ALGEBRAIC GEOMETRY (MATH 145) PROBLEM SET 3

This set is due by 4pm on Friday, February 3, in Laurent Côté's mailbox.

You are encouraged to talk to each other (in person, privately, on piazza, over pizza, etc.), and to me, about the problems. But write up your solutions separately, and also give credit for ideas and sources. If you don't know people, but would like to work together, just let me know, and I'll introduce you to others. Some of these problems require hints, and I'm happy to give them!

You are allowed to assume the answer to one problem in solving another (within reason — if in doubt, just ask).

Questions? Try asking on piazza (this is the first time I'm using it in a class, and so far I like it), and also feel free to email me or Laurent.

Reminder: My office hours are Wednesdays 9:15-11:15 am and Fridays 2:30-3:30 pm in 383-Q. Laurent's office hours are Wednesdays 3:30-4:15 pm and Thursdays 7-8:15 pm in 381-L.

0. (mandatory) Complete the online check-in on canvas (by the due date).

Also, do **nine** of the following problems. If you are ambitious (and have the time), go for more.

1. *(special case of Bezout)* We work over an algebraically closed field k. Suppose X is a hypersurface in $\mathbb{P}^n$ of degree d, and suppose ℓ is a line in $\mathbb{P}^n$ not contained in X. Show that ℓ meets X in d points, counted correctly. (What does "counted correctly" mean? How will you define multiplicity?) Possible hint: think first about the case where the line is a "coordinate line" spanned by $[1, 0, \dots, 0]$ and $[0, 1, 0, \dots, 0]$, i.e. given by $x_2 = x_3 = \cdots = 0$.

2. Show that $x^2 + y^2 = 13$ has a solution modulo 7^{100}.

3. We work over an arbitrary field k. Suppose X is a degree d hypersurface in $\mathbb{P}^n$, and $p \in X$. Show that the smoothness of a hypersurface at p is independent of coordinate choice. In other words, if a hypersurface is smooth, as checked on the standard coordinate charts, the same remains true after you act on it by a projective transformation. (Hint: suppose $x_0, \dots, x_n$ are one set of coordinates, and $y_0, \dots, y_n$ are another set of coordinates, and p is in the "zeroth" coordinate chart for both the x-coordinates and the y-coordinates. The coordinates are related by an $(n+1) \times (n+1)$ invertible matrix.)

4. We work over a field k. Suppose $a, b, c \in k$.
(a) Show that that $(x - a, y - b, z - c)$ is a maximal ideal of $k[x, y, z]$.

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(b) Suppose $f_1(x, y, z)$, $f_2(x, y, z)$, ... are polynomials (in x, y, z). Show that $f_1(a, b, c) = f_2(a, b, c) = \cdots = 0$ if and only if the ideal $(f_1, f_2, \dots)$ is contained in the maximal ideal $(x - a, y - b, z - c)$.

5. We work over a field k. Suppose $f(x, y)$ and $g(x, y)$ are polynomials such that $k[x, y]/(f, g)$ is a finite-dimensional vector space over k.

(a) Show that there is some nonzero polynomial $a(x) \in k[x]$ such that $a(x) \in (f(x, y), g(x, y))$.
(b) Show that there is some nonzero polynomial $b(y) \in k[y]$ such that $b(y) \in (f(x, y), g(x, y))$.
(c) Show that $f(x, y) = g(x, y) = 0$ has finitely many solutions in k^2 (indeed, at most $(\deg a)(\deg b)$).

6. Find all maximal ideals in $\mathbb{Q}[x, y]$ containing $x^2 + y^2 - 25$ and:

(a) y (b) $(y - 3)$ (c) $(y - 5)$ (d) $(y - 2)$

You do not need to prove that you have found them all. (Hint: draw a picture to find maximal ideals.)

7. In defining the group law on smooth cubic curves in $\mathbb{P}^2$ of the form $y^2 z = x^3 + ax^2 z + bxz^2 + cz^3$, where did we use the fact that we chose the identity to be the point $[0, 1, 0]$? (In other words, why couldn't we choose the identity to be some other point p, and instead of "reflection in the x-axis" in the definition, we send q to the third point of intersection of the line pq with the cubic?)

8. Suppose E is the smooth cubic curve $y^2 z = x^3 - xz^2$. Suppose also that p and q are two points of E. Define the map $f : E \to E$ given by: r maps to the "third point of intersection" of the line pr with E. Define the map $g : E \to E$ given by: r maps to the "third point of intersection" of the line qr with E. Let $h = f \circ g$. Suppose there is a point $s \in E$ such that $h^{100}(s) = s$. Show that $h^{100}(t) = t$ for *all* $t \in E$. (Hint: interpret f, g, and h in terms of the group law on E.)

9. Suppose C is a real ellipse. Define a map $f : C \to C$ as follows. To compute $f(p)$, we first take the "other" point of intersection of the horizontal line through p and call it q, then take the "other" point of intersection of the vertical line through q and call it $f(p)$. Suppose $f^{100}(s) = s$ for some $s \in C$. Show that $f^{100}(s) = s$ for all $s \in C$.

10. *(worth two)* A construction we have used often is the following. If E is a smooth cubic in $\mathbb{P}^2$, then there is a map of sets $E \times E \to E$ which sends p and q (not necessarily distinct!) to the third, and this map should "be algebraic" (something we haven't yet discussed). In this problem, we describe the map, and even make sure it is algebraic "near when $p = q$".

We know that the line pq meets E in 3 points counted correctly (problem 1). Pick a coordinate chart where all 3 points do not lie on the line at infinity. (Technically, we should assume the field is not $k = \mathbb{F}_2$ to make this possible. So make this assumption.) Also, choose this coordinate chart so that pq is not a vertical line, and that the tangent to p is not a vertical line. So now we have a cubic $f(x, y) = 0$, and two points $p = (p_1, p_2)$

and $q = (q_1, q_2)$. The nonvertical lines ℓ_m through p are given by $(y - p_2) = m(x - p_1)$, i.e.

$$y = m(x - p_1) + p_2.$$

as m runs through k. We plug this in to $f(x, y) = 0$.
(a) Show that $f(x, m(x - p_1) + p_2)$ is a *nonzero* polynomial in x of degree at most 3 (call it $g(x)$) (no matter what m is), and has $x = p_1$ as a solution.
(b) Show that $g(x)/(x - p_1)$ is a nonzero polynomial in x of degree at most 2.
(c) Show that if either (i) $p \neq q$ and $q \in \ell_m$, or (ii) $p = q$, and ℓ_m is the tangent line at p, then $(x - p_2)$ is a factor of $g(x)/(x - p_2)$. Thus $g(x)/((x - p_1)(x - q_1))$ is a nonzero polynomial of degree at most 1.
(d) Show that this polynomial is not constant (so $g(x)$ is a polynomial of degree precisely 3).
(e) Argue (convincingly, with proof) that you could then write an explicit formula for the third point of intersection!

11. Suppose $p_1, \ldots, p_8$ are distinct points in the plane, no four on a line and no six on a conic. Show that there is another point p_9 such that every cubic through $p_1, \ldots, p_8$ also passes through p_9.

12. Complete the proof of associativity of the group law on cubics over $\mathbb{C}$ or $\mathbb{R}$. (You can assume facts about continuity etc.)

13. Complete the proof of associativity of the group law over an arbitrary field k.

14. Suppose $f(x, y, z) = 0$ is a smooth conic in $\mathbb{P}^2$ over $\mathbb{F}_p$ (where p is an odd prime). Show that $f(x, y, z)$ has a point. (This is unlike the case for $k = \mathbb{R}$!) This is intended to be a challenge — I'm not sure how best to attack it given what you know, so for this problem I won't give any hints to start you off. But I'm happy to discuss any ideas you might have.

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