ALGEBRAIC GEOMETRY (MATH 145) PROBLEM SET 1

This set is due by 3pm on Friday, January 20, in Laurent Côté's mailbox.

You are encouraged to talk to each other (in person, privately, on piazza, etc.), and to me, about the problems. But write up your solutions separately, and also give credit for ideas and sources. If you don't know people, but would like to work together, just let me know, and I'll introduce you to others. Some of these problems require hints, and I'm happy to give them!

0. (mandatory) Complete the online check-in on canvas (by the due-date).

Also, do **seven** of the following problems. If you are ambitious (and have the time), go for more.

1. Suppose I have 200 linear equations in 17 unknowns $(x_1, \dots, x_{17})$, and all of these equations have integer coefficients. Suppose further that $(\pi, i, \dots)$ is a solution. Show that there is a rational solution as well.

2. (Practice with the language of categories.) Suppose we have a category C (for example, the category of vector spaces), with objects (for example, vector spaces) and morphisms (for example, linear maps). Suppose $V \in C$ is an object. Show that those morphisms from V to itself that are *isomorphisms* form a group. (The main part of this problem is thinking through definitions.)

3. In $\mathbb{P}^2$, show that any two distinct lines (those cut out by homogeneous degree 1 equations) meet in precisely one point. (Your argument should have no reference to the field.)

4. Find a linear fractional transformation $f(t) \in PGL(2)$ that satisfy the following, preferably over any field.

- $f(t) \neq t$ for some t.
- $f(f(f(t))) = t$ for all t.

In other words f has order precisely 3 (not 1) in $PGL(2)$.

5. Show that any two order 3 elements of $PGL(2)$ are conjugate. (Possible hint: use transitivity.)

Date: January 15, 2017.

For the next questions, in $\mathbb{P}^2_k$, we (as usual) take projective coordinates x_0, x_1, x_2. The big open set (or "coordinate chart") $U_0 = \{[x_0, x_1, x_2] : x_0 \neq 0\}$ has coordinates $x_{1/0}$ and $x_{2/0}$, which we interpret as x_1/x_0 and x_2/x_0. We have similar definitions for U_1 and U_2. (It can be convenient to define $x_{0/0}$ as 1.)

6. Describe $(x_{0/1}, x_{2/1})$ in terms of $(x_{1/0}, x_{2/0})$.

7. (a) Describe *all* homogeneous polynomials in x_0, x_1, x_2 that can be dehomogenized to $x_{1/0}^2 + x_{2/0}^3 = 1$. Explain which one is "best".
(b) In what points to these polynomials meet the "line at infinity" $x_0 = 0$?

8. Consider the parabola $x_{2/0} = x_{1/0}^2$ (or, if you prefer, $y = x^2$). How does it meet the line at infinity? What is its description in the different coordinate patches U_1 and U_2?

9. Suppose I have 4 numbers (a, b, c, d) such that the cross ratio of (a, b, c, d) is the same as (a, b, d, c). Then what is this cross ratio?

10. Show that the cross ratio of (a, b, c, d) is the same as the cross ratio of (b, a, d, c). The group S_4 acts on 4-tuples of numbers, and hence on the cross ratio. Show that the action of S_4 on the cross-ratio is trivial when restricted to the "Klein 4-group" $V_4 \subset S_4$, the 4-element abelian subgroup that can be described as $(abcd)$, $(badc)$, $(cdab)$, $(dcba)$. (Just ask if it isn't clear what I mean by this.)

11. Consider the group action of $PGL(3)$, acting on the projective plane $\mathbb{P}^2$.
(a) Show that this group action is 2-transitive.
(b) If p_1, p_2, and p_3 are distinct points of $\mathbb{P}^2$, and $g \in PGL(3)$, show that p_1, p_2, and p_3 are collinear if and only if $g(p_1)$, $g(p_2)$, and $g(p_3)$ are collinear.
(c) Show that $PGL(3)$ is not 3-transitive.

Any questions? Let's try asking on piazza (this is the first time I'm using it in a class), but also feel free to email me or Laurent.

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