ALGEBRAIC GEOMETRY (MATH 145) PROBLEM SET 7

This set is due by 4pm on Friday, March 3, in Laurent Côté's mailbox.

You are encouraged to talk to each other (in person, privately, on piazza, over pizza, etc.), and to me, about the problems. But write up your solutions separately, and also give credit for ideas and sources. If you don't know people, but would like to work together, just let me know, and I'll introduce you to others. Some of these problems require hints, and I'm happy to give them!

You are allowed to assume the answer to one problem in solving another (within reason — if in doubt, just ask).

Questions? Try asking on piazza (this is the first time I'm using it in a class, and so far I like it), and also feel free to email me or Laurent.

Reminder: My office hours are Wednesdays 9:15-11:15 am and Fridays 2:30-3:30 pm in 383-Q. Laurent's office hours are Wednesdays 3:30-4:15 pm and Thursdays 7-8:15 pm in 381-L.

0. (mandatory) Complete the online check-in on canvas (by the due date).

Also, do **four** of the following problems. If you are ambitious (and have the time), go for more. *You may assume Hilbert's Nullstellensatz in solving these problems.*

In all of these problems, assume the base field k is algebraically closed to avoid distractions. Sometimes this hypothesis is essential, and sometimes it is not.

1. Suppose R is a finitely generated algebra over an algebraically closed field k, that is *not necessarily* an integral domain, and suppose $f \in R \setminus 0$. Show that the sections of the sheaf of algebraic functions $\mathcal{O}_{\mathrm{m\,Spec}\,R}$ over the distinguished open set $D(f)$ are precisely R_f.

2. Suppose R is a commutative ring, and $z \in R$. Give an isomorphism $R_z \cong R[t]/(tz-1)$. Hence show that if R is a finitely generated algebra over $\overline{k}$, then R_z is as well. Show that if R is nilpotent-free, then R_z is as well

3. Suppose R is a ring, and $S \subset R$ is a multiplicative subset (including 1). In last week's problem set, some of you showed that if R is finitely generated over an algebraically closed field k, then the maximal ideals of $S^{-1}R$ can be identified with those maximal ideals of R containing no elements of S. This week, give a counterexample (with proof).

Date: February 26, 2017.

4. Suppose that $\pi : X \to Y$ and $\rho : Y \to Z$ are both morphisms of varieties. Show that $\rho \circ \pi : X \to Z$ is a morphism as well.

5. Make sense of the following statement, and prove it: $\mathbb{A}^{n+1} \setminus \{(0,0)\} \to \mathbb{P}^n$ given by $(x_0, \dots, x_n) \mapsto [x_0, \dots, x_n]$ is a morphism of varieties. (This jibes with our understanding of projective space as a quotient of affine space minus the origin.)

6. Make sense of the following statement, and prove it: the map $\mathbb{P}^1 \to \mathbb{P}^3$ given by $[x, y] \mapsto [x^3, x^2y, xy^2, y^3]$ gives an isomorphism of $\mathbb{P}^1$ with the variety in $\mathbb{P}^3$ cut out by $ad = bc$, $ac = b^2$, $bd = c^2$. (This curve in projective space is called the "twisted cubic".)

7. *(only if you didn't do this one last week)* Write a rough draft of your writing project, in latex. (Perhaps: make an overleaf document that you can share with Laurent.) If you do this, you also get the points for problem 7.

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