EQUIVARIANT ALGEBRAIC GEOMETRY

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1. Examples

1.1. **A geometric question.** Consider "$\mathbb{P}^1$ with four unordered points" up to automorphisms of $\mathbb{P}^1$. This is a little vague so far, so let's try to digest what it means.

Given four *ordered* points on $\mathbb{P}^1$, there exists a unique automorphism sending the first three to $0, 1, \infty$, and the fourth to λ. Therefore, "four ordered points" on $\mathbb{P}^1$ are parametrized by $\mathbb{P}^1$, via the *cross-ratio* λ (there are some automorphism issues at $\lambda = 0, 1, \infty$ which we'll ignore). So four unordered points are parametrized by "$\mathbb{P}^1/S_4$." This means that there is an S_4-action on the quotient space. But the subgroup $V_4 \subset S_4$ acts trivially, and we have an exact sequence

$$1 \to V_4 \to S_4 \to S_3 \to 1$$

so the space is really "$\mathbb{P}^1/S_3$," a curve admitting a degree 6 cover of $\mathbb{P}^1$, which is again $\mathbb{P}^1$.

If we want to study "line bundles on the quotient," then we need to understand the group action on line bundles. For instance, we have the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$.

Problem 1.1. When, and in how many ways, can one extend the S_4 action from $\mathbb{P}^1$ to $\mathcal{O}(n)$?

In particular, such an extension would make $\Gamma(\mathcal{O}(n))$ into an S_4-representation, and then the graded ring $\bigoplus_{d \geq 0} \Gamma(\mathcal{O}(n))$ is a representation of S_4. What does $\Gamma(\mathcal{O}(n))$ "look like" as n grows?

1.2. **An algebraic question.** Let's abstract this question a bit. Let G be a finite group (non-abelian) and let V be a finite-dimensional representation over $\mathbb{C}$. What representations occur in $\operatorname{Sym}^n V$? A related question: what representations occur in $V^{\otimes n}$?

Problem 1.2. Find estimates on "how often" a given irreducible representation of G occurs in $\operatorname{Sym}^n V$ and $V^{\otimes n}$.

Example 1.3 (Finite cyclic groups). The representation ring of $G = \mathbb{Z}/n$ is $A = \mathbb{C}[t]/(t^n - 1)$ (of course, there are small issues like why we choose coefficients in $\mathbb{C}$). Now Spec $A = \mu_n \cong \mathbb{Z}/n$ (over $\mathbb{C}$). So we have "recovered" the group in this funny way from its representation ring, and the eigenspace decomposition corresopnds to a decomposition as representations.

Example 1.4 ($\mathbb{G}_m$). A finite-dimensional $\mathbb{G}_m$-representation is the same as a finite-dimensional module over $\mathbb{C}[t, t^{-1}]$. But Spec $\mathbb{C}[t, t^{-1}] = \mathbb{G}_m$, so we again "recover" the group (but things gets more subtle over a non-abelian group.

1.3. **Combining the questions.** We have an S_4 action on $\mathbb{P}^1$, which in stacky terms means that we have a diagram

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] \arrow[d] & {[\mathbb{P}^1/S_4]} \arrow[r] \arrow[d] & \mathbb{P}^1 \\ pt \arrow[r] & BS_4 & \end{tikzcd}$$

A line bundle on BS_4 is just a k-module with S_4 action, i.e. a representation of S_4, so our earlier question can be thought of as asking how to "push forward" a line bundle on $\mathbb{P}^1$ to a line bundle on BS_4.

Exercise 1.5. Try to work out how Grothendieck-Riemann-Roch implies Hirzebruch-Riemann-Roch implies Riemann-Roch (say for surfaces).

1.4. **Weighted projective space.** We saw that the action of S_4 on $\mathbb{P}^1$ factors through S_3, so let's study $[\mathbb{P}^1/S_3]$.

First of all, what is the action of S_3 on $\mathbb{P}^1$? Explicitly, it can be described as the effect on the cross-ratio induced by permuting $0, 1, \infty$. Let's calculate the effect in coordinates. Let x, y be coordinates on $\mathbb{P}^1$ and $u = x/y$. Then the elements of S_3 are

$$\begin{aligned} u &\mapsto u \\ u &\mapsto 1 - u \\ u &\mapsto 1/u \\ u &\mapsto u/(u-1) \\ u &\mapsto 1/(u-1) \\ u &\mapsto (u-1)/u \end{aligned}$$

What are the points with non-trivial stabilizers? $\{0, 1, \infty\}$ and $\{1/2, 2, -1\}$ have stabilizer of size 2, and the roots of $u^2 - u + 1 = 0$ form an orbit of size 2. We claim that we are done.

Here is a slightly sketchy way to see this. $\mathbb{P}^1$ should be a "covering space" of $[\mathbb{P}^1/S_3]$, so the latter should have Euler characteristic $2/6 = 1/3$. On the other hand, we are "removing" each point x that has a non-trivial stabilizer and pasting back in a point with "weight" $\frac{1}{\mathrm{Stab}(x)}$, for a total of $2 - 1 - 1 - 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = 1/3$. The reasoning here is that a "stacky point" with stabilizer of order 2, is $B\mathbb{Z}/2$, since it admits a double cover by a point.

Let's try to rigorously justify the claim that the inclusion of a point with $\mathbb{Z}/2$ stabilizer corresponds to a closed embedding $B\mathbb{Z}/2 \to [\mathbb{P}^1/S_3]$. First of all, what *is* the map? By definition, this is defined in terms of X-points, so what is a map from a scheme to $[\mathbb{P}^1/S_3]$? That is just the data of a map

$$\begin{tikzcd} W \arrow[r, "\text{equiv.}"] \arrow[d, "S_3"] & \mathbb{P}^1 \arrow[d] \\ X \arrow[r] & {[\mathbb{P}^1/S_3].} \end{tikzcd}$$

Given this data, we should produce an X-point of $B\mathbb{Z}/2$, i.e. a diagram

$$\begin{tikzcd} Y \arrow[r, "\text{equiv.}"] \arrow[d, "\mathbb{Z}/2"] & \mathrm{pt} \arrow[d] \\ X \arrow[r] & B\mathbb{Z}/2. \end{tikzcd}$$

where $Y \hookrightarrow W$ is a closed embedding. Indeed, we can take Y to be the pre-image of $0 \in \mathbb{P}^1$ in W

Problem 1.6. It is a fact that a closed subscheme of $[X/G]$ is a closed-subscheme of X preserved by G.

Now let's try to produce an étale cover of $[\mathbb{P}^1/S_3]$. Consider

$$\begin{tikzcd} U \arrow[r, hook, "\text{open}"] \arrow[d, "S_3"'] & \mathbb{P}^1 \arrow[d, "S_3"'] \\ V \arrow[r, hook, "\text{open}"] & {[\mathbb{P}^1/S_3]} \end{tikzcd}$$

where we want U to be a judicious choice of S_3-invariant open affine. We can take U to be $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, which is Spec $\mathbb{C}[u]_{u(u-1)}$. A natural thing to do is to form Spec $(\mathbb{C}[u]_{u(u-1)})^{S_3}$, which will end up giving the *coarse* moduli space.

Exercise 1.7. Since we threw out one orbit, the coarse space should be $\mathbb{A}^1$. Show that $(\mathbb{C}[u]_{u(u-1)})^{S_3} \cong \mathbb{C}[t]$. What is t?

We have a cover Spec $\mathbb{C}[u]_{u(u-1)} \to [\text{Spec } \mathbb{C}[u]_{u(u-1)}/S_3]$. The latter has two "stacky" points, and the map should be an isomorphism away from the stacky points. How might one prove this? If you find the corresponding values of t, and poke them out, then you should find $\mathbb{A}^1 - \{\text{two points}\}$ with an honest S_3-cover.

Using the explicit data of this étale cover, we should be able to compute with quasicoherent sheaves, etc.