EQUIVARIANT ALGEBRAIC GEOMETRY

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1. Finishing GRR

We're going to finish the proof of Grothendieck-Riemann-Roch for schemes.

$$\begin{tikzcd} K_0(X)_{\mathbb{Q}} \arrow[r, "ch \cdot Td(X)"] \arrow[d, "\pi_!"'] & CH^*_{\mathbb{Q}}(X) \arrow[d, "\pi_*"'] \\ K_0(Y)_{\mathbb{Q}} \arrow[r, "ch \cdot Td(Y)"] & CH^*_{\mathbb{Q}}(Y) \end{tikzcd}$$

It is also true that the horizontal maps are isomorphisms, which we'll see come out of the proof.

We know the result when $X = \mathbb{P}^n \times_Y$ and the map is projection to Y. We want to know the case of a closed embedding (which is necessarily regular since X and Y are both assumed to be smooth).

We studied the special case where $X \hookrightarrow Y$ is the 0-section of a rank d vector bundle N $(= Y)$. We did this special case by studying the compactification $\mathbb{P}(N \oplus \mathcal{O})$.

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \mathbb{P}(N \oplus \mathcal{O}) \\ X & \end{tikzcd}$$

The point was that there is a tautological exact sequence

$$0 \to \mathcal{O}(-1) \to p^*(N) \oplus \mathcal{O} \to Q \to 0.$$

There is a section of the quotient Q (which we think of as the hyperplane clas) such that $V(s) = X$. Also, $Q|_X = \pi^* Q \cong N$ (since everything here is natural, you can check this in local coordinates).

In this case the Koszul complex is

$$\ldots \to \bigwedge^2 Q^\vee \to Q^\vee \xrightarrow{s^\vee} \mathcal{O}_Y \to \pi_* \mathcal{O}_X \to 0.$$

We saw a slick way of checking exactness last time, but we remark that another, less clever way is to just define the maps and use local coordinates.

Now we just compute the two paths of the diagram, and hope that they agree. Take $\mathcal{E}$ to be a vector bundle on X. We want $[\pi_*\mathcal{E}] \in K(Y)$. Since $\pi_*\mathcal{E}$ isn't a vector bundle, we need to resolve it by vector bundles. Fortunately, we have the Koszul complex on X and we can tensor with $p^*\mathcal{E}$: we get

$$\ldots \to p^*\mathcal{E} \otimes Q^\vee \to p^*\mathcal{E}|_Y \to \pi_*\mathcal{E} \to 0$$

so in $K(Y)$,

$$\pi_*\mathcal{E} = \sum_{i=0}^{d} (-1)^i [\bigwedge^i Q^\vee \otimes p^*\mathcal{E}].$$

Now applying the Chern character, we get

$$\mathrm{ch}(\pi_*\mathcal{E}) = \mathrm{ch}\left(\sum_{i=0}^{d} (-1)^i [\bigwedge^i Q^\vee]\right) \cdot \mathrm{ch}(p^*\mathcal{E}).$$

After some computation (which we did last time), we end up with

$$\frac{c_d(Q)}{\mathrm{Td}(Q)} \mathrm{ch}(p^*\mathcal{E}).$$

On the other hand, it is a general fact that if $X \hookrightarrow Y$ is a regular embedding with normal bundle N, then $c_d(N) \cap \beta = \pi_*\pi^*\beta$ (you could prove this using the universal case). Think of this as "intersecting with X."

Therefore,

$$\begin{aligned}\frac{c_d(Q)}{\mathrm{Td}(Q)} \mathrm{ch}(p^*\mathcal{E}) = \pi_* \frac{1}{\mathrm{Td}(N)} \pi^* \mathrm{ch}(p^*\mathcal{E}) \quad &= \pi_* \frac{\mathrm{Td}(T_X)}{\pi^* \mathrm{Td}(T_Y)} \mathrm{ch}(\mathcal{E}) \\ &= \frac{\pi_*(\mathrm{ch}(\mathcal{E}) \mathrm{Td}(T_X))}{\mathrm{Td}(T_Y)}\end{aligned}$$

by the projection formula (since π^* is multiplicative, $1/\pi^* \mathrm{Td}(T_y) = \pi^* \mathrm{Td}(T_y)^{-1}$.

Now we do the general case of closed embeddings. We use the algebro-geometric analogue of the "normal bundle," *deformation to the normal cone*. Let X be a closed subscheme of Y (in our case, even a regular embedding). To "get at" the tubular neighborhood, we form $Y \times \mathbb{P}^1$, which has the subscheme $X \times \mathbb{P}^1$. This maps to $\mathbb{P}^1$ by the obvious projection. We then blow up $X \times \{\infty\}$. What happens above ∞? We get two components: $\mathrm{Bl}_X Y$ and $\mathbb{P}(N \oplus \mathcal{O})$, which meet along $\mathbb{P}(N)$. This puts X inside the normal bundle inside $\mathbb{P}(N \oplus \mathcal{O})$. You can check all this locally. (A useful fact is that the exceptional divisor

is always the projectivization of the normal bundle, and the normal bundle of X in $Y \times \mathbb{P}^1$ is $N \oplus \mathcal{O}$.)

One way to visualize this is if we forget about the $\mathrm{Bl}_X Y$ then we are analytically "zooming in." Why is it flat? As the base is a smooth proper curve, one just has to see that there are no associated points, but that is a general property of any blowup.

We have

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\pi"'] & X \times \mathbb{P}^1 \arrow[d, "\Pi"'] & X \arrow[l] \arrow[d] \\
Y = M_0 \arrow[r] \arrow[d] & M = \mathrm{Bl}_{X\times\infty} Y \times \mathbb{P}^1 \arrow[d] & \mathbb{P}(N \oplus \mathcal{O}) \cup \mathrm{Bl}_X Y \arrow[l, "k"'] \arrow[d] \\
0 \arrow[r] & \mathbb{P}^1 & \infty \arrow[l]
\end{tikzcd}$$

Let $\mathcal{E}$ be a vector bundle on X. Let's find $[\pi_* \mathcal{E}]$. We instead resolve $\Pi_* \mathrm{pr}^* \mathcal{E}$, the pushforward of a vector bundle on $X \times \mathbb{P}^1$.

$$G_\bullet \to \Pi_* \mathrm{pr}^* \mathcal{E} \to 0.$$

Since everything is flat over $\mathbb{P}^1$, this remains exact upon restricting to a fiber ♠♠♠ TONY: [blowup of flat along flat is flat?]. Since 0 and ∞ are "homologous," we get the same class upon restriction to 0 and ∞. When we restrict $\Pi_* \mathrm{pr}^* \mathcal{E}$ to 0 we just ge precisely $\pi_* \mathcal{E}$. So $j_0^* \mathcal{G}_\bullet$ is a resolution of $\pi_* \mathcal{E}$, so

$$\mathrm{ch}(\pi_* \mathcal{E}) = \mathrm{ch}(j_0^* \mathcal{G}_\bullet) = j_0^* \,\mathrm{ch}(\mathcal{G}_\bullet).$$

This is on Y, but since we want to transfer things along M, we push it forward to M:

$$j_{0*} \,\mathrm{ch}(\pi_* \mathcal{E}) = j_0^* j_0^* \,\mathrm{ch}(\mathcal{G}_\bullet) = \mathrm{ch}(\mathcal{G}_\bullet) \cap M_0 = \mathrm{ch}(\mathcal{G}_\bullet) \cap M_\infty.$$

But we know $[M_\infty] = [\mathrm{Bl}_X Y] + [\mathbb{P}(N \oplus \mathcal{O})]$. But as $\mathcal{G}$ resolved something living on $X \times \mathbb{P}^1$, it is exact on $[\mathrm{Bl}_X Y]$ so it is 0 there. But this is just $k_* \,\mathrm{ch}(\pi_{\infty *} \mathcal{E})$, and by the special case we did, this is

$$= k_*(\mathrm{Td}(T_Y)^{-1}(\pi_{\infty *}(\mathrm{ch}(\mathcal{E}))\,\mathrm{Td}(T_X)).$$

We need to push this back to Y, and when we do that we get

$$PR_* j_{0*} \mathrm{ch}(\pi_* \mathcal{E}) = \mathrm{ch}(\pi_* \mathcal{E})$$

and we get the right thing on the right hand side too.