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We first construct a map $Coh(X) \to K_0(X)$.
loc. free K-theory

$P_\bullet \to A$ two loc free res.

$P'_\bullet \to$

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & K' & = & K' & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & P \times_A P' & \to & P'_0 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & P_0 & \to & A & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Check: kernels of the exact sequences agree.

$$P'' \twoheadrightarrow P_0 \times_A P'_0 = \{ (m,n) \in P_0 \oplus P'_0 \mid \bar{m} = \bar{n} \text{ in } A \}.$$

$$0 \to K \times_{P \times_A P} P'' \to P'' \to P'_0 \to 0$$

$$K \times_{P \times_A P} P'' = K_0$$

K_0 is locally free, since $P'' \twoheadrightarrow P'$ a surjection (resp K'_0) of locally free sheaves.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Q & \to & K'_0 & \to & K' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_0 & \to & P'' & \to & P'_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & P_0 & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Check: $\exists$ maps $K_0 \to K$, $K'_0 \to K'$ whose kernels then agree.

We have two resolutions of K; K'

$P_{\bullet+1} \to K$ resp $P'_{\bullet+1} \to K'$

and

$K_\bullet \to K$ $\quad K'_\bullet \to K'$

and two resolutions of Q

$K_{\bullet+1}, K_{\bullet+1}$.

By induction on lengths of locally free resolutions,

(by general results we should be able to ensure length of $K_\bullet \leq$ length of $P_{\bullet+1}$)

we may assume that these resolutions give the same objects in locally free K-theory.

Thus $[K], [K'], [Q]$ are ~~defined~~ defined in K-theory, and we have the relations

$$[K] = [K_0] - [Q]$$
$$[K'] = [K_0'] - [Q]$$

, using resolutions $K_\bullet \to K$, $K_{\bullet+1} \to Q$.

Using the two SES of loc. frees in the "cross" of the above diagram, we obtain

$$[K_0] + [P_0] = [P''] = [K_0'] + [P_0'] .$$

Thus

~~[illegible]~~

~~[illegible]~~

$$[P_0] - [K] = [P_0'] - [K'] .$$

Using the resolutions $P_{\bullet+1} \to K$, $P'_{\bullet+1} \to K'$, we obtain

$$\sum(-1)^i [P_i] = \sum(-1)^i [P_i'] .$$

Thus $\mathrm{Coh}(X) \to K_0(X)$ is well-defined.

Note that we proved, along the way, that given

$$0 \to C \to P \to A \to 0 \quad \text{w/ } P \text{ loc. free},$$

$$[A] + [C] = [P] \text{ in } K_0(X).$$

For ~~[illegible]~~ general SES of coh. sheaves $0 \to C \to B \to A \to 0$, and a ~~projective~~ loc. free $P \twoheadrightarrow A$, ~~[illegible]~~ the same "~~projective~~ loc. free over fiber product" trick shows $\exists$ Q loc. free s.t. $\longrightarrow$ holds.

$$\begin{array}{ccccccccc} 0 & \to & R & \to & Q & \to & P & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C & \to & B & \to & A & \to & 0 \end{array}$$

Thus, as before, R is locally free, and this diagram exists:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C' & \to & B' & \to & A' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R & \to & Q & \to & P & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C & \to & B & \to & A & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

By induction, we may assume

$$[C'] + [A'] = [B'],$$

and we have proven, since P, Q, R loc. free, that the additivity also holds in the columns.

This implies $[C] + [A] = [B]$, as desired.

Thus $Coh(X) \to K_0(X)$ factors through $G_0(X)$.

$$
\begin{array}{ccc}
Coh(X) & \longrightarrow & K_0(X) \\
 & \searrow & \nearrow \exists \\
 & G_0(X) &
\end{array}
$$