ALGEBRAIC GEOMETRY (MATH 145) PROBLEM SET 2 (REVISED)

This set is due by 3pm on Friday, January 27, in Laurent Côté's mailbox.

You are encouraged to talk to each other (in person, privately, on piazza, over pizza, etc.), and to me, about the problems. But write up your solutions separately, and also give credit for ideas and sources. If you don't know people, but would like to work together, just let me know, and I'll introduce you to others. Some of these problems require hints, and I'm happy to give them!

0. (mandatory) Complete the online check-in on canvas (by the due-date).

Also, do **nine** of the following problems. If you are ambitious (and have the time), go for more.

1. Suppose the cross ratio of the four numbers are (a, b, c, d) is λ. If you rearrange these four numbers, what different cross ratios can you get (in terms of λ)? Hint: you won't get 24.

2. Show that the homogeneous polynomials in $n+1$ variables $x_0, \dots, x_n$ of degree d form a vector space of dimension $\binom{n+d}{d}$.

3. For this question, we work over $\mathbb{C}$.

(b) What are the asymptotes of the degree 4 equation $1+3x+4y^2+3x^2+xy^3-x^3y=0$?
(a) How do you want to define asymptote (for a polynomial in $\mathbb{C}^2$?

Obviously, you can't answer (b) without knowing the answer to (a). On the other hand, you should figure out (b) before (a).

4. For this question, we work over an algebraically closed field of characteristic not 2. Notice that any quadratic form in n variables can be written as

$$\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where M is a symmetric matrix.

(a) Describe how change of coordinates corresponds to conjugation of M.
(b) Use this to define the **rank** of a quadratic form.

Date: January 22, 2017.

5. For this question, we work over $\mathbb{R}$. Show that any quadratic form in n variables can be written, after change of coordinates, as

$$y_1^2 + y_2^2 + \cdots + y_a^2 - y_{a+1}^2 - \cdots - y_{a+b}^2,$$

and that a and b are independent of your change of basis. (Such a quadratic form is said to be of **index** (a, b).) In $\mathbb{P}^2_{\mathbb{R}}$, quickly describe what all quadric surfaces can look like, characterized by index.

6. Find all rational solutions of $x^4 + y^3 + x^2y + xy^2 = 0$.

7. (In this problem, we work over any field of characteristic not 2. You might want to keep the rationals in mind.) In class, we used projection from the point $[1, 0, 1]$ in order to find all solutions to $x^2 + y^2 = z^2$ in $\mathbb{P}^2$ — we ended up with a bijection of the points to $\mathbb{P}^1$. If instead we used $[0, 1, 1]$, we would have gotten a *different* bijection. Composing these bijections we would get a bijection of the points of $\mathbb{P}^1$ with itself. Show that the resulting bijection is a projective transformation (with as little algebraic suffering as possible).

8. (a) For this problem, we work over $\mathbb{C}$. If we picked four random points and one random line in $\mathbb{P}^2$, how many conics would pass through those four points and are tangent to the line? (There are some undefined words in this question...)
(b) What other fields can your argument work for? (This is also slightly open-ended.)

9. For this problem, we work over any field k.

(a) Show that the lines in $\mathbb{P}^2$ through a point p are in some natural way in bijection with $\mathbb{P}^1$. Draw a picture of this.
(b) Suppose we have a curve C in the projective plane given by $f(x, y, z) = 0$. Suppose $p = [a, b, c]$ is a smooth point on C. Explain how we get a map from $C - p$ to $\mathbb{P}^1$, given by mapping a point q to the line pq through p.

10. (following up on problem 9) Show that there is one and only one way to extend the map $C - p \to \mathbb{P}^1$ to a map $C \to \mathbb{P}^1$, and that this map sends p to the tangent line at p. (You are not allowed to use limits! We are doing this over an arbitrary field. No deltas and epsilons allowed!) This is a good *definition* of **tangent line**.

11. Show that $x^2 + y^2 = -2$ has solutions modulo 7^{100}.

12. For this problem we work over a field of characteristic 0 for convenience. Suppose we have a plane curve $f(x, y) = 0$, such that $(0, a)$ is a smooth point whose tangent line is nonvertical. Show that there is precisely one power series solution $y = p(x)$ such that $f(x, p(x)) = 0$ (as a power series), and $p(0) = a$. Do not worry about convergence — you can't because we don't have any deltas and epsilons in the picture!) If m is the slope of the tangent line at $(0, a)$, show that the power series begins $a + mx + \cdots$.... (This then gives a new definition of tangent line. Do you see how it compares to the definition in problem 10?)

13. Find the singular points of the following curves. (We work over $\overline{\mathbb{Q}}$.)

(a) $x^n + y^n = 1$ in $\mathbb{A}^2$
(b) $y^2 = x^4 - x^3 - x^2 + x$ in $\mathbb{A}^2$
(c) $xy^2 = z^3$ in $\mathbb{P}^2$
(d) $xy^2 = z^3$ in $\mathbb{P}^3$

14. Consider the polynomial $y^2 = x^3 + 2017x^2 - 2014$. Notice that $(1, 2)$ and $(1, -2)$ are two rational solutions. Find another one.

Any questions? Let's try asking on piazza (this is the first time I'm using it in a class), but also feel free to email me or Laurent.

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