ALGEBRAIC GEOMETRY (MATH 145) PROBLEM SET 5

This set is due by 4pm on Friday, February 17, in Laurent Côté's mailbox.

You are encouraged to talk to each other (in person, privately, on piazza, over pizza, etc.), and to me, about the problems. But write up your solutions separately, and also give credit for ideas and sources. If you don't know people, but would like to work together, just let me know, and I'll introduce you to others. Some of these problems require hints, and I'm happy to give them!

You are allowed to assume the answer to one problem in solving another (within reason — if in doubt, just ask).

Questions? Try asking on piazza (this is the first time I'm using it in a class, and so far I like it), and also feel free to email me or Laurent.

Reminder: My office hours are Wednesdays 9:15-11:15 am and Fridays 2:30-3:30 pm in 383-Q. Laurent's office hours are Wednesdays 3:30-4:15 pm and Thursdays 7-8:15 pm in 381-L.

0. (mandatory) Complete the online check-in on canvas (by the due date).

Also, do **six** of the following problems. If you are ambitious (and have the time), go for more. *You may assume Hilbert's Nullstellensatz in solving these problems. Remember that the Nullstellensatz has the hypothesis that the "base field" k is algebraically closed!*

1. Suppose $f : R \to S$ is a map of finitely-generated nilpotent-free algebras over $\overline{k}$.

(a) Describe an induced map $f^\sharp : \mathfrak{m}\operatorname{Spec} S \to \mathfrak{m}\operatorname{Spec} R$ that sends maximal ideals of S to maximal ideals of R.
(b) Describe an induced map (let's call it $f^\sharp$ as well) from the set of prime ideals of S to the set of prime ideals of R.
(c) Then the map (a) describes the maps of the corresponding affine algebraic varieties (or affine algebraic sets) as points. Give a geometric interpretation of map in (b), keeping in mind that prime ideals correspond to irreducible closed subsets.

2. Consider the ring map $\overline{k}[x, y] \to \overline{k}[t]$ given by $x \mapsto t^2$, $y \mapsto t^3$.

(a) (easy) Show that this corresponds to a map from $\mathbb{A}^1_{\overline{k}}$ (with coordinate t) to the the cuspidal curve in $\mathbb{A}^2$ cut out by $y^2 = x^3$.
(b) Show that this is a bijection on points.
(c) On the other hand, show that this is not an isomorphism of algebraic sets.

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Hence there is an "inverse map of sets", but this map is not "algebraic".

3. Consider the ring map $\overline{k}[x,y,z] \to \overline{k}[t]$ given by $x \mapsto t^2+t^3$, $y \mapsto t+t^3$, $z \mapsto t+t^2$.
(a) Show that this gives an isomorphism of $\mathbb{A}^1$ (with coordinate t) with the algebraic set in $\mathbb{A}^3$ cut out by

$$\begin{aligned} x &= \left(\frac{y+z-x}{2}\right)^2 + \left(\frac{y+z-x}{2}\right)^3 \\ y &= \left(\frac{y+z-x}{2}\right) + \left(\frac{y+z-x}{2}\right)^3 \\ z &= \left(\frac{y+z-x}{2}\right) + \left(\frac{y+z-x}{2}\right)^2. \end{aligned}$$

(b) Show that the equations

$$\begin{aligned} x &= \left(\frac{y+z-x}{2}\right)^2 + \left(\frac{y+z-x}{2}\right)^3 \\ y &= \left(\frac{y+z-x}{2}\right) + \left(\frac{y+z-x}{2}\right)^3 \\ z &= \left(\frac{y+z-x}{2}\right) + \left(\frac{y+z-x}{2}\right)^2. \end{aligned}$$

cut out an irreducible closed subset of $\mathbb{A}^3$.

4. Show that (the data of) a function on an affine variety X is the "same" as a map to $\mathbb{A}^1$.

5. (a) Prove "Fermat's last theorem for polynomials": let n be an integer greater than 2. If there are polynomials $f, g, h \in \mathbb{C}[t]$, and $f(t)^n + g(t)^n = h(t)^n$, then there is some polynomial $j(t)$ (possibly constant) such that f, g, and h are all scalar multiples of $j(t)$. (Translation: the only solutions to $F(t)^n + G(t)^n = 1$, where $F, G \in \mathbb{C}(t)$, are constants.) Possible hint: reduce to the case where f, g, and h have no common factor. Factor $h(t)^n - f(t)^n$. Show that your factors are all perfect nth powers. Find some linear combination among three of them, giving a counterexample of smaller degree.
(b) Show that part (a) is false if $n = 2$.
(c) Describe all maps from $\mathbb{A}^1$ to the affine variety in $\mathbb{A}^3$ cut out by $x^n + y^n = z^n$, where $n > 2$.

6. Suppose we have an isomorphism of finitely-generated nilpotent-free k-algebras

$$k[x,y,z]/(f(x,y,z)) \cong k[s,t,u]/(g(s,t,u)).$$

Then the corresponding algebraic sets Y in $\mathbb{A}^3$ (with coordinates x,y,z) and Z in $\mathbb{A}^3$ (with coordinates s,t,u) are isomorphic. Show that Y is smooth if and only if Z is smooth. (We will soon see that smoothness depends only on the variety, not on its choice of embedding in affine space. But right now, we have only defined smoothness for hypersurfaces.)

7. How many affine varieties (over $\overline{k}$, up to isomorphism) have precisely 17 points?

8. Show that $\mathbb{A}^2_{\overline{k}}$ and $\mathbb{A}^3_{\overline{k}}$ are not isomorphic affine varieties.

9. Show that those polynomials $f(t) \in \mathbb{C}[t]$ such that $f'(1) = f''(1) = 0$ form a ring R. Thus $\mathfrak{m}\operatorname{Spec} R$ must be an affine variety. Give an explicit description (with proof!) as an algebraic set — give some affine space, and some equations, which cut out this algebraic set.

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