ALGEBRAIC GEOMETRY (MATH 145) PROBLEM SET 4 (REVISED)

This set is due by 4pm on Friday, February 10, in Laurent Côté's mailbox.

You are encouraged to talk to each other (in person, privately, on piazza, over pizza, etc.), and to me, about the problems. But write up your solutions separately, and also give credit for ideas and sources. If you don't know people, but would like to work together, just let me know, and I'll introduce you to others. Some of these problems require hints, and I'm happy to give them!

You are allowed to assume the answer to one problem in solving another (within reason — if in doubt, just ask).

Questions? Try asking on piazza (this is the first time I'm using it in a class, and so far I like it), and also feel free to email me or Laurent.

Reminder: My office hours are Wednesdays 9:15-11:15 am and Fridays 2:30-3:30 pm in 383-Q. Laurent's office hours are Wednesdays 3:30-4:15 pm and Thursdays 7-8:15 pm in 381-L.

0. (mandatory) Complete the online check-in on canvas (by the due date).

Also, do **nine** of the following problems. If you are ambitious (and have the time), go for more. *You may assume Hilbert's Nullstellensatz in solving these problems. Remember that the Nullstellensatz has the hypothesis that the "base field" k is algebraically closed!*

1. Suppose S is a subset of $\mathbb{A}^n_k$. Show that $\overline{S} = V(I(S))$.

2. Show that prime ideals are radical.

3. Show that a closed subset Z in $\mathbb{A}^n_k$ is irreducible if and only if $I(Z)$ is prime. If $k = \overline{k}$, show that a radical ideal J is prime if and only if $V(J)$ is irreducible.

4. Let $J = (x^2 + y^2 - 1, y - 1)$. Find, with proof, an element of $I(V(J)) \setminus J$.

5. Show that $V(IJ) = V(I \cap J) = V(I) \cup V(J)$. Hint: show that $(I \cap J)^2 \subset IJ \subset I \cap J$.

6. Suppose $p(x, y)$ and $q(x, y)$ are polynomials over an algebraically closed field $k = \overline{k}$. Show that they cut out the same sets (i.e., $V(p) = V(q)$) if and only if p divides q^n and q divides p^m for some positive integers m, n. (Equivalently, $V(p) = V(q)$ if and only if they have the same irreducible factors, possibly with different multiplicities. Hence given the zero-set of a polynomial, we can recover the polynomial up to irreducible factors, and a nonzero scalar coefficient.) Show that the result can be false if k is not algebraically closed.

Date: February 3, 2017. Revised Feb. 8, 2017.

7. *(uniqueness of decomposition into irreducible components)* Suppose $X \subset \mathbb{A}^n_k$ is an algebraic subset. We say that a closed subset $Y \subset X$ is an **irreducible component** of X if it is a maximal irreducible closed subset of X, i.e. if any irreducible closed subset of X contains Y, then it must *be* Y. In class, we showed that X can be written as a finite union of irreducible components, with no repeated terms. Show that this decomposition is unique: if $X = Y_1 \cup \cdots \cup Y_m$ one such decomposition, and $X = Z_1 \cup \cdots \cup Z_n$ is another such decomposition, then $m = n$, and the Y_i's are a rearrangement of the Z_j's.

8. What are the irreducible components of $V(xy, yz, zx)$? (Hint: draw a picture.)

9. A prime ideal I of a ring A is said to be a **minimal prime ideal** if there is no prime ideal of A strictly contained in I. For example, if A is an integral domain, then there is only one minimal prime ideal — the 0 ideal. If J is any ideal of $R = k[x_1, \dots, x_n]$, show that R/J has finitely many minimal prime ideals. (Hint: translate this into "finitely many irreducible components".)

10. We work over an algebraically closed field $k = \overline{k}$. Suppose I is a radical ideal of $A := \overline{k}[x_1, \dots, x_m]$ corresponding to the algebraic set $X \subset \mathbb{A}^m_{\overline{k}}$, and J is a radical ideal of $B := \overline{k}[y_1, \dots, y_n]$ corresponding to the algebraic set $Y \subset \mathbb{A}^n_{\overline{k}}$. Show that any polynomial map $X \to Y$ induces a map of rings $B/J \to A/I$.

11. We continue the notation of problem 10. Show that any map of rings $B/J \to A/I$ induces a polynomial map $X \to Y$.

12. Show that the maps in problems 10 and 11 are inverses.

13. Consider the curve C given by $y^2 z = x^3$ in $\mathbb{P}^2_k$. Show that $[0, 0, 1]$ is the only non-smooth point of C. Show that the constructions used making the group law on a smooth cubic of the form $y^2 z = x^3 + ax^2 + bx + c$, taken without change, define a group law on the *smooth points* of C. Show that this group is isomorphic to the group $(k, +)$ (the group k under addition), sending $[x, y, z]$ to x/y, with the reverse map $t \mapsto [t, 1, t^3]$.

14. Consider the curve C given by $y^2 z = x^2(x + z)$ in $\mathbb{P}^2_k$. Show that $[0, 0, 1]$ is the only non-smooth point of C. Show that the constructions used making the group law on a smooth cubic of the form $y^2 z = x^3 + ax^2 + bx + c$, taken without change, define a group law on the *smooth points* of C. Show that this group is isomorphic to the group $(k^*, \times)$ (the group $k \setminus \{0\}$ under multiplication). What is the isomorphism? (Possible hint: from earlier in the quarter, you have shown that this nodal cubic can be mostly identified with $\mathbb{P}^1_k$, by considering lines through the origin.)

15. Suppose A is an arbitrary ring. Define the set $\operatorname{Spec} A$ (called the **spectrum of** A) as the set of prime ideals of A. For notation, we say the element of $\operatorname{Spec} A$ corresponding to $\mathfrak{p}$ is $[\mathfrak{p}]$. We say that an element $f \in A$ **vanishes at** $[\mathfrak{p}]$ if $f \in \mathfrak{p}$. For any subset of A, say $S = \{f_1, f_2, \dots\}$, define $V(S) \subset \operatorname{Spec} A$ to be the subset of points where the elements of S vanishes. Show that if we take these as "closed subsets" of $\operatorname{Spec} A$, this defines a topology on the set $\operatorname{Spec} A$.

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