1. LOOSE ENDS

Last time we tried to give an example of a Deligne-Mumford stack that comes up “in nature.” We gave the example of 4 “unmarked points” in \( \mathbb{P}^1 \) up to automorphisms, and we realized this as the quotient of 4 “marked points” by \( S_4 \). We “retracted” this as a quotient of \( \mathbb{P}^1 \) by \( S_3 \), which can be thought of in two ways: either by quotienting out the data of 1 marked point and 3 unmarked (which we labeled as \( 0, 1, \infty, \lambda \)) or by realizing that the Klein quartic \( V_4 \subset S_3 \) acts trivially.

There were three special orbits with non-trivial stabilizers: \( \{0, 1, \infty\} \), \( \{-1, \frac{1}{2}, 2\} \), and \( \{1 \pm \sqrt{3}/2\} \).

First of all, let’s mention another way of accounting for all special orbits. Our previous argument exploited the idea of “Euler characteristic.” For a complex manifold, the (compactly supported) Euler characteristic is

\[
\chi_e(X) = \sum_i (-1)^i h^i_c(X).
\]

This is additive in nice situations, e.g. if \( X = \bigcup_{\text{open}} Y \cup \bigcup_{\text{closed}} Z \), then \( \chi_e(X) = \chi_e(Y) + \chi_e(Z) \). It is easy to check that under a finite étale map \( f: X \to Y \) of degree \( d \), \( \chi_e(X) = d\chi_e(Y) \). For stacks, we apply this formula to some étale cover. Of course, one has to check that it is well-defined (i.e. independent of cover). To do this, you can take the fibered product of any two étale covers by schemes (using that the diagonal is representable for Deligne-Mumford stacks):

\[
X \times_{\mathcal{X}} X' \longrightarrow X
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
X' \longrightarrow \mathcal{X}
\]

In general one may not have a finite étale cover of \( \mathcal{X} \), but it has some étale cover. If \( \mathcal{X} \) is quasicompact, then one can choose an étale cover comprised of finitely many finite étale morphisms \( \{U_i \to \mathcal{X}\} \), and
one can stratify \( \mathcal{X} \) by the number of pre-images, apply our previous definition for finite étale covers, and then add them up.

What does it even mean to stratify a stack? We say that \( \mathcal{X} = \mathcal{Y} \coprod \mathcal{Z} \) if any “point” \( \text{Spec } k \rightarrow \mathcal{X} \) factors through \( \mathcal{Y} \) or \( \mathcal{Z} \). We’ve basically just sketched an argument for the following result.

**Theorem 1.1.** Suppose \( \mathcal{X} \) is a finite type Deligne-Mumford stack with atlas \( \mathcal{U} \xrightarrow{\pi} \mathcal{X} \). Then there exists a finite collection of locally closed substacks \( Z_1, \ldots, Z_n \) that are equidimensional, such that \( \mathcal{X} = Z_1 \coprod \cdots \coprod Z_n \), and such that \( \pi^{-1}(Z_i) \rightarrow Z_i \) is finite étale.

The proof should just involve the following fact about schemes: given a finite type scheme \( X \) and a surjective étale map \( \mathcal{U} \rightarrow X \), the result is true, with the stratification being determined by fiber degrees.

This means that we can make sense of the Euler characteristic.

Here’s a second way to think about this. The map \( \mathbb{P}^1 \) to the “coarse quotient” \( \mathbb{P}^1 \) will be branched precisely over the points with non-trivial stabilizer. This is a degree 6 map, and we found three points such that two have 3 point above and one has 2 points above, and we can use Riemann-Hurwitz to see that this must be everything.

One more comment. We suggested picturing this quotient stack as \( \mathbb{P}^1 \) minus three points with two \( \mathbb{BZ}/2 \) and one \( \mathbb{BZ}/3 \) glued in. We basically already constructed the universal family over this.

In this case, we have an étale cover \( \mathbb{P}^1 \xrightarrow{s} [\mathbb{P}^1/S_3] \) so we can push and pull sheaves to heart’s content, and do all calculations on the honest scheme \( \mathbb{P}^1 \). For instance, what’s the degree of a point?

\[
\begin{array}{ccc}
\mathbb{P}^1 & \xrightarrow{s} & [\mathbb{P}^1/S_3] \\
\downarrow\text{deg 6} & & \downarrow \\
\text{Spec } k & & \\
\end{array}
\]

Let \( d \) be the degree of the point in \( [\mathbb{P}^1/S_3] \) corresponding to \( \{0, 1, \infty\} \) consists of 3 reduced points, which has deg so the degree of this point must be \( \frac{3}{6} = \frac{1}{2} \).

**2. Riemann-Roch**

The Grothendieck-Riemann-Roch theorem describes a relationship between two maps.
On the “sheaf” side, we have a map $K_0 \to G_0$. We think of $K_0$ as being like cohomology, and $G_0$ as being like homology, and this map as being analogous to cupping with the fundamental class.

On the “Chow side,” we have a map $\text{CH}^* \to \text{CH}_*$. In the smooth case these “coincide”, as do $K_0$ and $G_0$, but in general one is an “operation” on the other.

Grothendieck-Riemann-Roch says that there is the following compatibility:

$$
\begin{array}{ccc}
K_0 & \longrightarrow & G_0 \\
\downarrow c_i & & \downarrow c_i \\
\text{CH}^* & \longrightarrow & \text{CH}_*
\end{array}
$$

Observe that $K_0$ and $\text{CH}^*$ as “the same” for $X = \mathbb{P}^n_k$, for instance. We can write $X = U \cup Z$ where $U$ is open (e.g. $\mathbb{A}^n$) and $Z$ is closed (e.g. $\mathbb{P}^{n-1}$), and we have an exact sequence

$$
G_0(Z) \to G_0(X) \to G_0(U) \to 0.
$$

For $\mathbb{P}^n_k$, the first map is actually injective.

There is a similar sequence for projective spaces, and then one can see that $K_0$ and $\text{CH}^*$ are both $n+1$-dimensional projective spaces (using homotopy invariance).

OK, what about projective space over an arbitrary base, $\mathbb{P}^n_B$? Then one identifies $n+1$ copies of $K_0(B)$ and $n+1$ copies of $\text{CH}^*(B)$, so if you believe the theorem for $B$ then you get it for $\mathbb{P}^n_B$ as well.

One more example: consider an elliptic curve $E$. Then $A^0(E) = \mathbb{Z}[E], A^1(E) = A_0(E) = \text{Pic}(E)$, and the same holds for $K$-theory. If you think about the proof that deg 0 line bundles are parametrized by $E$, then you see that they are “the same” on both sides.

Ok, well if we believe that $K_0$ and $\text{CH}^*$ are “the same” then we should be able to produce a map $K_0 \xrightarrow{\text{ch}} \text{CH}^*$. Recall that the Chern character of a vector bundle $V$ is defined by

$$
\text{ch}(V) = \sum_i e^{\alpha_i}
$$

where the $\alpha_i$ are the “Chern roots.”

Grothendieck would complain that this map is not functorial under proper pushforwards. (The motivation for considering proper pushforwards goes back to Hirzebruch.) In other words, given $X \to \mathbb{P}^n_k$
Y proper pushforward then the diagram

\[
\begin{array}{c}
K_0(X) \xrightarrow{\text{ch}} CH^*(X) \\
\downarrow \\
K_0(Y) \xrightarrow{\text{ch}} CH^*(Y)
\end{array}
\]

does not commute.

It was Hirzebruch who realized that to get commutativity, one has to “twist” by the Todd class.

\[
\begin{array}{c}
K_0(X) \xrightarrow{\text{ch} \cdot Td(X)} CH^*_Q(X) \\
\downarrow \\
K_0(Y) \xrightarrow{\text{ch} \cdot Td(X)} CH^*_Q(Y)
\end{array}
\]

Example 2.1. Let’s see that this really follows from GRR. Let \(X\) be a curve and \(F\) a vector bundle of rank \(r\). Let \(Y\) be a point. Then the diagram becomes

\[
\begin{array}{c}
K_0(X) \xrightarrow{\text{ch} \cdot Td(X)} CH^*_Q(X) \\
\downarrow \pi_* \\
K_0(Y) \xrightarrow{\text{ch} \cdot Td(X)} CH^*_Q(Y)
\end{array}
\]

The pushforward is then the alternating sum of the cohomology groups, so

\[
\pi_*(F) = h^0(X, F) - h^1(X, F).
\]

We have to figure out the Chern character of \(F\) and the Todd class of \(X\). The Chern character is just \(r[\mathcal{O}X] + c_1(F)\), and the Todd class is

\[
\prod \left( \frac{\alpha_i}{1 - e^{-\alpha_i}} \right) = 1[X] + \frac{1}{2} c_1(TX)
\]

Thus, because the codimension 0 stuff does under pushforward,

\[
h^0(X, F) - h^1(X, F) = (r[\mathcal{O}X] + c_1(F))(1[X] + \frac{1}{2} c_1(TX)) = \deg c_1(F) + r(1-g).
\]

This is Riemann-Roch for curves!

Of course, Riemann wouldn’t have said things this way. Riemann observed that \(h^0(\mathcal{L}) \geq \deg L\), and the difference was realized as \(h^1(\mathcal{L})\).

For surfaces, one gets \(h^0 + h^1 + h^2 = \ldots\)
We then want to go in the “Grothendieckan” direction, i.e. the case where \( X \rightarrow Y \) is a “family” of curves. There should be some recipe to produce, from a vector bundle on \( X \) (i.e. a family of vector bundles over \( Y \)) a vector bundle on \( Y \). One knows that the “vector bundle” isn’t well-behaved but the “virtual vector bundles” (i.e. alternating sum) will be. Then we can still ask for the Chern classes, and so the Grothendieck-Riemann-Roch formula doesn’t seem so unnatural.

Once one has the statement of GRR

\[
\begin{array}{ccc}
K_0(X) & \xrightarrow{\text{ch} \cdot \text{Td}(X)} & CH^*_Q(X) \\
\downarrow & & \downarrow \\
K_0(Y) & \xrightarrow{\text{ch} \cdot \text{Td}(X)} & CH^*_Q(Y)
\end{array}
\]

then it becomes clear that the formula is true under composition. Therefore, one only has to prove it for projective space and closed embeddings (which are regular embeddings in the smooth case).

One final comment. It is “easy” to prove Riemann-Roch in the Euler characteristic version, but Serre duality is the “hard ingredient.” Therefore, it is trickier depending on the definition of the genus.