, but this is not the case. If we differentiate again we obtain a homogeneous linear ODE in which x enters only through its derivative: so constant functions are solutions, and $\{\mathrm{ca}, \mathrm{sa}, \mathrm{ha}, 1\}$ forms a linearly independent set of solutions. Thereafter, differentiating further just adds another power of t as a new independent solution.