EQUIVARIANT ALGEBRAIC GEOMETRY

JANUARY 30, 2015

1. Why the Todd class?

If we seek a characteristic class satisfying

$$\int_X \mathrm{Td}(T_X) = \chi(X, \mathcal{O}_X)$$

then perhaps one is led naturally to the Todd class(?)

2. The splitting principle

Let $V \to X$ be a vector bundle. We "wish" that $V = L_1 \oplus L_2 \oplus \ldots \oplus L_n$. One can always "achieve" this in topology by an appropriate pullback, but it is too ambitious to demand in algebraic geometry. Instead, we "wish" that there were a *filtration*

$$V \supset V_{n-1} \supset V_{n-2} \supset \ldots \supset 0$$

with each successive quotient $V_i / V_{i-1} \cong L_i$ a line bundle.

To "achieve" this we solve the corresponding *universal bundle*. For $V \to X$, we create a *flag bundle* $\mathrm{Fl}(V)$ equipped with a map $\pi \colon \mathrm{Fl}(V) \to X$, which is basically a tower of projective bundles. The key point is that the map $CH^*X \hookrightarrow CH^*(\mathrm{Fl}(V))$ is *injective*, so we can (for instance) prove identities on CH^*X bull packing back and working in $CH^*(\mathrm{Fl}(V))$.

So why is this injective? This is a general property of projective bundles, as you can create a "section" by pulling back a class and intersecting with an appropriate power of the hyperplane class coming from the tautological bundle.

3. Chern classes

We'll see shortly how to use this to define Segre classes, Chern classes, etc. Given a vector bundle $V \to X$, we form the projective bundle $\mathbb{P}(V \oplus \mathcal{O})$. The rough outline is this: to define a Segre class, we need to tell you how it acts on a cycle. So we pull it back to $\mathbb{P}(V \oplus \mathcal{O})$, and then cut it down with powers of the first Chern class of the tautological bundle until we get something of the right dimension, and then push this forward.

Before giving a technical definition, let's try to get an overview of "what are" Chern classes. A vector bundle $V \to X$ is a map $X \to BGL_n$. Now, BGL_n has a canonical flag bundle $\mathrm{Fl}(BGL_n) \to BGL_n$, and the cohomology of BGL_n embeds into that of the flag bundle. We can compute the generators of the cohomology of BGL_n, and these we will call $c_1, \dots, c_n$. Now, $\mathrm{Fl}(BGL_n)$ has a vector bundle with a canonical filtration with associated graded $\mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_n$, and the cohomology ring is $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ where $\alpha_i = c_1(\mathcal{L}_i)$. The inclusion of the cohomology of BGL_n is the inclusion of the elementary symmetric polynomials in the α_i.

One could then define the Chern classes purely in terms of c_1 (defined as a Chow class of $B\mathbb{G}_m$).

So how do you compute the Chow groups? We have a map

$$B\mathbb{G}_m \times \dots \times B\mathbb{G}_m \to B(\mathcal{B}).$$

We claim that this is an $\mathbb{A}^n$-bundle. The content of this is that the pull-back to a scheme is the association of a product of line bundles to a Borel-bundle.

By the way, why does an $\mathbb{A}^n$-bundle have the same Chow? First, take the projective completion to get a $\mathbb{P}^n$-bundle. Then we use the same cutting-with-hyperplane trick.

So we have to somehow show that the map $\mathrm{Fl}(BGL_n) \to BGL_n$ induces the claim inclusion of Chow rings. Let's come back to this issue.

Let's talk more about the first Chern class. Let $\mathcal{L} \to X$ be a line bundle corresponding to an effective Cartier divisor D and $s\colon X \to \mathcal{L}$ a section, then we claim that

$$c_1(\mathcal{L})[X] = [D].$$

Furthermore, if $Z \subset X$ represents a cycle intersecting D "nicely enough," then we want

$$c_1(\mathcal{L})[Z] = [Z \cap D].$$

We'll then use this to justify the intuitive description of Chern classes from last time: if $V \to X$ is a rank n vector bundle and s is a section, then $\{s = 0\}$ is codimension n and $c_n(V)[X] = \{s = 0\}$. How might one prove this? By the splitting principle, it suffices to do this assuming we have a filtration, and then reduce it to the case of line bundles.

Furthermore, if s, t are two sections then the locus where s and t are linearly dependent has codimension $n-1$, then $c_{n-1}(V)$ is that locus. Presumably we will prove this again by a similar reduction.

Exercise 3.1. Try to work this out for rank 2 vector bundles to see why it is reasonable.

4. Back to Grothendieck-Riemann-Roch

We have to finish off the proof of GRR, i.e. for smooth varieties X, Y and a projective morhipsm $\pi \colon X \to Y$ then we have a commutative diagram

$$\begin{tikzcd} K_0(X) \arrow[r, "\mathrm{ch} \cdot \mathrm{Td}(X)"] \arrow[d, "\pi_!"'] & CH^*_{\mathbb{Q}}(X) \arrow[d, "\pi_*"'] \\ K_0(Y) \arrow[r, "\mathrm{ch} \cdot \mathrm{Td}(Y)"] & CH^*_{\mathbb{Q}}(Y) \end{tikzcd}$$

We already did the case $\mathbb{P}^n \times Y \to Y$. We're in the middle of proving it for $X \hookrightarrow V$ as the 0-section of a vector bundle, which is a precursor to the case of a regular embedding.

We had a construction

$$\begin{tikzcd} V \arrow[r] \arrow[d, shift left, "\pi"] & \mathbb{P}(V \oplus \mathcal{O}) \\ X \arrow[u, shift left, "p"] & \end{tikzcd}$$

On the projective closure, we have the tautological sequence $0 \to \mathcal{O}(-1) \to p^*(V) \oplus \mathcal{O} \to Q \to 0$. So we have a section $\mathcal{O} \to Q$, vanishing on the $\pi(X) \subset V \subset \mathcal{P}(V \oplus \mathcal{O})$.

We want to prove the commutativity of the diagram, so let's check that for vector bundles (since those generate). Let $\mathcal{E} \to X$ be a vector bundle. Then $[\pi_* \mathcal{E}] \in K(Y)$ (an honest pushforward since the morphism is affine, being a closed embedding). Now this fellow isn't a vector bundle, and we only really know how to compute chern classes of vector bundles, so we want to resolve this by vector bundles.

First, let's give a resolution of the structure sheaf. First let's do a special case - the Koszul complex for the origin in $\mathbb{A}^n$. Suppose X is the origin and $\mathbb{A}^n$ is the trivial vector bundle. Then we want to resolve $k[x_1, \dots, x_n]/(x_1, \dots, x_n)$. Let $R = k[x_1, \dots, x_n]$. The natural thing to do is take the free guy on the generators:

$$Rx_1 \oplus \dots \oplus Rx_n \to R \to k[x_1, \dots, x_n]/(x_1, \dots, x_n) \to 0.$$

But there are obvious relations here, as Rx_1 and Rx_2 cancel out an $x_2x_1 - x_1x_2$. So we have to put stuff in for more terms, blah blah.

Globally, this looks like

$$\dots \to \bigwedge^2 Q^\vee \to Q^\vee \to \mathcal{O}_Y \to \pi_* \mathcal{O}_X \to 0.$$

(The dual is necessary because the x_i are considered as *functions*.) Then tensor with $p^* \mathcal{E}$ to get a resolution of $\pi_* \mathcal{E}$.

Let $\alpha_1, \dots, \alpha_n$ be the Chern roots of Q. The chern character is multiplicative, so

$$\mathrm{ch}(\pi_* \mathcal{E}) = \mathrm{ch}(\bigwedge^\bullet Q^\vee \otimes p^*(\mathcal{E})) = \mathrm{ch}(\bigwedge^\bullet Q^\vee) \cdot \mathrm{ch}(p^* \mathcal{E}).$$

The first guy is

$$\sum_p (-1)^p \,\mathrm{ch}(\bigwedge^p Q^\vee).$$

The chern roots of $Q^\vee$ are $-\alpha_1, \dots, -\alpha_n$, so the chern roots of $\bigwedge^p Q^\vee$ are p-fold products of the Chern roots of $Q^\vee$, and so in the end one gets

$$\sum_p (-1)^p \sum e^{-\alpha_{i_1}} \dots e^{-\alpha_{i_p}}$$

hence

$$\mathrm{Td}(N)^{-1} \,\mathrm{ch}_n(Q) = \prod_{i=1}^{p} \frac{(1 - e^{-\alpha_i})}{\alpha_i} \prod \alpha_i.$$

Here we are using $\pi^* Q \cong V$. So from $[\mathcal{E}]$ we get $\frac{c_n(N)}{\mathrm{Td}(N)} \mathrm{ch}(p^* \mathcal{E})$. (N is the normal bundle is V). In the GRR diagram, we get

$$\begin{tikzcd} {[\mathcal{E}]} \arrow[d] & \\ {} \arrow[r] & \pi_* \left(\frac{c_n(V)}{\mathrm{Td}(V)} \right) \mathrm{ch}(p^* \mathcal{E}) \cdot \mathrm{Td}(Y) = ? = \pi_* \,\mathrm{Td}(X)\, \mathrm{ch}(\mathcal{E}). \end{tikzcd}$$

Remark 4.1. If we think of how one might arrive at the GRR formula, we see that there should be a Todd class and a chern class, one multiplicative and one additive, and perhaps understanding this for projective space gives what we want.

So why are these two things equal? $\mathrm{Td}(N_{X/Y}) = \frac{\pi^* \mathrm{Td}(T_y)}{\mathrm{Td}(T_x))}$ just from multiplicativity and the short exact sequence for the normal bundle. Using this to compare the expressions we will get what we want.