ALGEBRAIC GEOMETRY (MATH 145) PROBLEM SET 8

This set is due by 4pm on Wednesday, March 15, in Laurent Côté's mailbox.

You are encouraged to talk to each other (in person, privately, on piazza, over pizza, etc.), and to me, about the problems. But write up your solutions separately, and also give credit for ideas and sources. If you don't know people, but would like to work together, just let me know, and I'll introduce you to others. Some of these problems require hints, and I'm happy to give them!

You are allowed to assume the answer to one problem in solving another (within reason — if in doubt, just ask).

Questions? Try asking on piazza, and also feel free to email me or Laurent.

Reminder: My office hours are Wednesdays 9:15-11:15 am and Fridays 2:30-3:30 pm in 383-Q. Laurent's office hours are Wednesdays 3:30-4:15 pm and Thursdays 7-8:15 pm in 381-L.

0. (mandatory) Complete the online check-in on canvas (by the due date).

Also, do **five** of the following problems. If you are ambitious (and have the time), go for more. *You may assume Hilbert's Nullstellensatz in solving these problems.*

In all of these problems, assume the base field k is algebraically closed to avoid distractions. Sometimes this hypothesis is essential, and sometimes it is not.

1. Show that the variety $x + y + z = 0$ in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. ("A line in the projective plane is really a projective line.")

2. Recall that a prevariety was defined to be a *quasicompact* topological space, with a sheaf of rings, that is locally isomorphic to $\mathrm{mSpec} R$ for some nilpotent-free ring R finitely generated over an algebraically closed field k. Show that every open subset of a prevariety is also quasicompact. (Hence every open subset of a prevariety is a prevariety.)

3. Show that the automorphisms of $\mathbb{P}^n$ are precisely the projective linear transformations.

4. Show that the nodal cubic $y^2 = x^3 - x$ (in the plane $\mathbb{A}^2$) minus the origin is isomorphic to $\mathbb{P}^1$ minus 3 points. Hint: project from the origin.

5. Show that every smooth quadric surface over an algebraically closed field of characteristic not 2 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Hint: recall that any such surface can be written,

Date: March 7, 2017.

in suitable coordinates, as $ad - bc = 0$. Consider "the map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ given by $([x,y],[u,v]) \mapsto [xu, xv, yu, yv]$".

6. Suppose X is a variety over a field k, and $Y = \mathfrak{m}\operatorname{Spec} R$ is an affine variety. Suppose A is the ring of (global) functions on X. Identify the morphisms $X \to Y$ with the maps (of k-algebras) $R \to A$. (We already know this when X is affine. The upshot is that we now know this without that assumption, which makes morphisms easier to check — we no longer have to cut the source up into affine pieces, although we can cut the target into affine pieces.)

7. Show that projective space is separated. Hint: Show that the diagonal is closed when restricted to the subsets of the form $U_i \times U_i$ and $U_i \times U_j$.

8. Show that the "line with the doubled origin" (when we glued $\mathbb{A}^1$ to $\mathbb{A}^1$ along $\mathbb{A}^1 - \{0\}$ in the way that didn't produce $\mathbb{P}^1$) is not separated.

9. Show that the old definition of Hausdorff (for topological spaces) is the same as separated, assuming the topology on the product *is* the product topology.

E-mail address: `vakil@math.stanford.edu`